Massively Parallel Algorithms for Relaxations of MIS

Shreyas Pai
IIT Madras

Algorithms for Massive Graphs 2024

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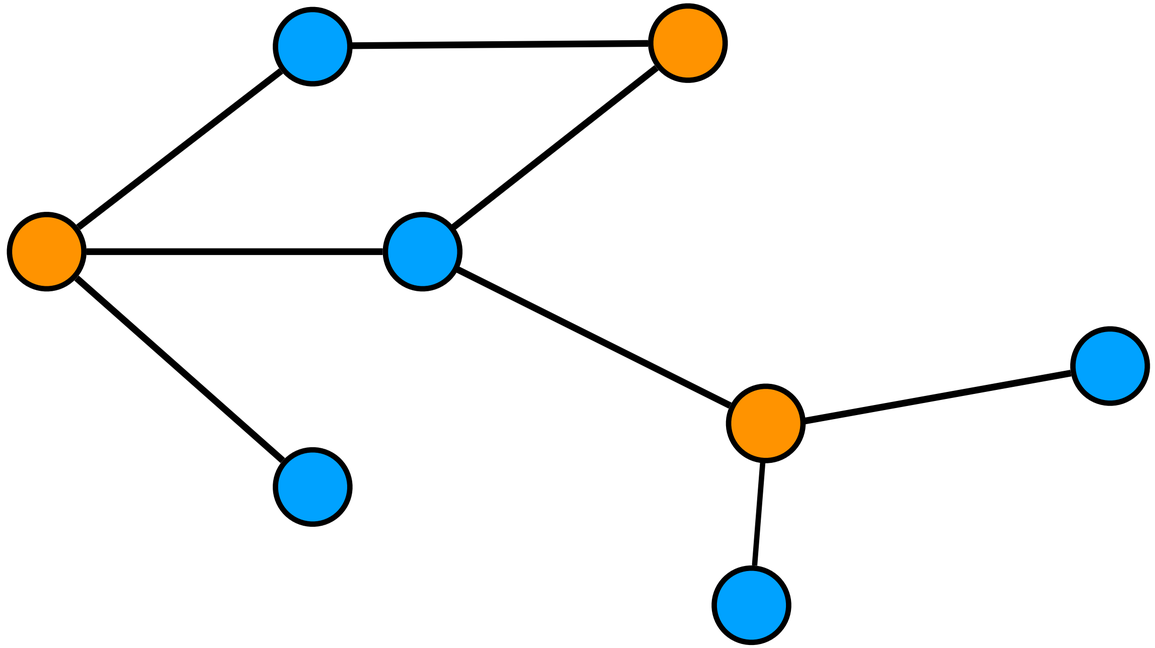

Massively Parallel Computing

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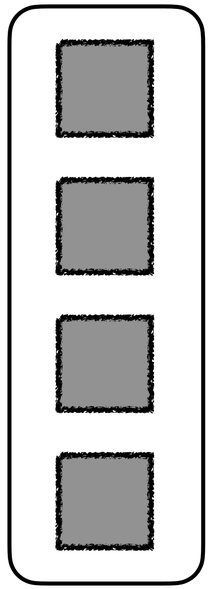

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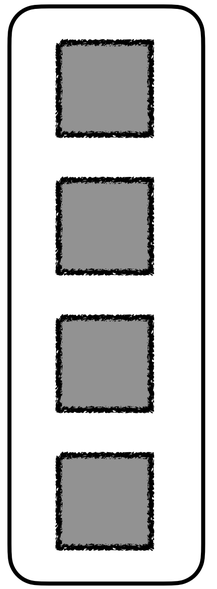

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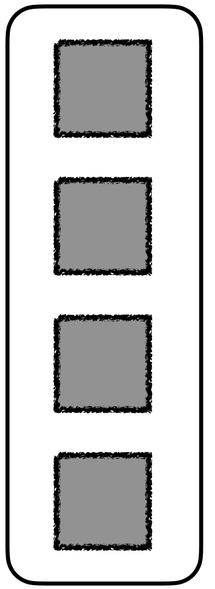

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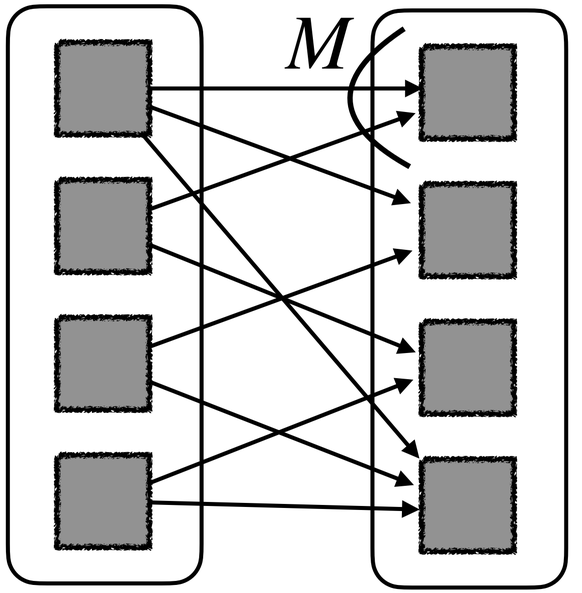

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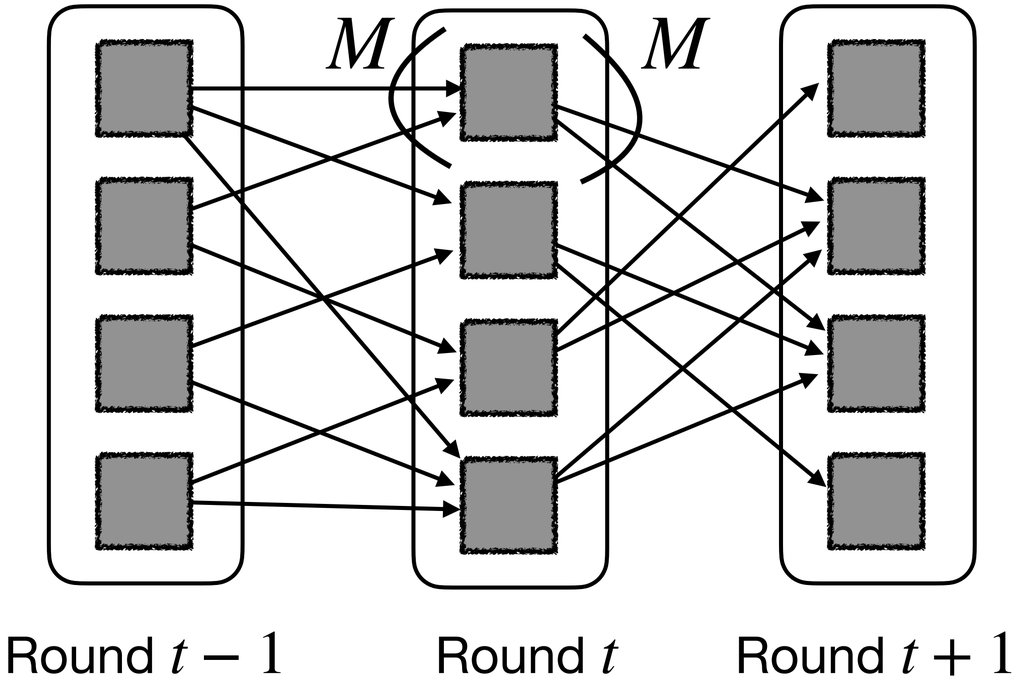

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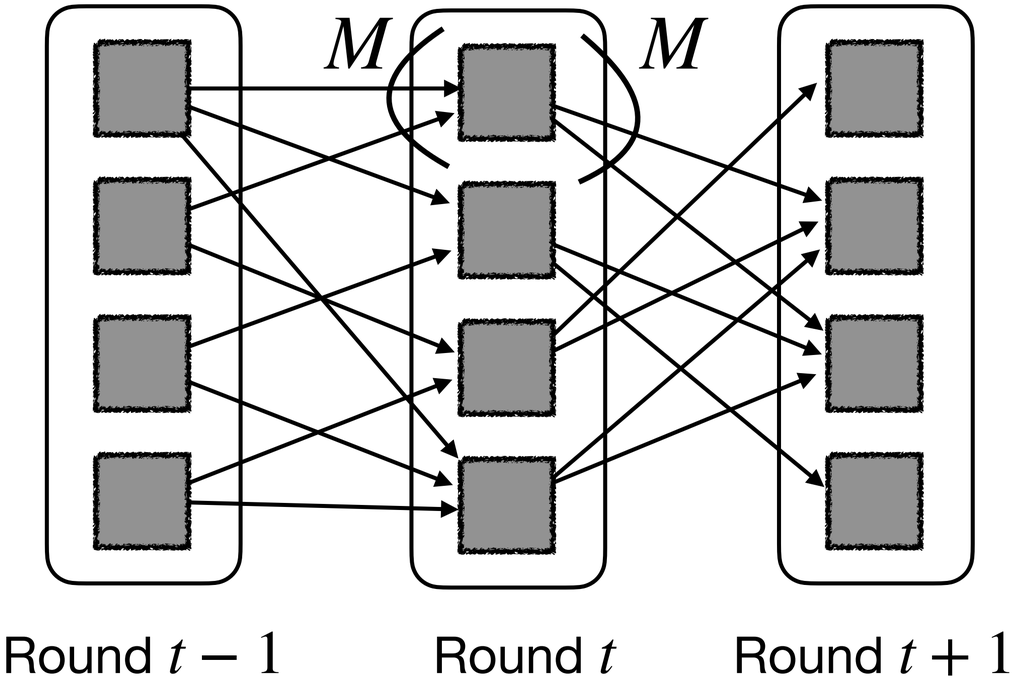
- Process input in $\tilde{O}(n)$ size chunks at a time.



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  - **Correlation Clustering**: Cambus, Kuhn, Lindy, P., and Uitto SODA 2024.

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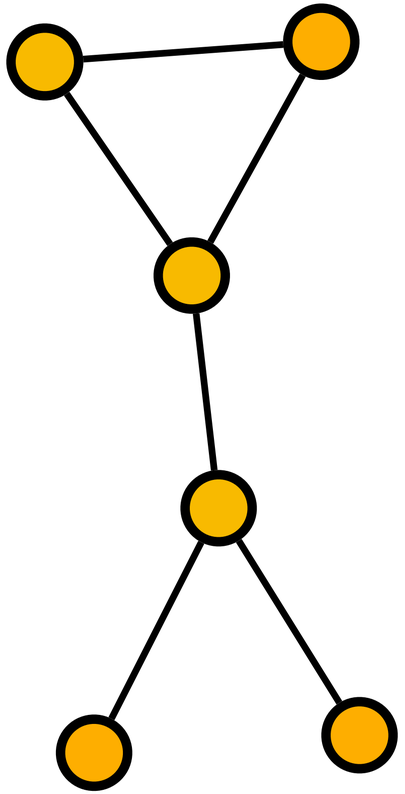

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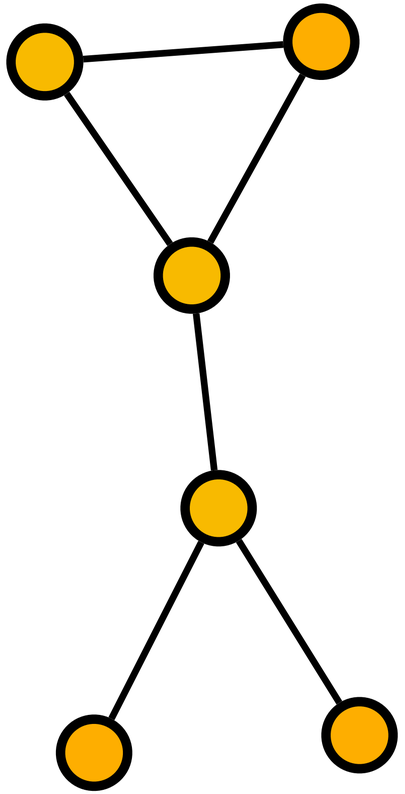

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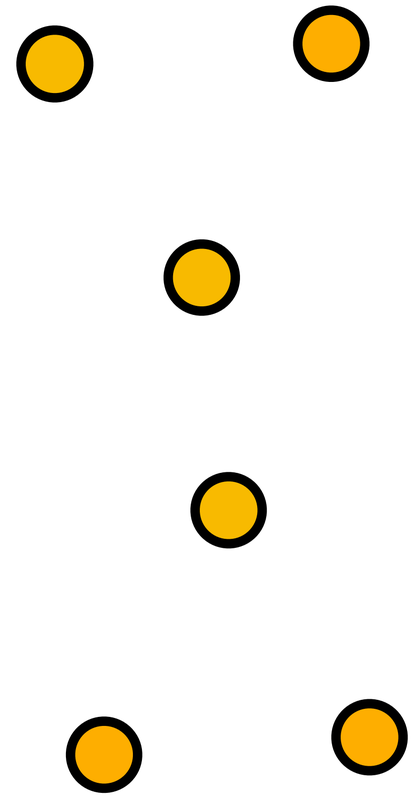

