Computationally Efficient Waveform Design in Spectrally Dense Environment

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Abstract—The problem of unimodular radar waveform design with similarity constraint in spectrally dense environment is considered. The corresponding optimization problem is non-convex and additionally large-scale. Indeed, the waveform length can be of several thousands and the waveform has to be designed in every coherent processing interval. Since the problem is non-convex, the majorization-minimization (MaMi) method is used for addressing it, and since it is large-scale, the alternating direction method of multipliers (ADMM) is adopted. Thus, we develop a computationally efficient algorithm for solving the problem by using MaMi with proper design of majorization function together with alternating ADMM with newly proposed computationally efficient proximal projections. We evaluate the computational cost of the proposed algorithm and show its fast convergence unmatched by existing approaches in terms of simulations.

I. INTRODUCTION

Waveform design is one of the classical topics in radar and communications [1], [2], [3], [4], [5], [6], [7]. Recently, the problem of spectrally constrained radar waveform design [8] has generated noticeable interest due to its practical significance for spectrally dense environments and thanks to the advances in optimization methods. The renewed interest was triggered as well thanks to the introduction of the so-called similarity constraint that ensures the similarity of a designed waveform to a waveform with desirable properties, for example, desirable ambiguity function [9], [10], [11]. The corresponding optimization problem of unimodular waveform design with similarity constraint for spectrally dense environments can be described as the maximization of radar signal to interference plus noise ratio (SINR) over a non-convex region [10], [11]. Despite the problem in practice belongs to the class of large-scale optimization problems, the solution given in [10], [11] has prohibited large complexity. Indeed, the radar signal is sampled at GHz level which implies that fast-time radar code vector has large dimension even for short pulses. In order to design fast-time radar code vector with large dimension, efficient solution methods, e.g., [12], [13], [14], need to be considered.

In this paper, a new computationally efficient (suitable for large-scale problems) method for solving the optimization problem of unimodular radar waveform design with similarity constraint for spectrally dense environments is developed.

The method uses convex surrogate function to upper-bound objective, and optimizes such upper bound, i.e., perform majorization minimization (MaMi) [15], [16], [17], [18], [19], using efficient alternating direction method of multipliers (ADMM) approach [20], [21], [22]. ADMM is a popular and efficient optimization approach that is widely used in a number of applications, including the unimodular waveform design [23]. The computational performance of the proposed method is tested by simulations, it is unmatched by the existing approaches.

II. PROBLEM FORMULATION

Let \( \mathbf{c} \triangleq (c[1], c[2], \ldots, c[K])^T \subset \mathbb{C}^K \) denote the fast-time \( K \)-dimensional complex valued baseband radar code waveform where \( \{c[k]\}_{k=1}^K \) are the fast-time radar code elements and \( (\cdot)^T \) stands for transpose. The fast-time received signal \( \mathbf{v} \) is an attenuated, time- and frequency shifted, and noisy version of the fast-time radar waveform given as \( \mathbf{v} = \alpha \mathbf{c} + \mathbf{n} \), where \( \alpha \in \mathbb{C} \) is the parameter that captures the attenuation, backscattering, and channel propagation effects, \( \mathbf{n} \in \mathbb{C}^N \) is the noise component usually modelled as Gaussian distributed, zero-mean random vector with covariance matrix \( \mathbf{M} \triangleq \mathbb{E} \{ \mathbf{n} \mathbf{n}^H \} \), and \((\cdot)^H \) is Hermitian transpose.

Matched filtering the fast-time radar waveform at the receiver, the SINR is maximized by choosing the filter as \( \mathbf{h} = \mathbf{M}^{-1} \mathbf{c} \). The maximum SINR value can be then expressed in terms of the following quadratic form

\[
\alpha^2 \mathbf{c}^H \mathbf{M}^{-1} \mathbf{c} = \alpha^2 \mathbf{c}^H \mathbf{R} \mathbf{c}. 
\]

(1)

The energy radiated by the fast-time radar waveform over normalized bandwidth \( \Omega_k = \{f_1^k, f_2^k\} \) can be written as the following quadratic form

\[
\sum_{k=1}^{K} w_k \int_{\Omega_k} |F_k(\mathbf{c})|^2 df = \mathbf{c}^H \mathbf{R}_k \mathbf{c} 
\]