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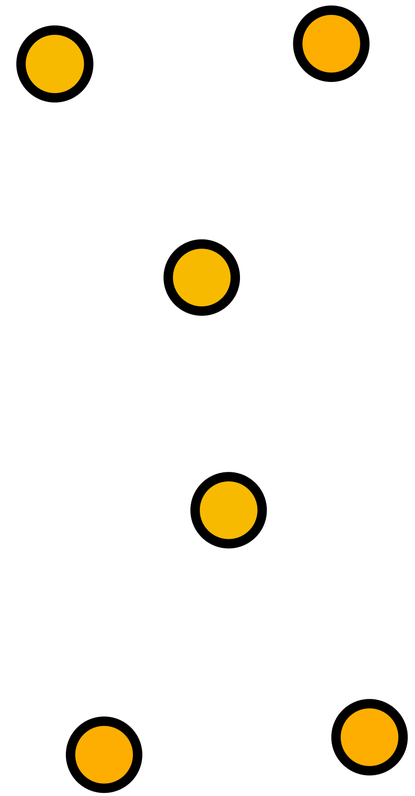

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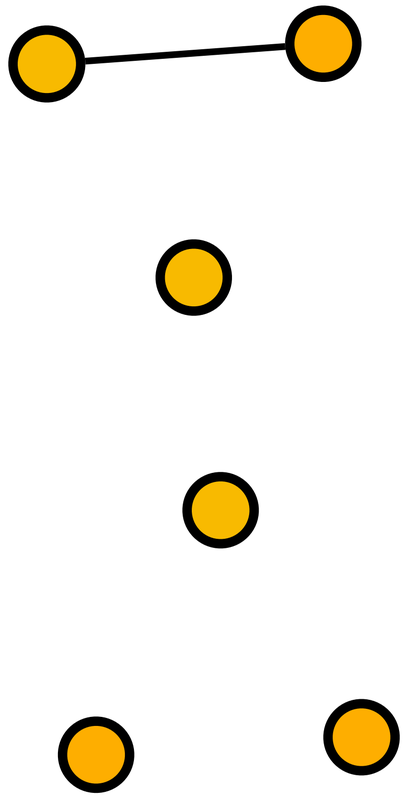

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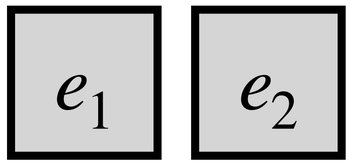

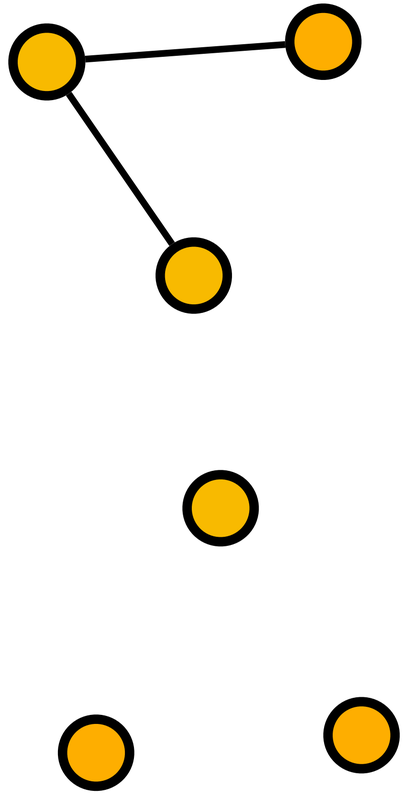

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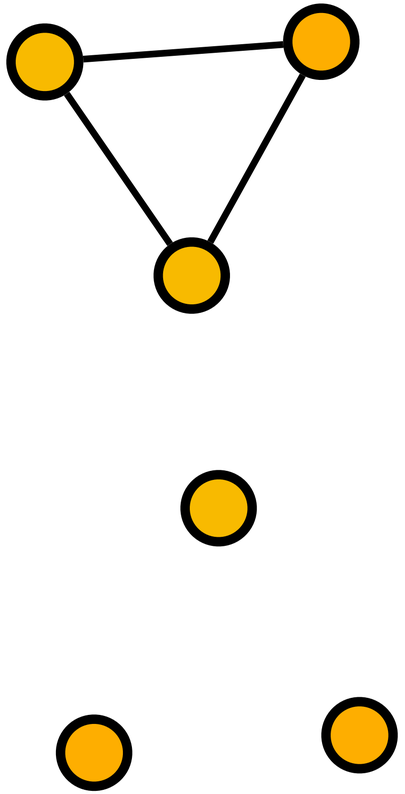

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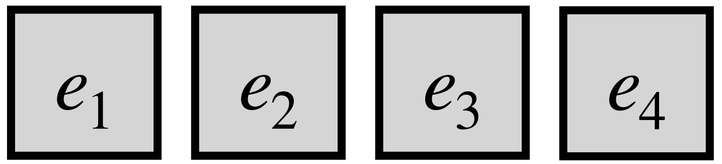

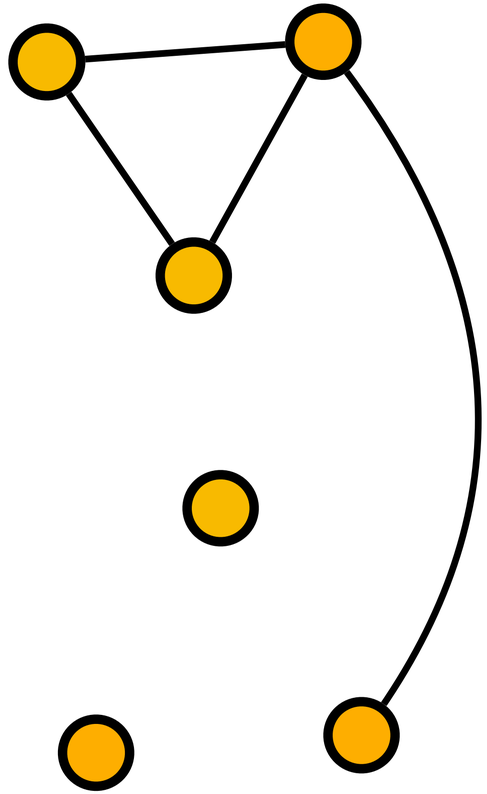

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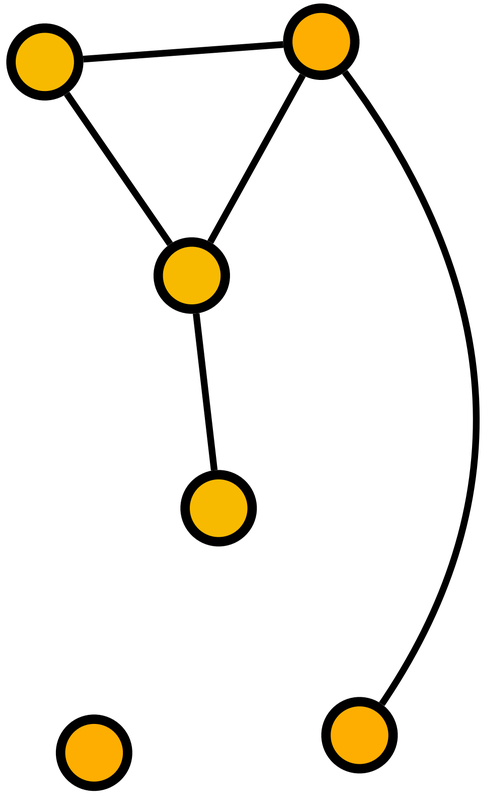

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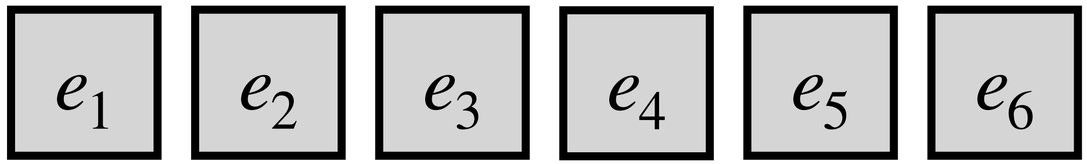

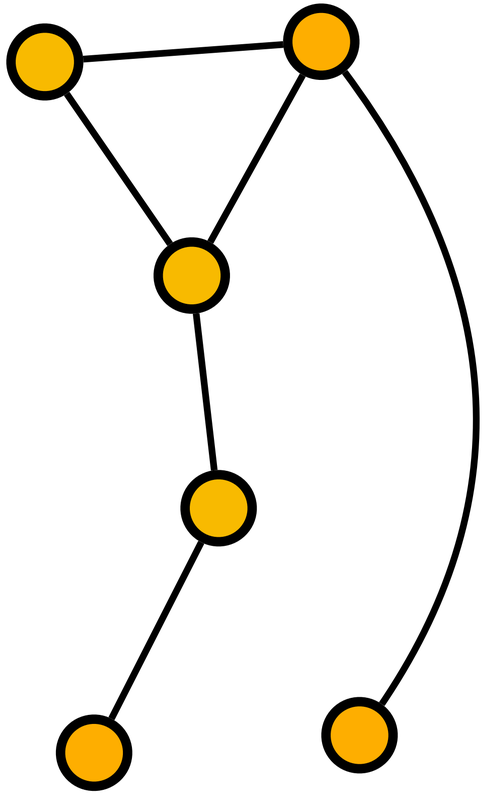

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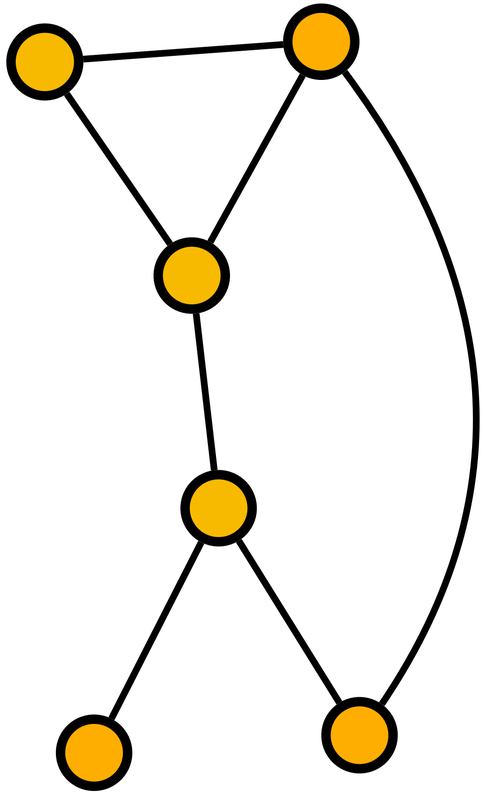

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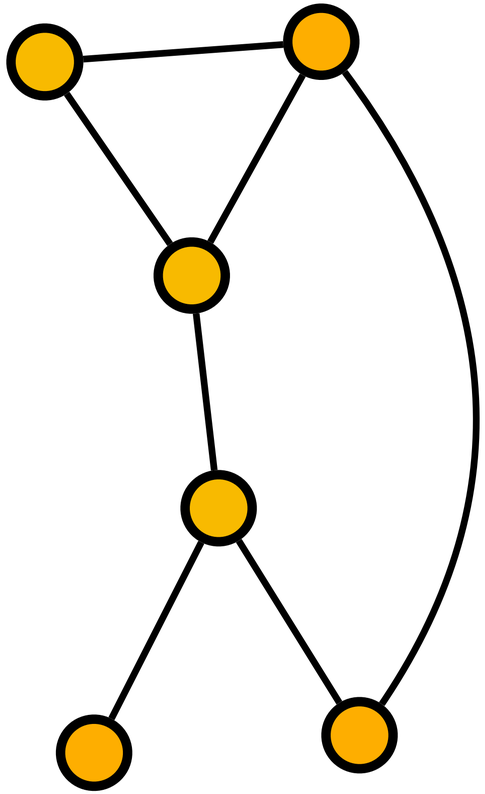

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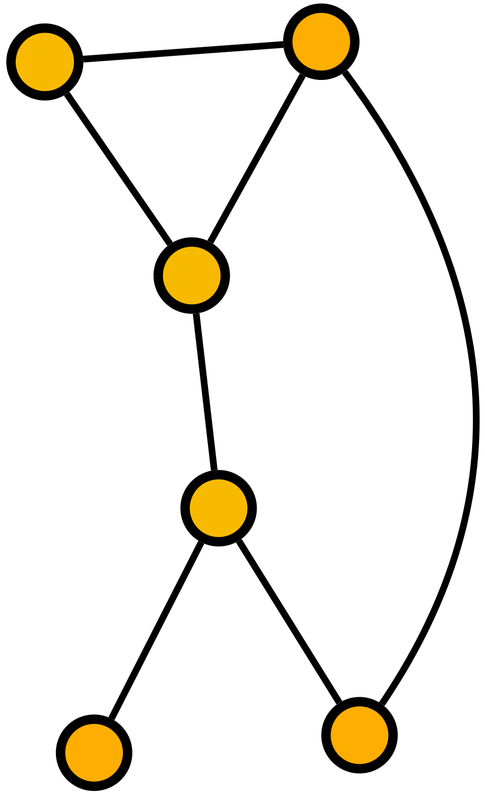

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  - Can we get $\tilde{O}(n)$ space with fewer passes for the relaxed problems?

[ACG+15] Ahn, Cormode, Guha, McGregor, and Wirth., ICML 2015
[AKNS24] Assadi, Konrad, Naidu, Sundaresan, STOC 2024

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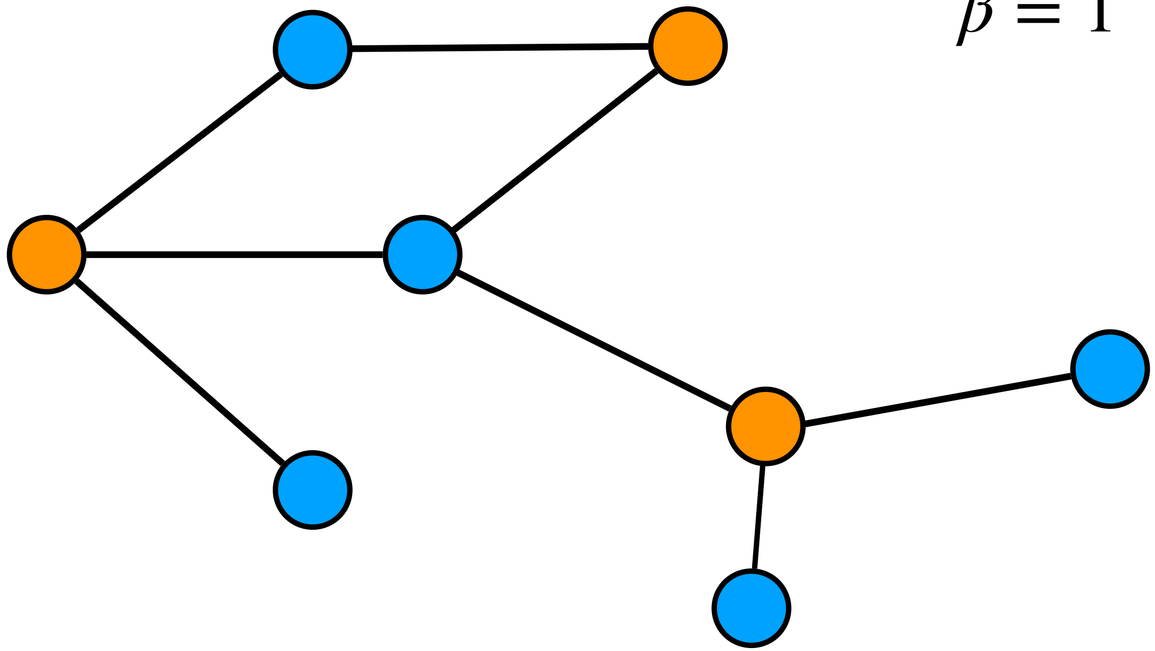

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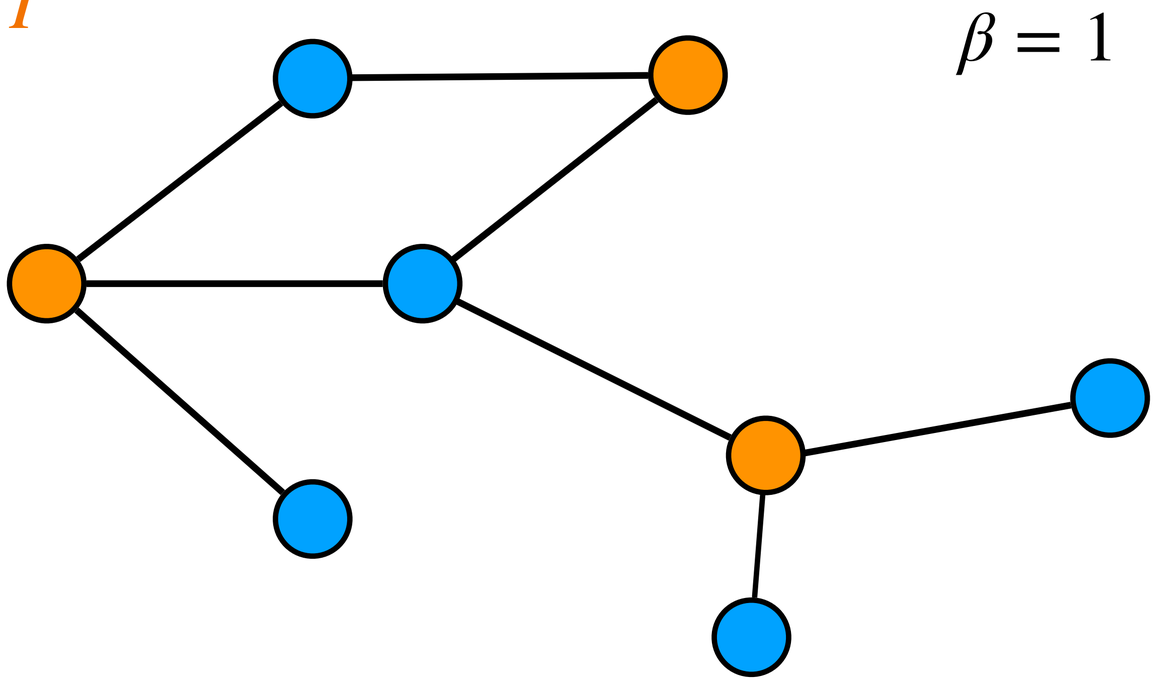

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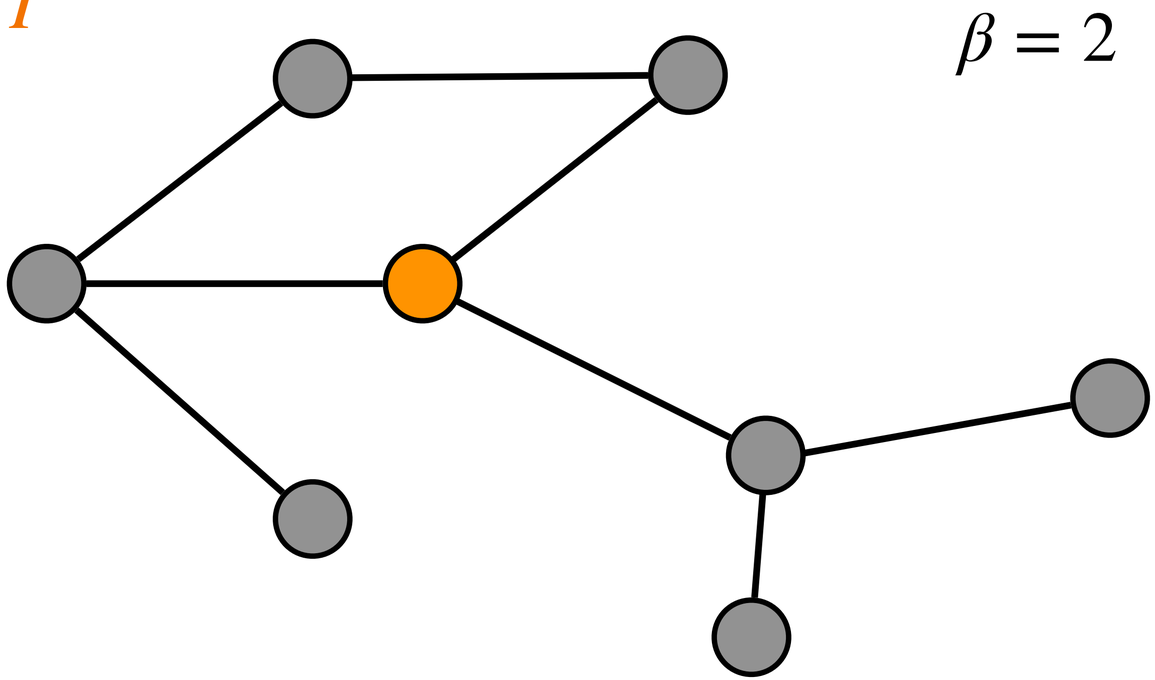

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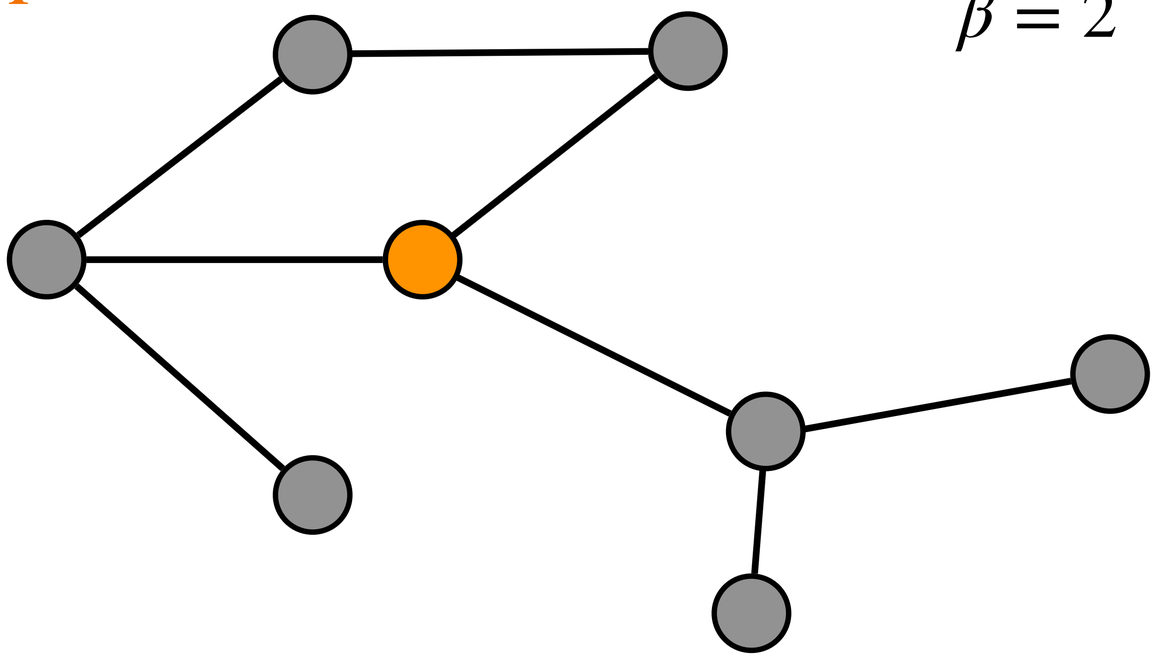

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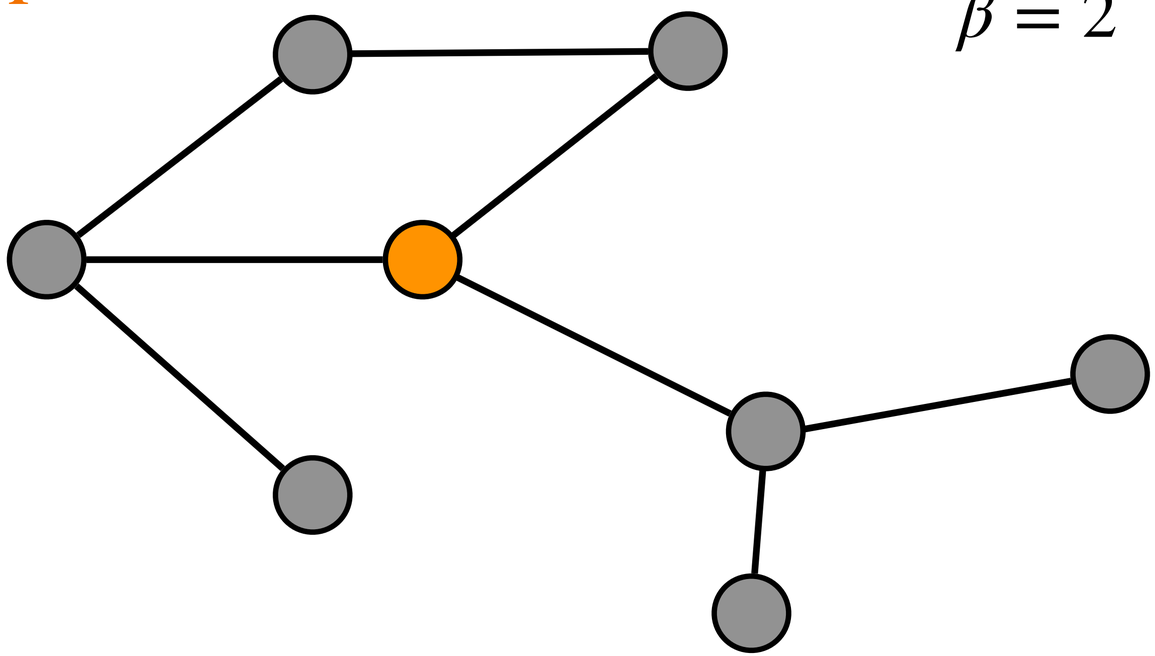
  - 2-RS is the “hardest” relaxation.



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    - Compute a β-ruling set on the conflict graph.
    - Gives an $O(\beta)$-approximation.

Main Result for Ruling Sets

A randomized las-vegas algorithm that computes a 2-ruling set.

Can be implemented in:

- Linear-memory MPC: $O(1)$ rounds whp.
- Congested Clique: $O(1)$ rounds whp.
- Streaming: $O(1)$ passes whp, with $O(n)$ space (insertion-only).