(2)

where \( \{w_k\}_{k=1}^K \) are non-negative weights, \( F_k(\mathbf{c}) \) stands for the discrete-time Fourier transform of \( \mathbf{c} \) given as \( \mathcal{F}_k(\mathbf{c}) \triangleq \sum_{k=1}^{K} c[k] e^{-j2\pi k f} \), and \( \mathbf{R}_k \triangleq \sum_k w_k \mathbf{R}_k^k \) with \( [\mathbf{R}_k^k]_{m,l} = (e^{j2\pi f_2^k (m-l)} - e^{j2\pi f_1^k (m-l)}) / e^{j2\pi (m-l)} \), if \( m \neq l \), and \( [\mathbf{R}_1^k]_{m,l} = f_2^k - f_1^k \) if \( m = l \).
Using (1) and (2) and constraining \( c \) to be a unit vector close to the known reference waveform \( c_0 \) with desired ambiguity function, the SINR-maximizing optimization problem for radar waveform design can be written as [10]

\[
\mathcal{P}_1 : \begin{cases}
\max_{\mathbf{c}} & \alpha \mathbf{c}^H \mathbf{R} \mathbf{c} \\
\text{s.t.} & \|\mathbf{c}\|^2 = 1 \\
& \mathbf{c}^H \mathbf{R} \mathbf{c} \leq E_t \end{cases}
\]

where \( E_t \) is the energy limit, \( \epsilon \) is the similarity accuracy bound, and \( \| \cdot \| \) denotes the Euclidian norm of a vector. The problem \( \mathcal{P}_1 \) is non-convex and NP-hard, in general, but needs to be efficiently and possibly suboptimally solved.

### III. Efficient Algorithm

Using the independence of real and imaginary parts, the complex valued matrix \( \mathbf{R} \in \mathbb{C}^{K \times K} \) and vectors \( \mathbf{c}, \mathbf{c}_0 \in \mathbb{C}^K \) can be equivalently written in real-valued notations as \( \mathbf{R} \triangleq \left( \begin{array}{cc} \text{Re} (\mathbf{R}) & -\text{Im} (\mathbf{R}) \\ \text{Im} (\mathbf{R}) & \text{Re} (\mathbf{R}) \end{array} \right) \), \( \mathbf{c} \triangleq \left( \begin{array}{c} \text{Re} (\mathbf{c}) \\ \text{Im} (\mathbf{c}) \end{array} \right) \) and \( \mathbf{c}_0 \triangleq \left( \begin{array}{c} \text{Re} (\mathbf{c}_0) \\ \text{Im} (\mathbf{c}_0) \end{array} \right) \). Because \( \mathbf{R} \) is the inverse of the noise covariance matrix, and therefore positive definite, the objective in \( \mathcal{P}_1 \) is convex. When \( \mathcal{P}_1 \) is written as minimization problem, namely \( \mathcal{P}_1^{\text{min}} \), objective function is concave, and need to be convexified. Using MaMi, i.e., majorizing the objective in \( \mathcal{P}_1^{\text{min}} \) by replacing \( \mathbf{Q} \) with \( \mathbf{Q} = \mu \mathbf{I} - \mathbf{R} \), where \( \mu \in \mathbb{R}_+ \) is chosen such that \( \mathbf{Q} \succeq 0 \), the optimization problem \( \mathcal{P}_1^{\text{min}} \) can be majorized by the following real-valued problem

\[
\mathcal{P}_2 : \min_{\mathbf{c}} \mathbf{c}^T \mathbf{Q} \mathbf{c} \quad \text{s.t.} \quad (3b), (3c), (3d)
\]

where the real-valued versions of the constraints (3b), (3c), and (3d) are used. The problems \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) have the same optimal point and are equivalent in this sense.

Our objective is to solve \( \mathcal{P}_2 \) efficiently and at least sub-optimally. Among existing accelerated methods, the ADMM method is known to be efficient and appropriate for addressing non-convex problems [22]. Introducing the slack variable \( \mathbf{z} \) such that \( \mathbf{c} = \mathbf{z} \), the augmented Lagrangian \( L_\rho (\mathbf{c}, \mathbf{z}, \lambda) \) for the problem \( \min_{\mathbf{c}} \mathbf{c}^T \mathbf{Q} \mathbf{c} \) s.t. \( \mathbf{c} = \mathbf{z} \) i.e., the problem \( \mathcal{P}_2 \) without the constraints, can be written as

\[
L_\rho (\mathbf{c}, \mathbf{z}, \lambda) = \mathbf{c}^T \mathbf{Q} \mathbf{c} + \lambda^T (\mathbf{c} - \mathbf{z}) + \frac{\rho}{2} \| \mathbf{c} - \mathbf{z} \|^2
\]