High Level View

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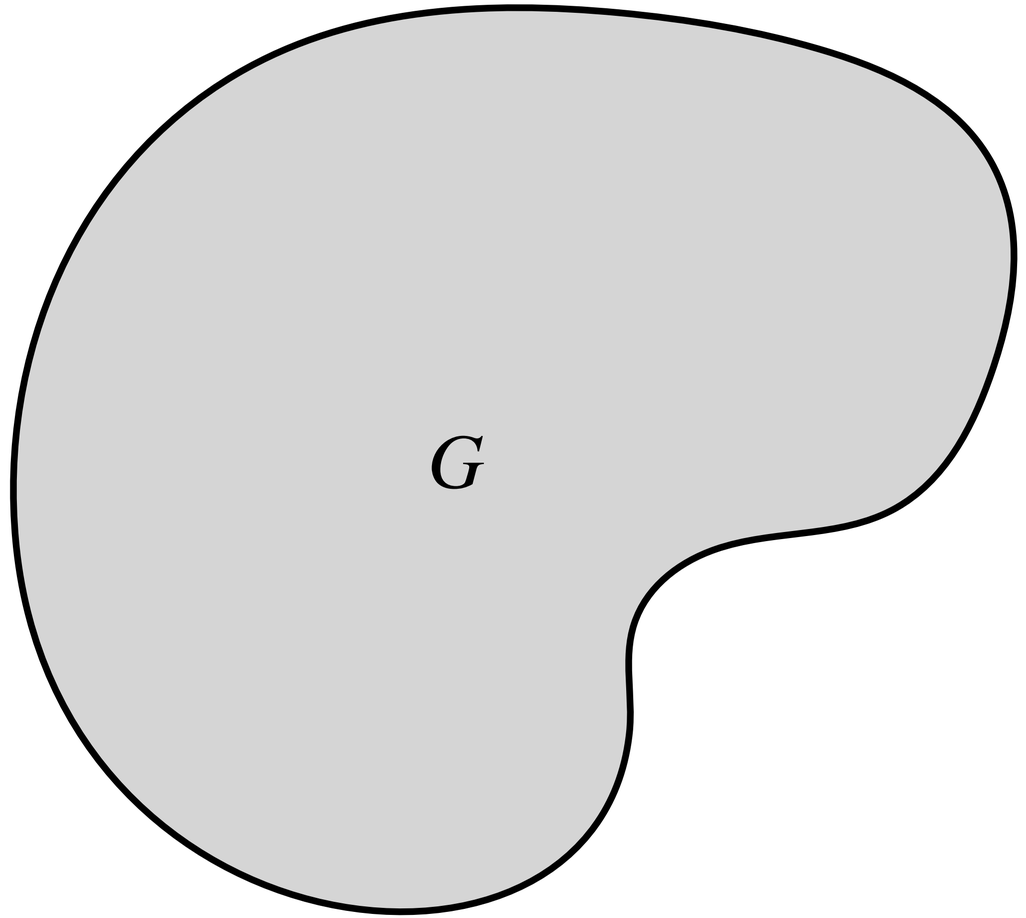

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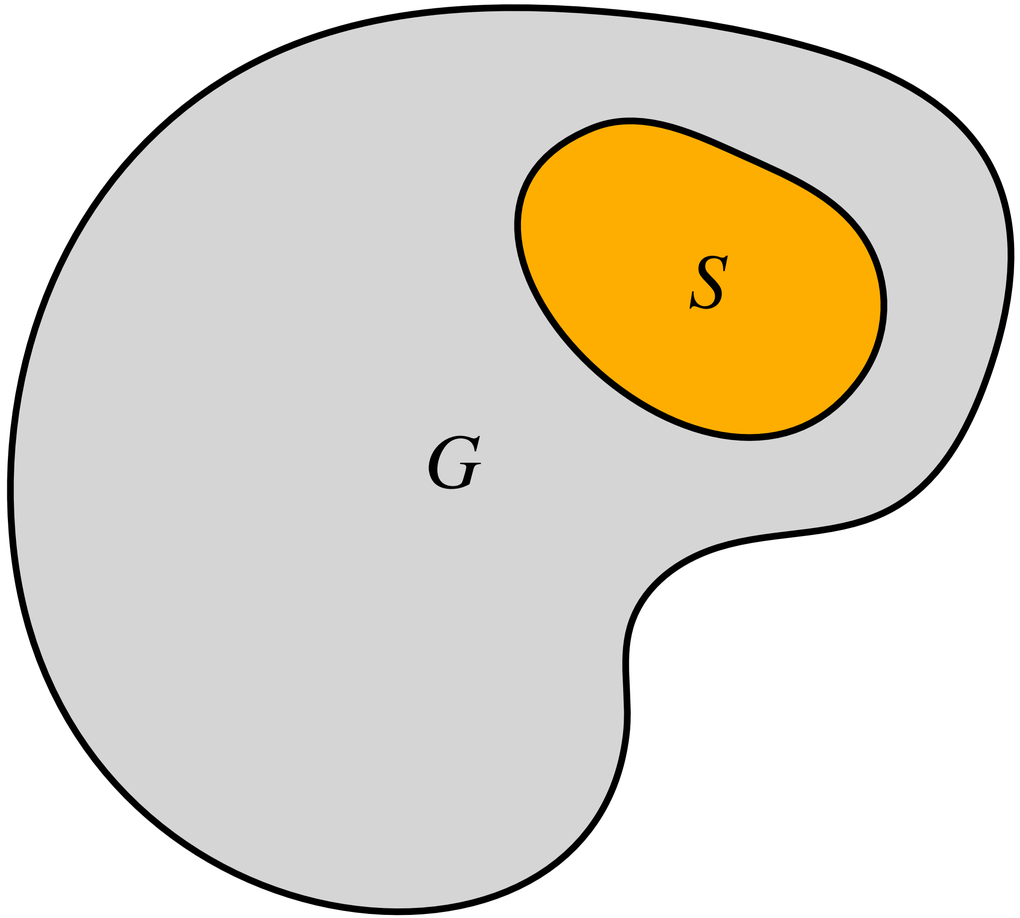
- Choose a set S of nodes such that $G[S]$ has $O(n)$ edges.



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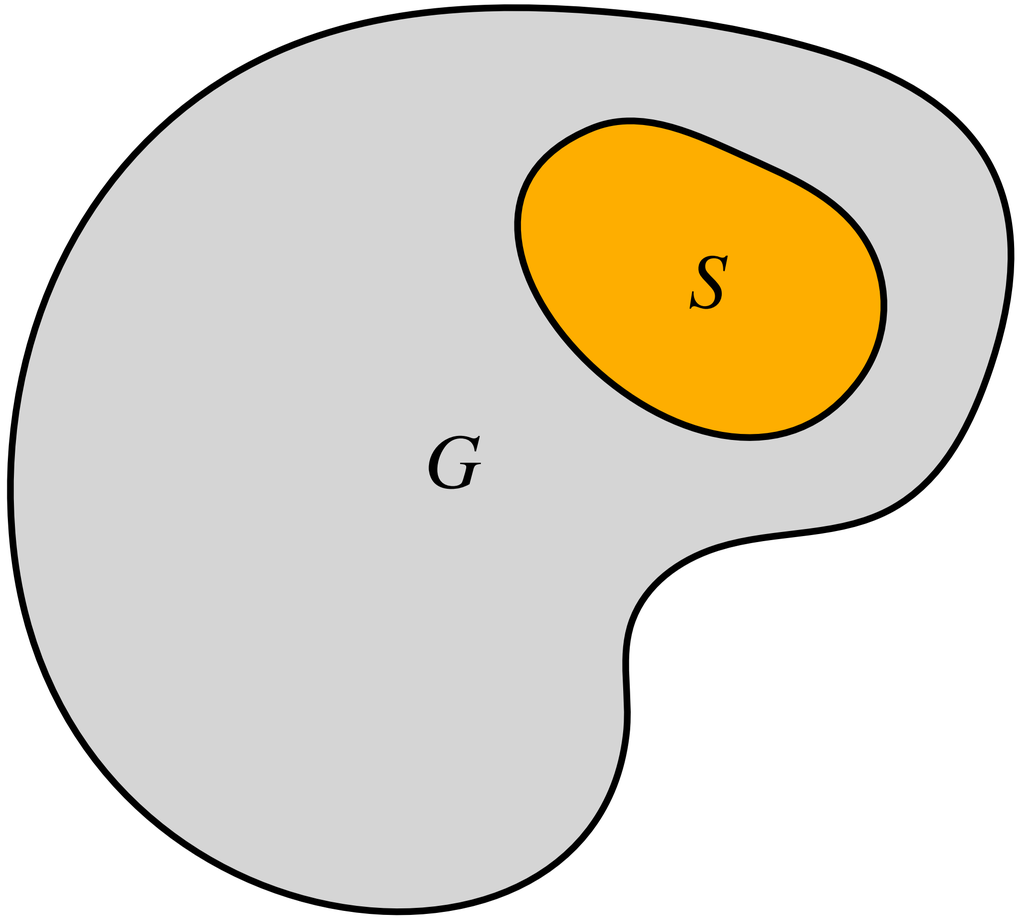

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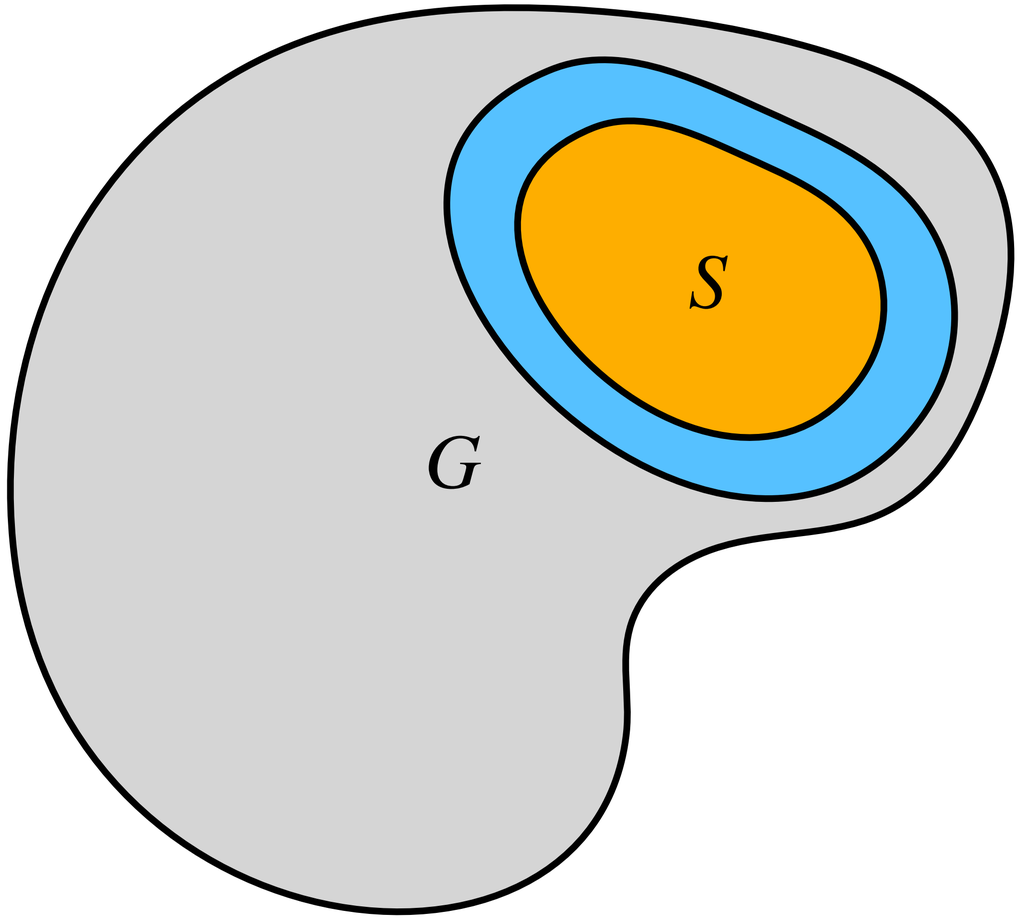

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    - Send $G[S]$ to a single machine which computes an MIS.
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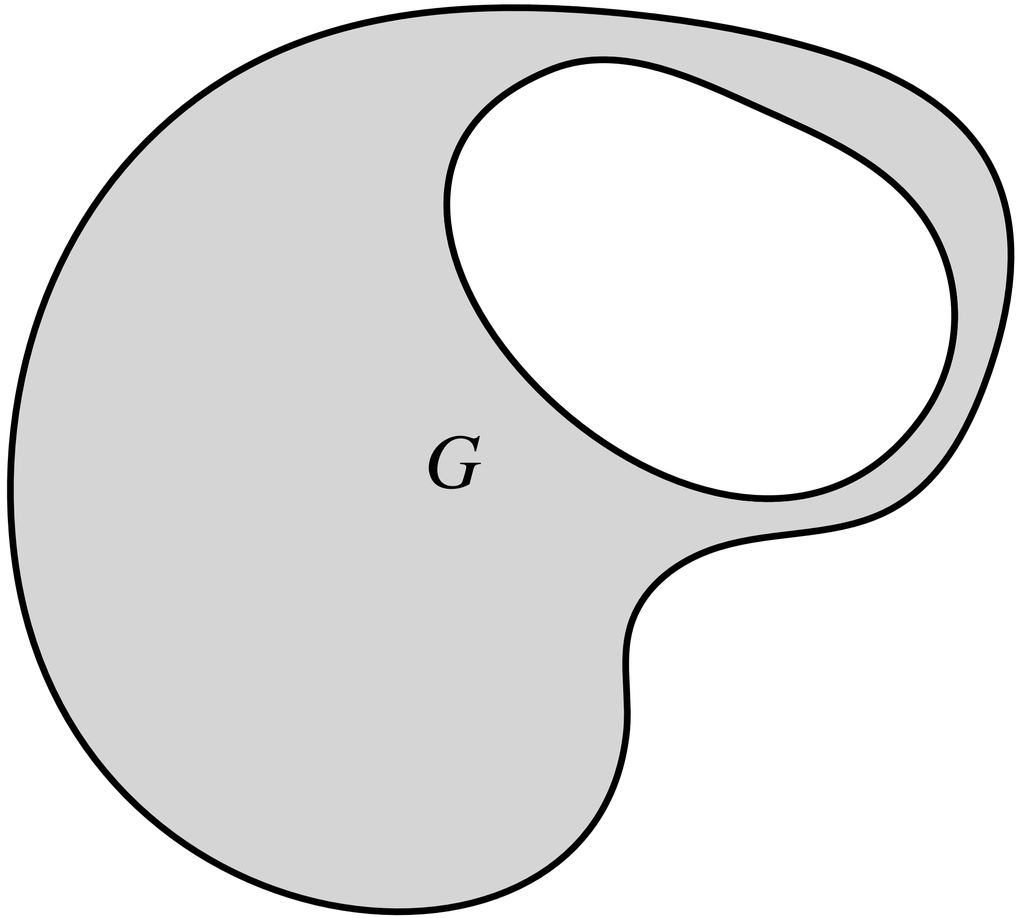
- Rinse and repeat until all nodes are covered.
- Repeatedly identify and process **linear sized** subgraphs.



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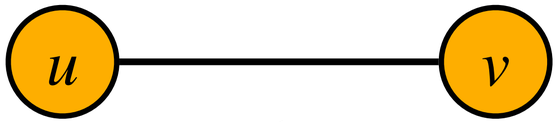

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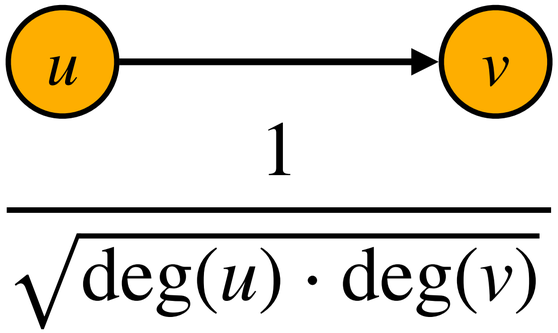

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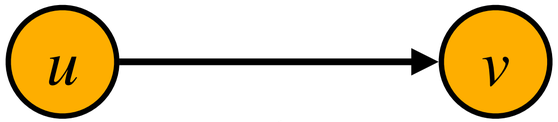

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Sampling a Sparse Hitting Set

Add each vertex u to S with probability $\frac{1}{\sqrt{\deg(u)}}$

$u \to v$ iff
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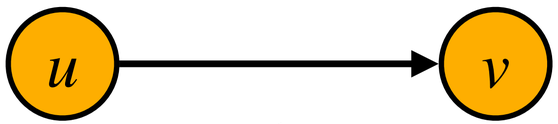

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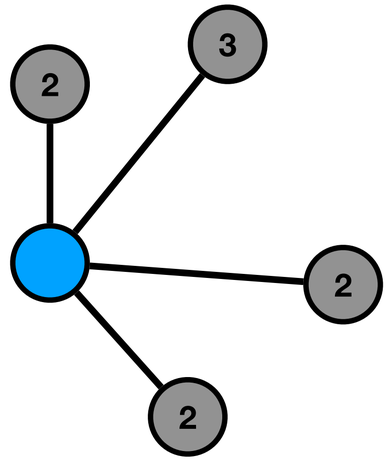

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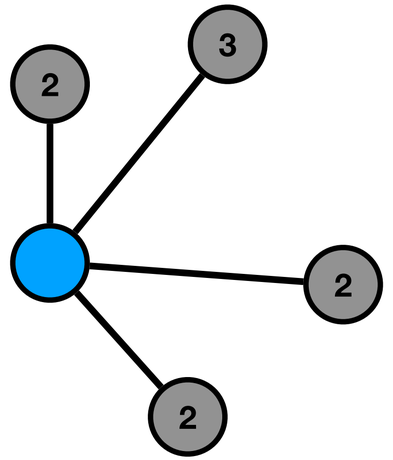

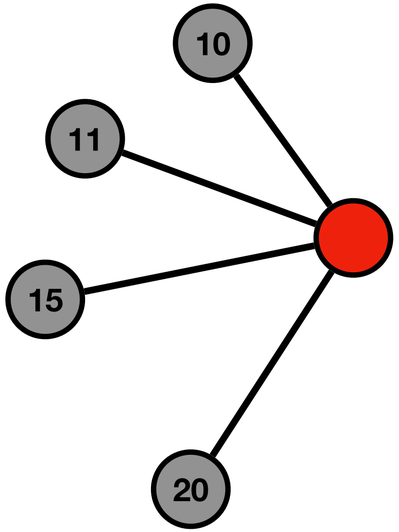

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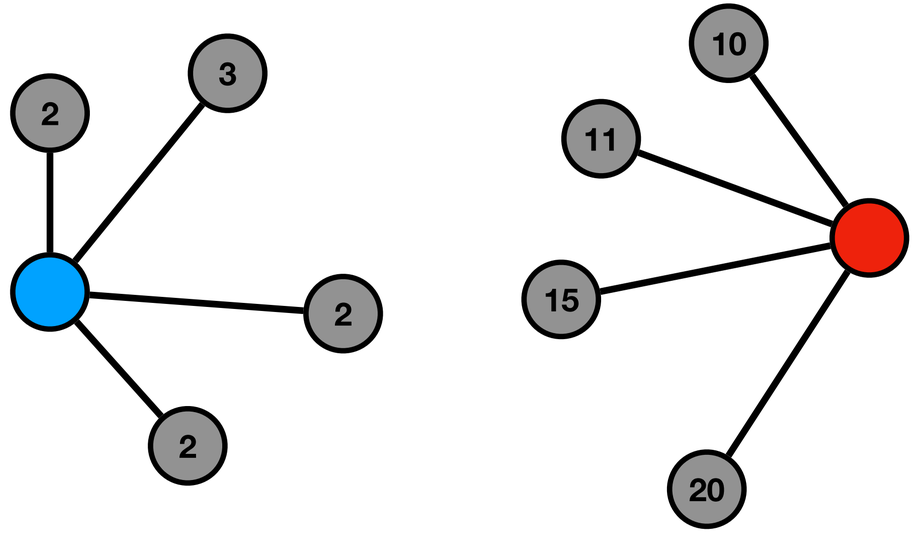

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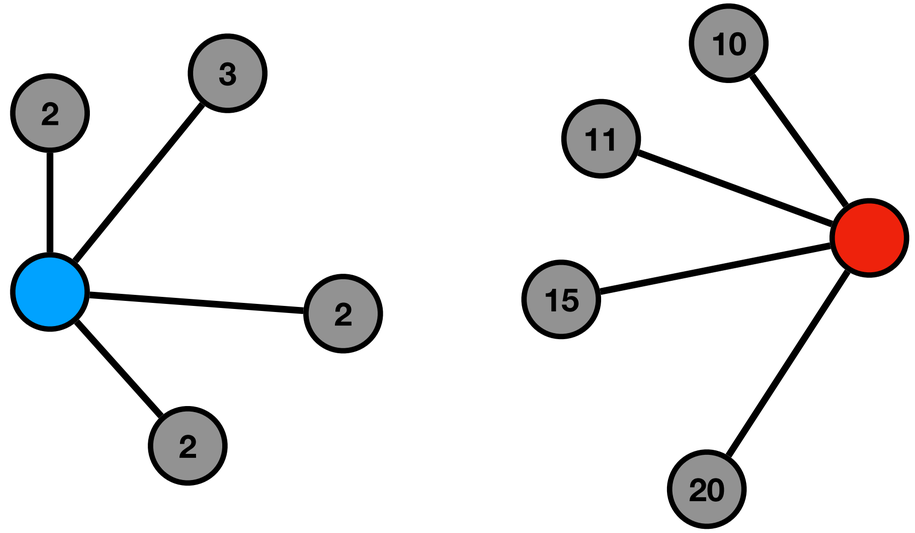

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Otherwise v is **bad**

Classification So Far

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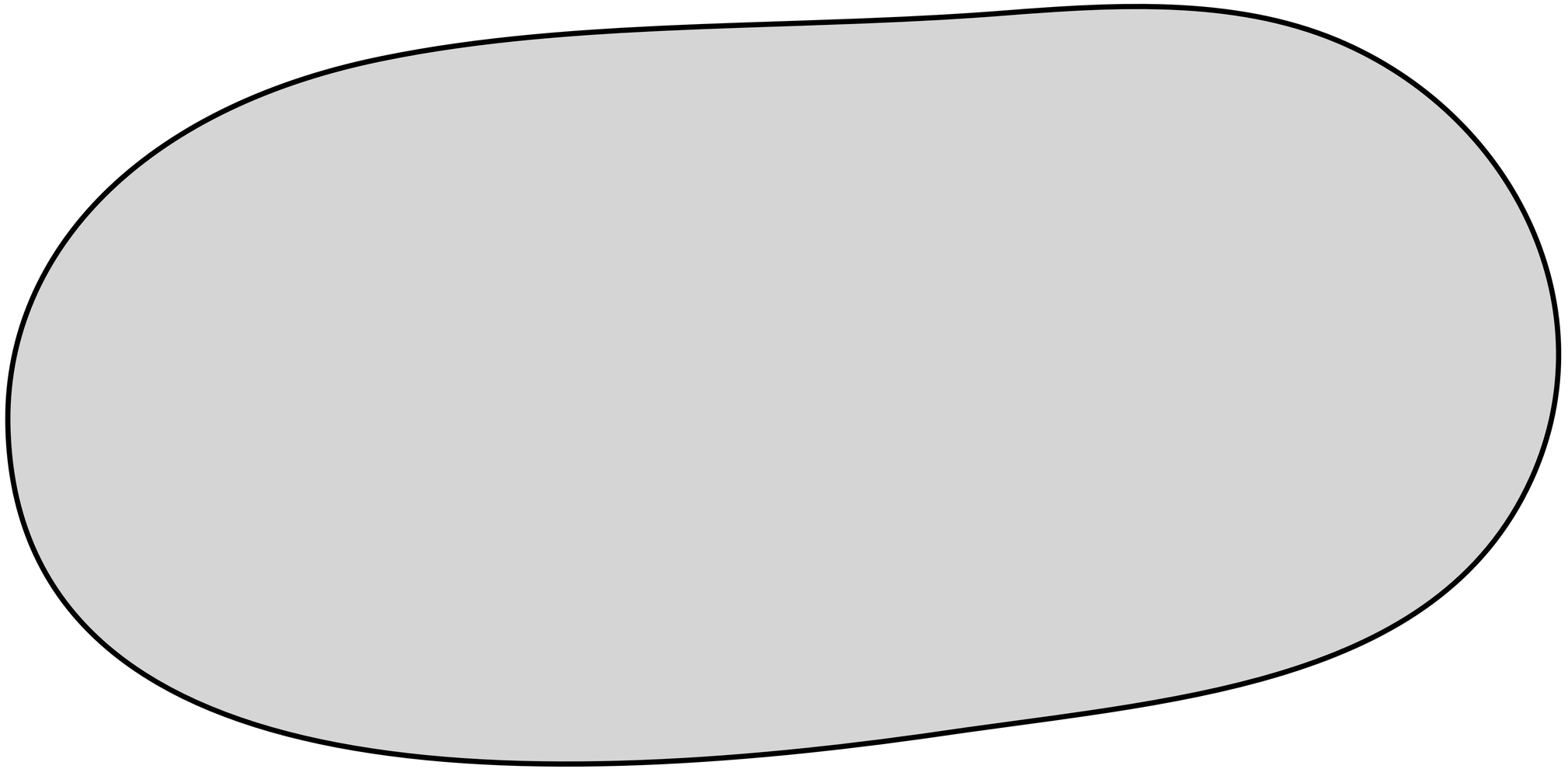

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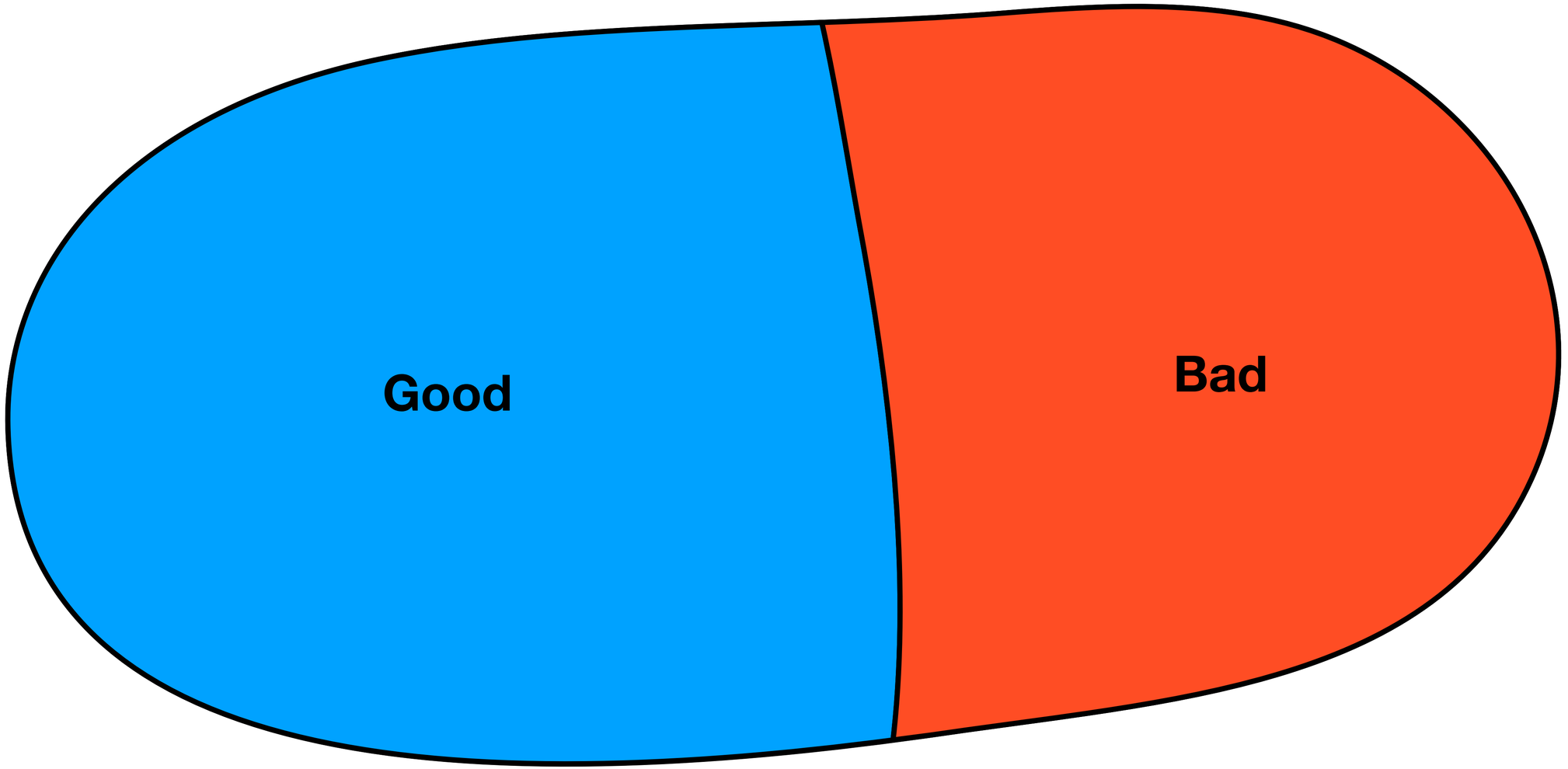