and the ADMM-updates for \( \mathcal{P}_2 \) with constraints enforced while updating corresponding optimization variables are

\[
\begin{aligned}
\mathbf{c}_{l+1} &= \arg \min_{\|\mathbf{c}\|^2 = 1, \|\mathbf{c} - \mathbf{c}_0\| \leq \epsilon} L_\rho (\mathbf{c}, \mathbf{z}_l, \lambda_l) \\
\mathbf{z}_{l+1} &= \arg \min_{\mathbf{z}^H \mathbf{R} \mathbf{z} \leq E_t} L_\rho (\mathbf{c}_{l+1}, \mathbf{z}, \lambda_l) \\
\lambda_{l+1} &= \lambda_l + \rho (\mathbf{c}_{l+1} - \mathbf{z}_{l+1})
\end{aligned}
\]

where \( \lambda \) is the vector of Lagrange multipliers, \( \rho \) is a positive constant, \( i \) is the iteration index, and the constraints of \( \mathcal{P}_2 \) will be addressed by computationally simple projections.

### c-variable update: Constraints (3b) and (3d) need to be enforced during update step (6a). Then this update can be written as

\[
\mathbf{c}_{l+1} = \arg \min_{\|\mathbf{c}\|^2 = 1, \|\mathbf{c} - \mathbf{c}_0\| \leq \epsilon} L_\rho (\mathbf{c}, \mathbf{z}_l, \lambda_l)
\]

where \( h(\mathbf{c}) \triangleq \mathbf{c}^T \mathbf{Q} \mathbf{c} + (\mathbf{\lambda} - \mathbf{\rho} \mathbf{z})^T \mathbf{c} \). Since the objective \( h(\mathbf{c}) \) is continuously differentiable and \( \nabla h(\mathbf{c}) \) is \( L \)-Lipschitz continuous, we find that

\[
\nabla h(\mathbf{c}) = \left( \mathbf{Q} + \mathbf{Q}^T \right) \mathbf{c} + (\mathbf{\lambda} - \mathbf{\rho} \mathbf{z})
\]

\[
\left| \nabla h(\mathbf{c}) - \nabla h(\mathbf{\lambda}) \right| = \left| \left( \mathbf{Q} + \mathbf{Q}^T \right) (\mathbf{\lambda} - \mathbf{\rho} \mathbf{z}) \right| \leq L |\mathbf{\lambda} - \mathbf{\rho} \mathbf{z}|
\]

\[
\Rightarrow \left| \left( \mathbf{Q} + \mathbf{Q}^T \right) \mathbf{\lambda} \right| \leq L, \quad i = 1, \cdots, 2K
\]

Moreover, \( \hat{\mathbf{c}}_{l+1} \) needs to be rotated and projected back to the region \( \Theta \). The corresponding low complexity algorithm is summarized in Algorithm 1. Indeed, the combination of the update (8) followed by normalization and Algorithm 1 solves the problem \( \min_{\mathbf{c}} h(\mathbf{c}) \) s.t. \( \mathbf{c} \in \Theta \).

The update can be expressed as

\[
\begin{aligned}
\mathbf{y}_{l+1} &= \mathbf{c}_l - \frac{1}{L} \nabla h(\mathbf{c}_l) \\
\mathbf{c}_{l+1} &= \arg \min_{\mathbf{c} \in \Theta} \| \mathbf{y}_{l+1} - \mathbf{c} \|
\end{aligned}
\]
The region $\Theta$ is a part of $2K$-dimensional sphere surface. It implies that $c \in \Theta$ can be expressed in spherical angular coordinates $\phi = [\phi_1, \ldots, \phi_{2K-1}]$, and (9b) written as

$$c_{l+1} = \arg \min_{\phi \in \Omega} \| y_{l+1} - c(\phi) \| (10)$$

where $\Omega = \{ \phi \in \mathbb{R}^{2K-1} | \| c(\phi) - c_0(\phi) \|^2 \leq \epsilon \}$.

Let $\phi' \in \mathbb{R}^{2K-1}$ denote the optimal $\phi$, i.e., $h(c(\phi')) \leq h(c(\phi))$, and $\phi'$ is reached by projection $y_{l+1} = \arg \min_{\phi \in \mathbb{R}^{2K-1}} \| y_{l+1} \| = \phi^*$. It can be then shown that $h(c(\phi))$ is convex in $\Omega$ by writing the Taylor expansion of $h(c(\phi))$ as follows

$$h(c+r) - h(c) = r^T \nabla_c h(c) + \frac{1}{2} r^T \nabla^2 h(c) r, \forall r \in \mathbb{R}^{2K} (11)$$

where $r$ is a sufficiently small step in any direction. It can be seen from (11) that $h(c(\phi))$ satisfies the convexity property $h(c+r) - h(c) \geq r^T \nabla_c h(c)$. Choosing $r$ such that $|c+r-c_0| \leq \epsilon$, it becomes obvious that $h(c(\phi))$ is convex in $\Theta$. The operation $c+r \in \Theta$ is equivalent to the rotation $c(\phi + \phi')$ for some $\phi' \in \mathbb{R}^{2K-1}$ such that $\phi + \phi' \in \Omega$. Since $h(c(\phi))$ is convex and the Hessian $\nabla^2 h(c) \geq 0$, $h(c_0) \leq h(\phi_0)$ if $\| \phi_0 \| = \| \phi_0 - \phi' \| < \| \phi_0 - \phi' \|$. Indeed, $r_1$ and $r_2$ can be chosen such that $|r_1| < |r_2|$, $c(\phi') + r_1 \in \Theta$, and $c(\phi') + r_2 \in \Theta$, and (10) rewritten as

$$\phi_{l+1} = \arg \min_{\phi \in \Omega} \| \phi' - \phi \| (12a)$$

$$c_{l+1} = c(\phi_{l+1}) (12b)$$

Finally, it is important to notice that although (9b) is not convex, the problem (12a) is convex, and it can be solved by simple rotation introduced in Algorithm 1.

**z-variable update:** Constraint (3c) needs to be enforced during update step (6b). Then the update can be written as

$$z_{l+1} = \arg \min_{z} L_p(c_{l+1}, z, \lambda_I)$$

$$= \arg \min_{z} u(z), \text{ s.t. } z^T R_I z \leq E_1 (13)$$

where $u(z) \triangleq \lambda^T (c - z) + \frac{\rho}{2} \| c - z \|^2$.

Rewriting the objective function $u(z)$, the minimization problem (13) can be written as

$$z_{l+1} = \arg \min_{z} \| z - (c + \frac{1}{\rho} \lambda) \|^2, \text{ s.t. } z^T R_I z \leq E_1 (14)$$

The Lagrangian for (14) is given as

$$L(z, \gamma) = \| z - (c + \frac{1}{\rho} \lambda) \|^2 + \gamma (z^T R_I z - E_1) (15)$$

where $\gamma$ is the Lagrange multiplier. Then the Karush-Kuhn-Tucker (KKT) conditions are

$$\nabla z L(z^*, \gamma^*) = 0 \Rightarrow (I + \gamma^* R_I) z^* = c + \frac{1}{\rho} \lambda (16)$$

$$\gamma^* (z^*^T R_I z^* - E_1) = 0 (17)$$

where critical points of the Lagrangian $L(z, \gamma)$ are denoted as $(z^*, \gamma^*)$ and $I$ is the identity matrix. Using (16) and (17), the update for $z_{l+1}$ can be obtained as

$$z_{l+1} = \left( I + \frac{\gamma_{l+1} \sigma_k}{1 + \gamma_{l+1} \sigma_k} p_k^T p_k^T \right) \left( c + \frac{1}{\rho} \lambda \right) (18)$$

where $\sigma_k$ is $k$th eigenvalue and $p_k$ is the corresponding eigenvector of $R_I$, which can be precomputed for given $R_I$. Now only $\gamma_{l+1} > 0$ needs to be found as the solution to (17), i.e., by solving the equation

$$z_{l+1}^T R_I z_{l+1} = E_1 \Leftrightarrow \sum_{k=1}^{2K} \frac{a_k \sigma_k}{(1 + \gamma_{l+1} \sigma_k)^2} - E_1 = 0 (19)$$

where $a_k \triangleq (p_k^T (c + \frac{1}{\rho} \lambda))^2$.

Finally, all steps (6a)–(6c) can be summarized in one procedure (see Algorithm 2) where the operation with the highest complexity is the matrix-vector product.

**IV. SIMULATION RESULTS**

In our simulation example, we aim at evaluating the performance of Algorithm 2 in terms of its time complexity, convergence speed, and quality of the designed signal. The latter is evaluated in terms of the designed signal frequency spectrum and ambiguity function.