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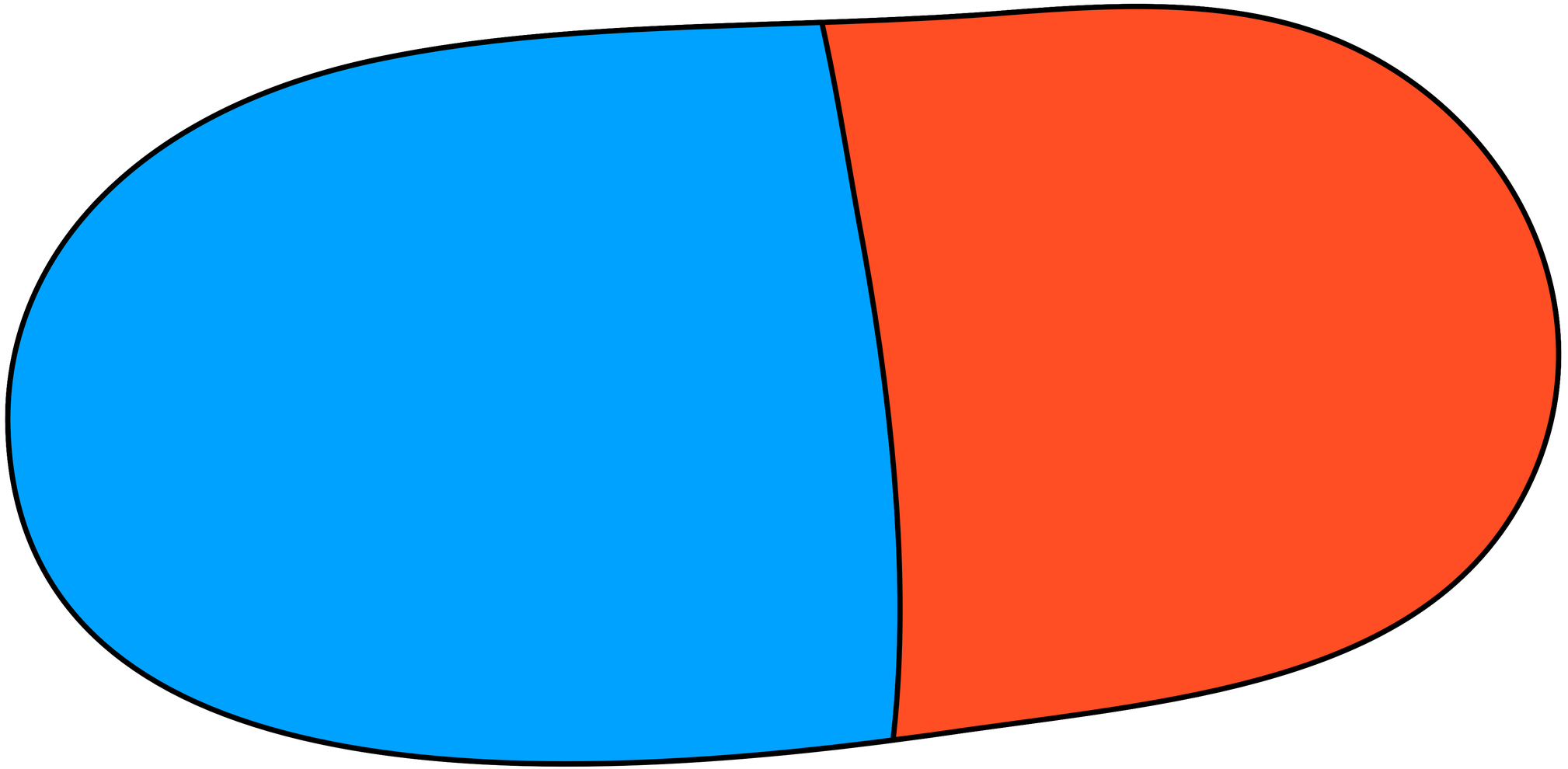

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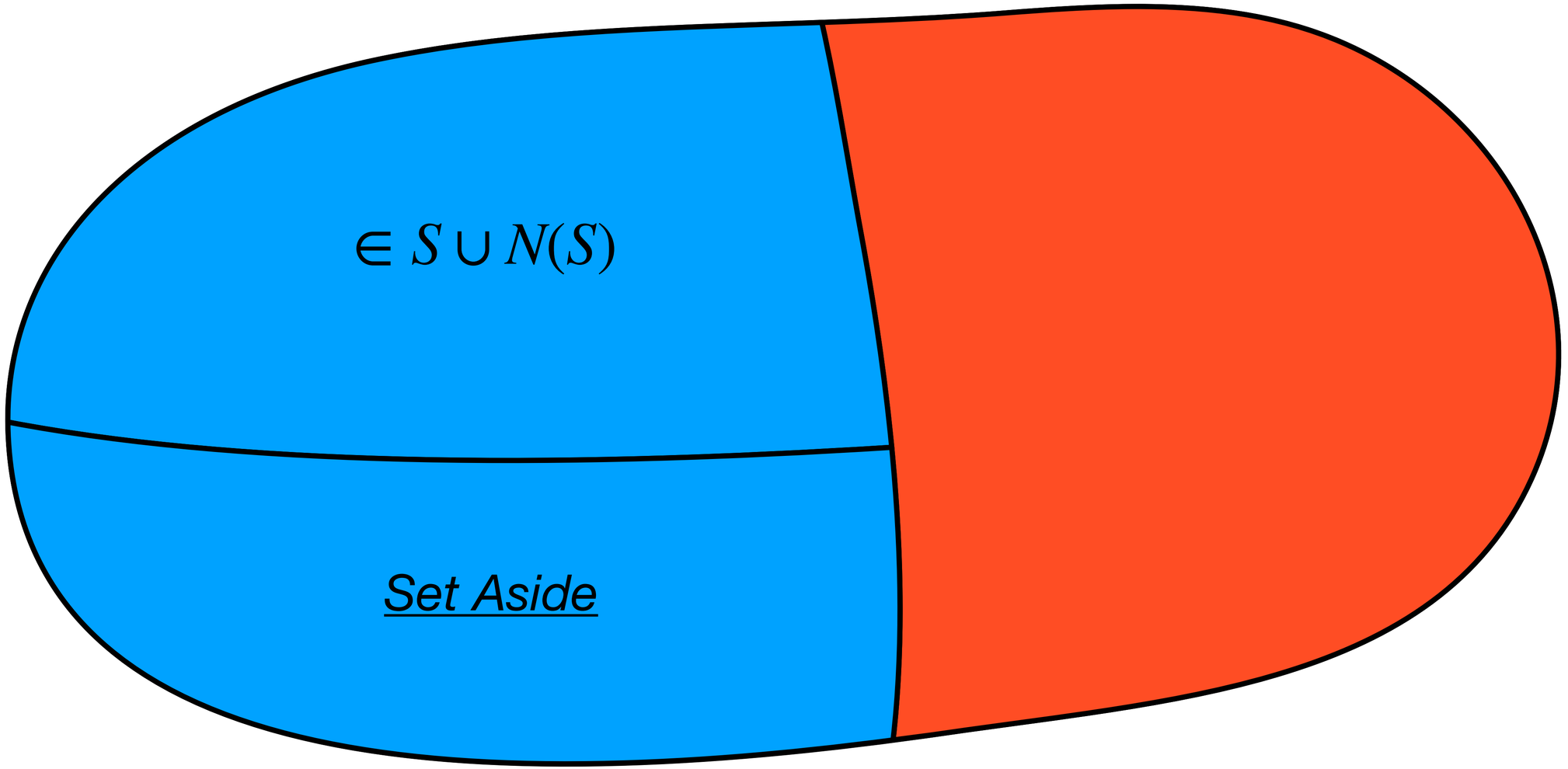

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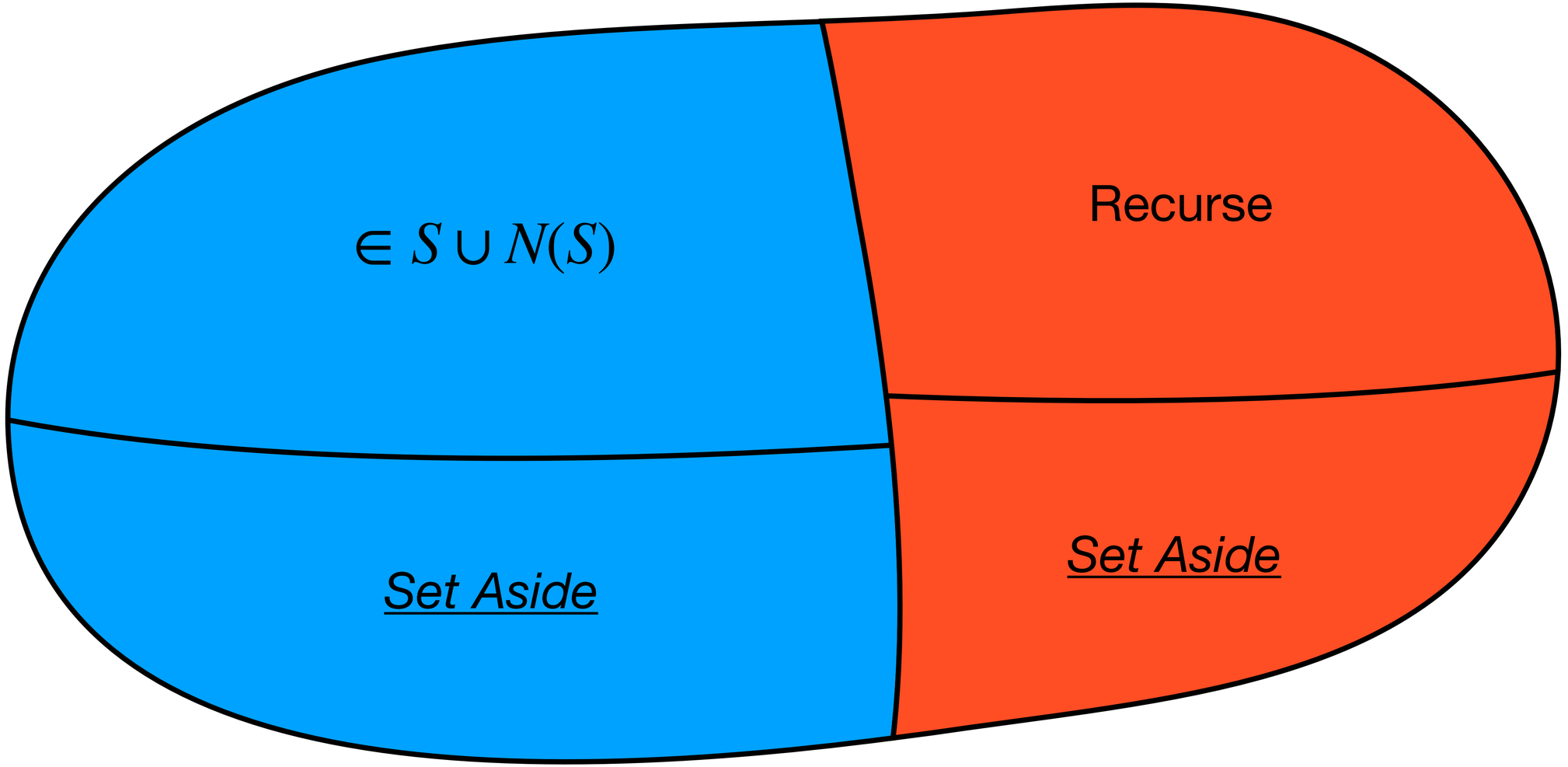