Consider a radar system with 6 GHz bandwidth that has been argued to be important for 5G systems [24]. Then the radar code/waveform design problem in spectrally busy environment appears, for example, in joint vehicular communication-radar systems. The sampling frequency is $f_s = 12$ GHz. The fast-time radar waveform has length $T = 1\mu s$ that is equivalent to $K = 12000$. The radar operates in the environment with seven constrained bandwidths $\Omega_k = \{0, 0.0017, 0.07, 0.1247, 0.01526, 0.254, 0.3086, 0.3827, 0.4074, 0.4938, 0.6185, 0.76, 0.82, 0.95\}$.

**Algorithm 2:** MaMi Algorithm

1. function MaMi($Q, c_0, R_I, E_1, \epsilon, K$);
2. Input $Q, c_0, R_I$ or $\sigma_k, p_k$\textsuperscript{2K} $k=1$, $E_1$, $\epsilon$ and $K$
3. Output $c$
4. Initialize $c$, $z$ and $\lambda$;
5. for $l = 0, l \leq \text{max number}$, $l = l + 1$
6. $c_{l+1} = c_l - \frac{1}{2} \left( Q + Q^T \right) c_l - (\lambda - \rho z_l)$;
7. $c_{l+1} = \frac{c_{l+1}}{\| c_{l+1} \|}$;
8. $c_{l+1} = \text{RotateVector}(c_{l+1}, c_0, \alpha, \epsilon)$;
9. Solve (20) for $\gamma_{l+1} > 0$;
10. $z_{l+1} = (I - \sum_{k=1}^{2K} \frac{\gamma_{l+1} \sigma_k}{1 + \gamma_{l+1} \sigma_k} p_k^T p_k^T) (c + \frac{1}{\rho} \lambda)$;
11. $\lambda_{l+1} = \lambda_l + \rho (c_{l+1} - z_{l+1})$;
12. end
The noise and jammer covariance matrix is modeled as
\[ \mathbf{M} = \sigma_0 \mathbf{I} + \sum_{k=1}^{J} \sigma_{J,k} \mathbf{R}_k + \sum_{k=1}^{K} \sigma_{J,k} \mathbf{R}_{J,k} \] with exactly same parameter values as in [10]. The following linearly modulated signal
\[ \mathbf{c}_0 = e^{j2\pi(f_\Delta n/f_s + f_0)n/f_s} \] with frequency range \( f_\Delta = 3.6 \text{GHz}/\mu s \) and carrier frequency \( f_0 = 1.8 \text{GHz} \) is used as the reference signal, while the similarity region is given by \( \epsilon = 0.9 \). The total energy is constrained by \( E_I = 0.87 \).

Fig. 1 demonstrates the time-complexity of Algorithm 2 when it runs on HP Z240 Tower Workstation with Xeon E3-1230v5 3.40 GHz 8 MB processor. The actual runtime is shown for Algorithm 2 in the figure, while the other plots demonstrate different complexity orders for comparison. It can be seen by comparing slopes of the plots that Algorithm 2 has at most quadratic complexity. The existing method of [11] is based on semidefinite programming (SDP) and has dramatically higher complexity (\( n^{3.5} \) in Fig. 1) than the proposed algorithm. In Fig. 2, convergence speed of Algorithm 2 is shown alongside waveform SINR value with \( \alpha = 1 \). In Fig. 3, frequency spectrum of the designed waveform is shown. It can be seen that the waveform uses most of the available bands, except for two narrowest. The magnitude of the waveform is higher by 20 dB for available bands as compared to the constraint bands – \( E_I \). Finally, the ambiguity function of the designed waveform is shown in Fig. 4. It has good (narrow) autocorrelation properties with small Doppler leakages. These ambiguity characteristics are typical for linearly modulated signals.

V. CONCLUSION

An efficient algorithm for unimodular radar code/waveform design in spectrally dense environment has been developed. The algorithm consists of MaMi step, in which a non-convex objective function is upper-bounded by convex surrogate function, and ADMM steps, in which convex surrogate function is made separable by introducing a slack variable. The corresponding ADMM subproblems are efficiently solved using newly developed computationally efficient proximal projections, so that the complexity order of the proposed algorithm is at most quadratic. The algorithm has been used for waveform design in an example environment which included several constrained frequency bands. The designed fast-time radar waveform satisfies the constraints on frequency bands and its ambiguity function is similar to that of the reference signal. The algorithm has low complexity and fast convergence speed, which makes it practically applicable since it can easily execute in a duration of single CPI.
VI. REFERENCES