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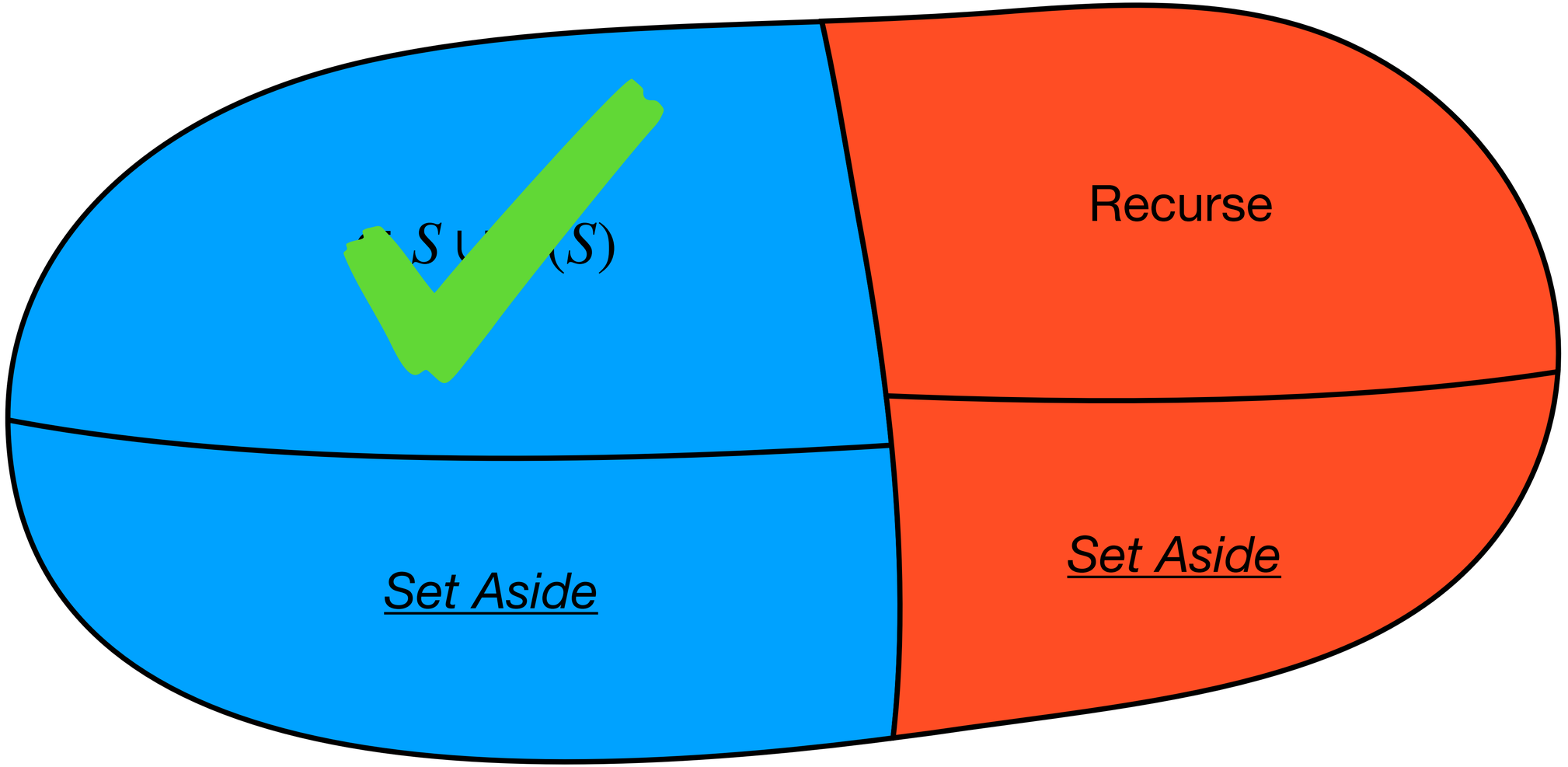

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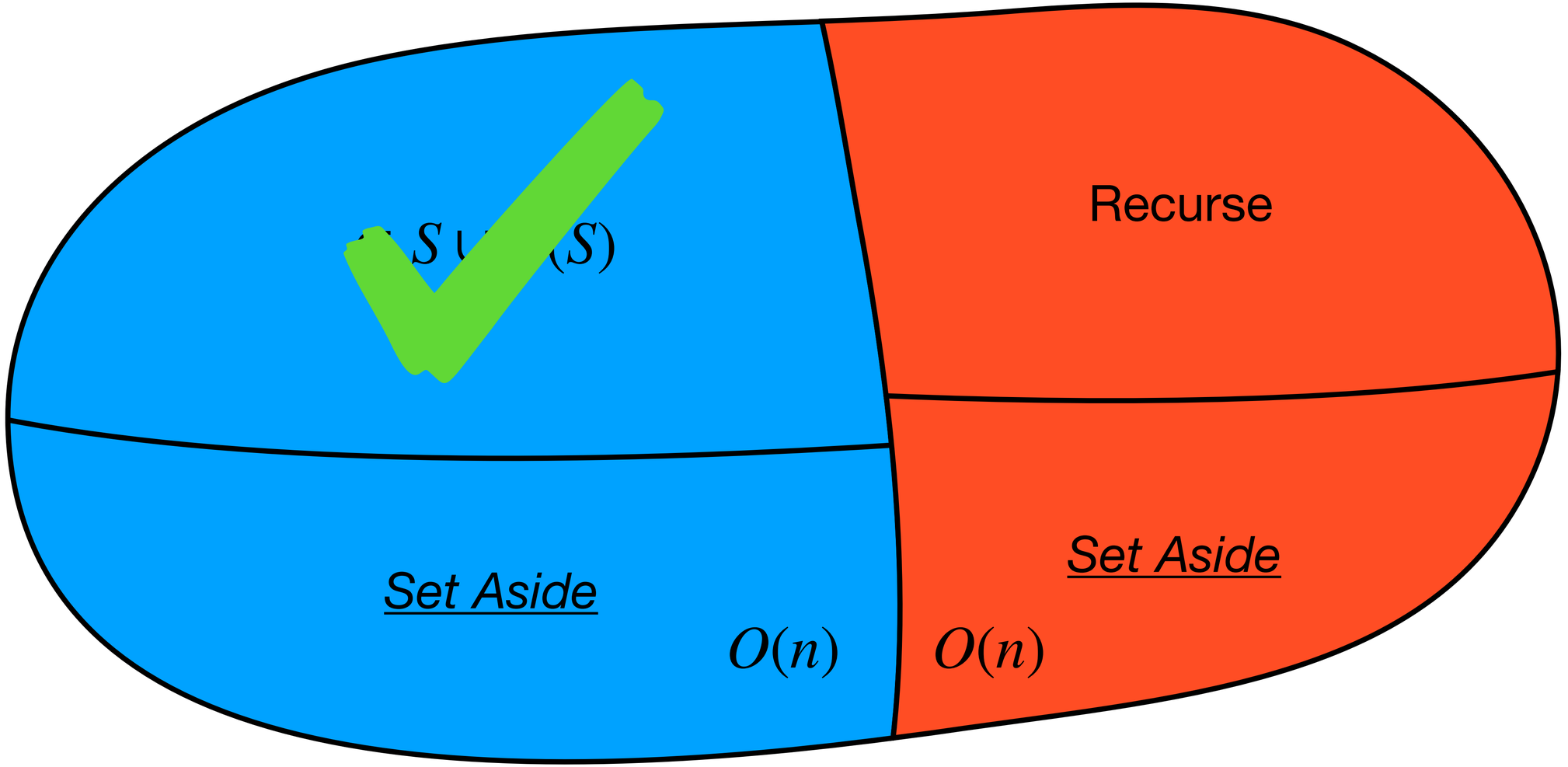

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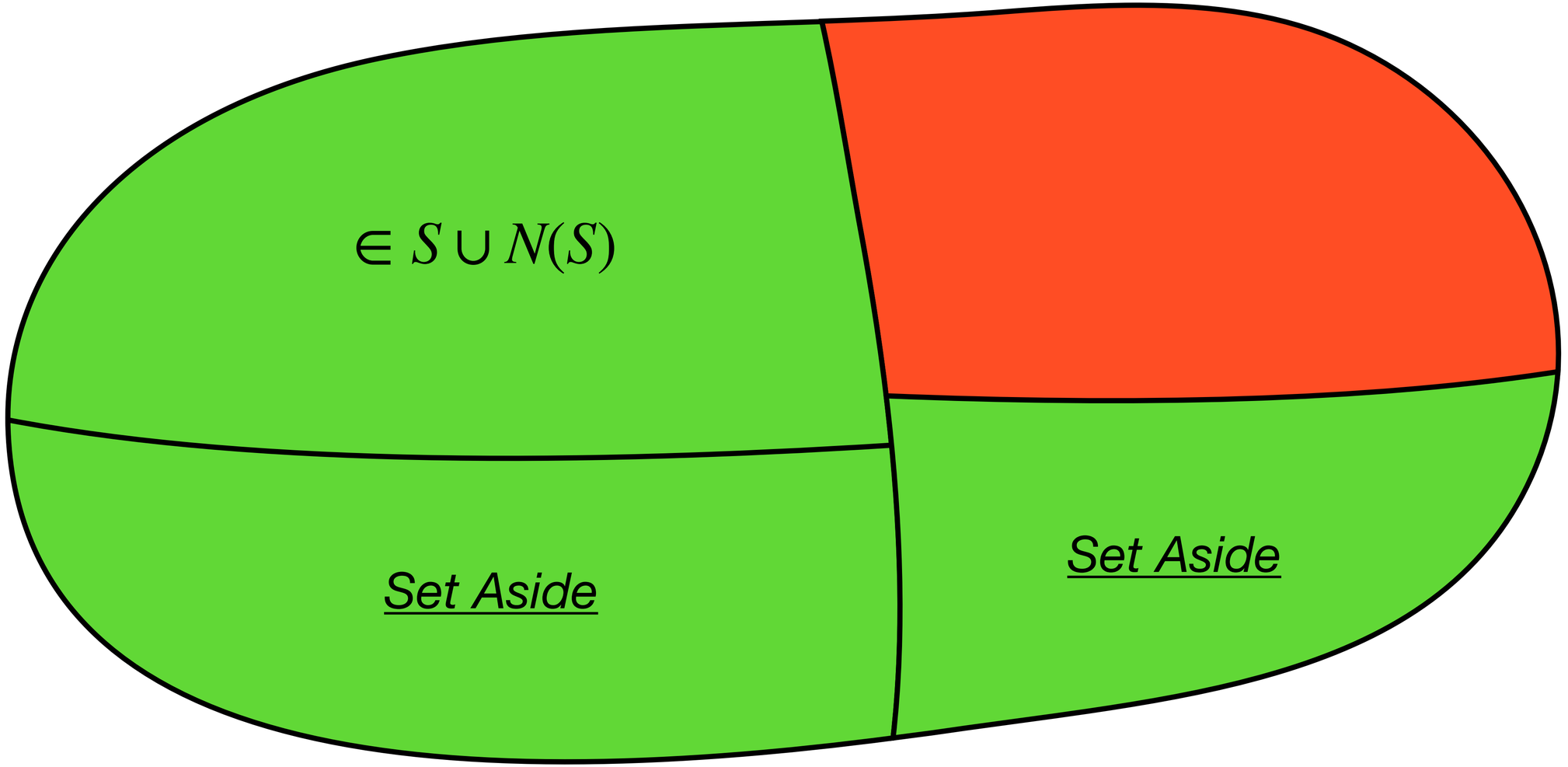

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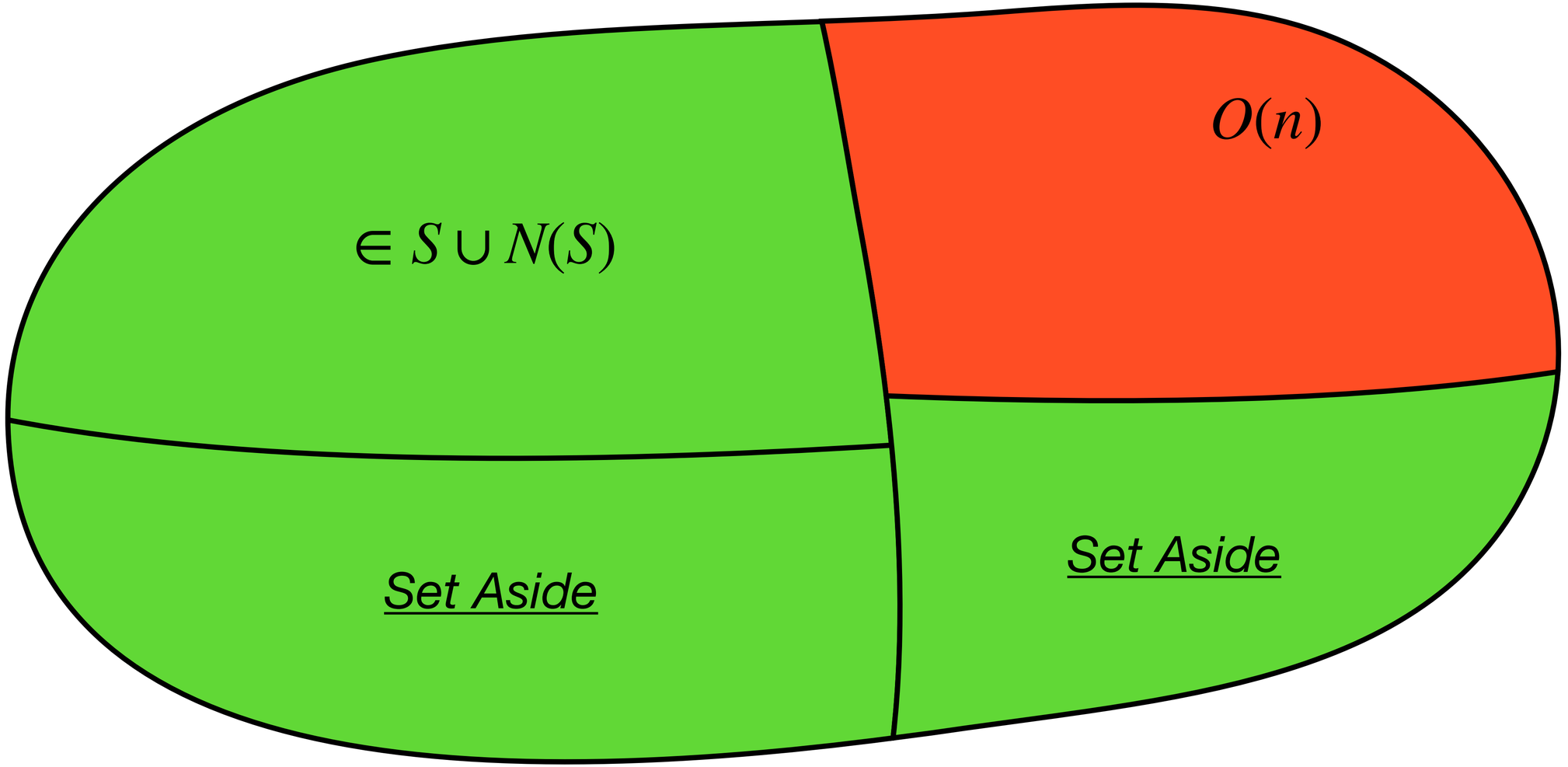

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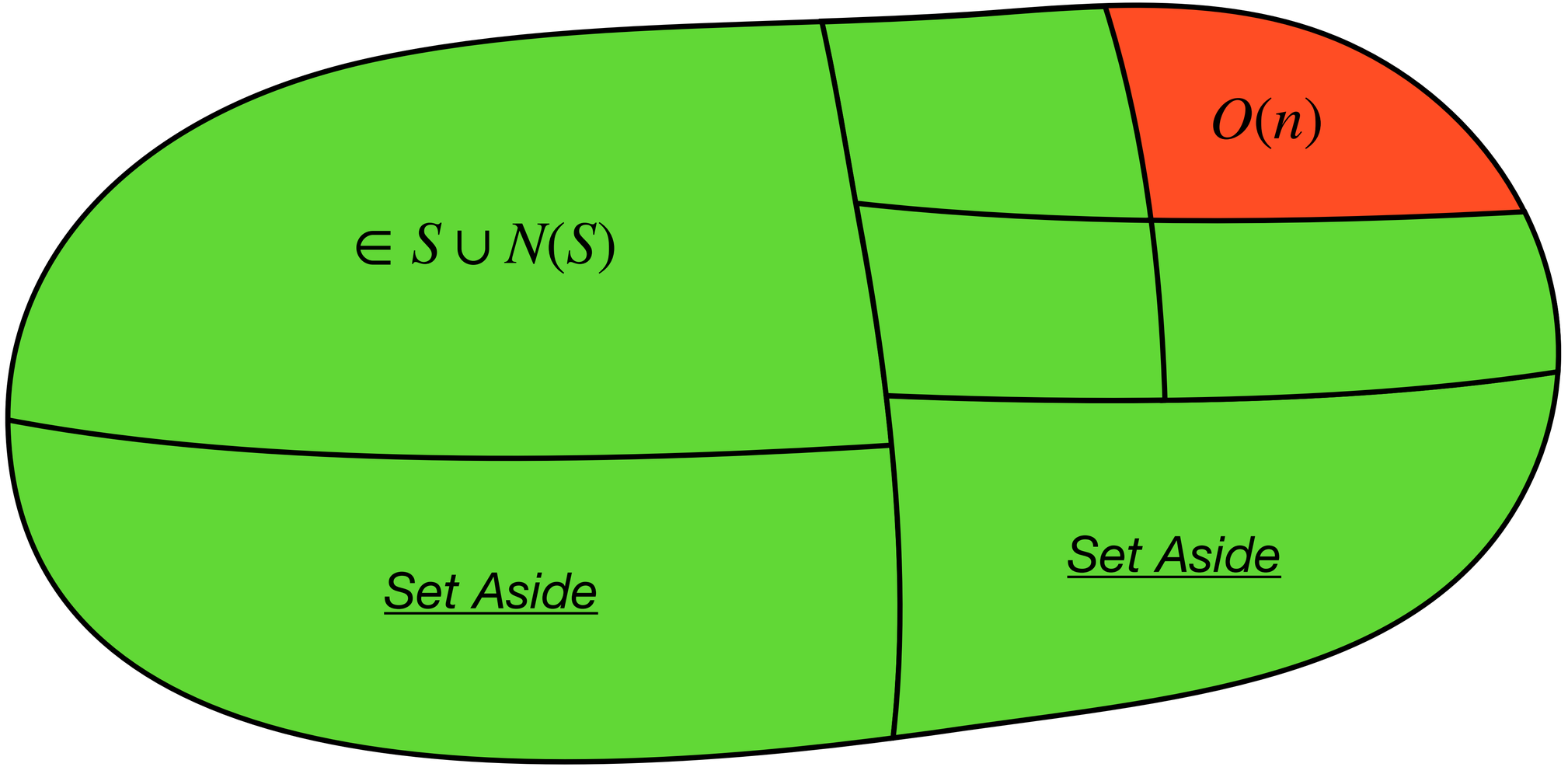

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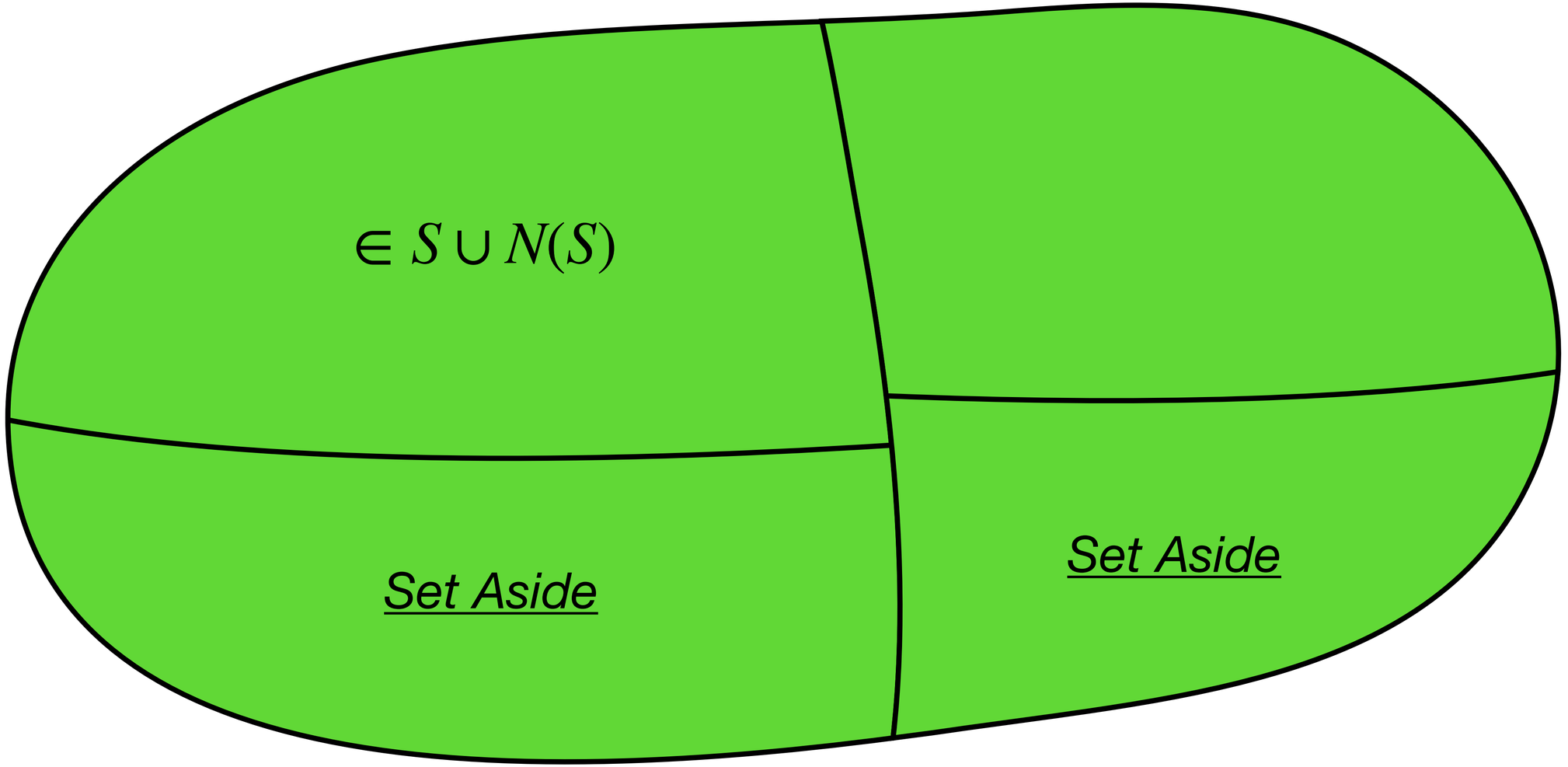

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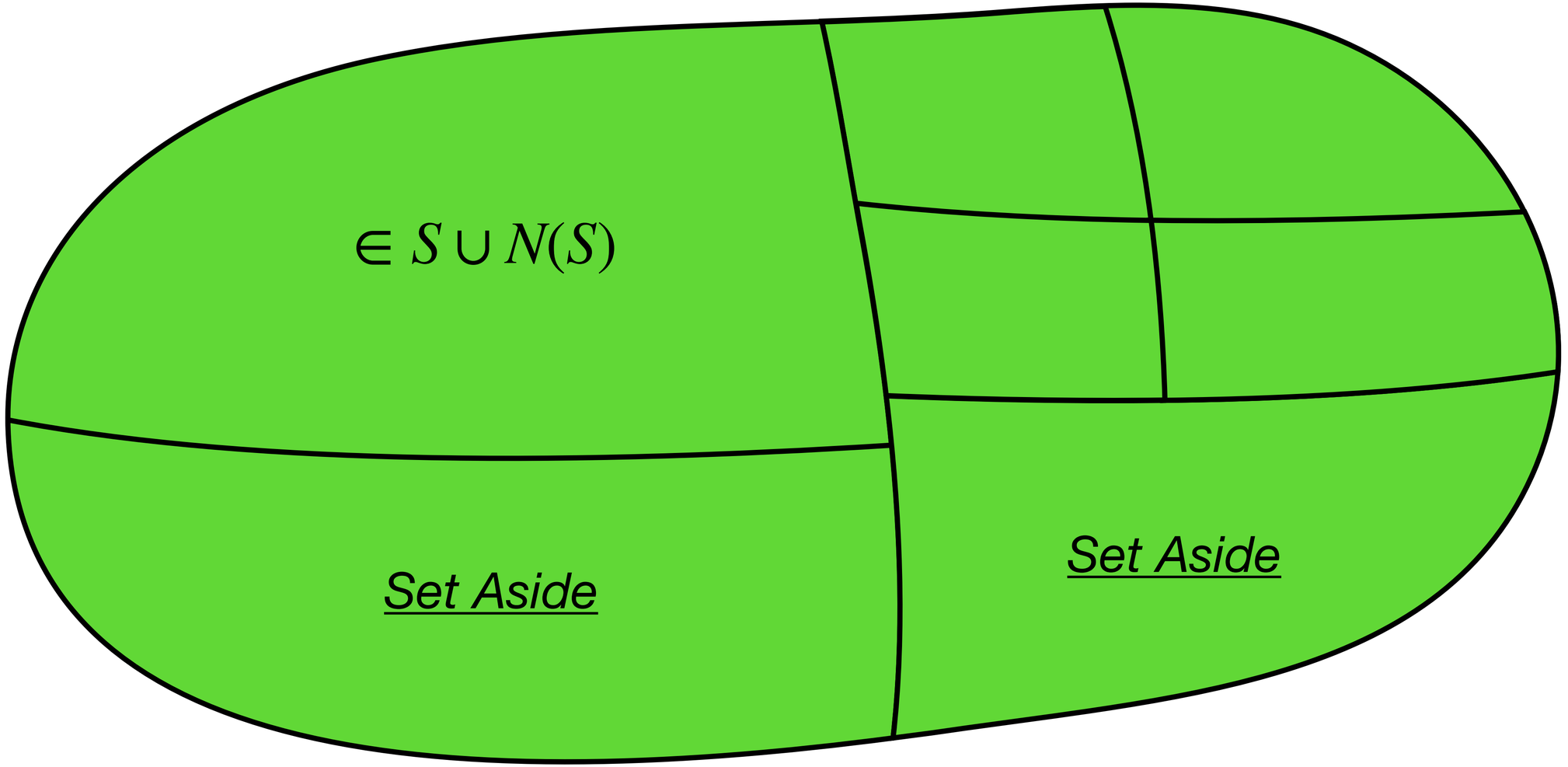

Ruling Set Summary

A randomized las-vegas algorithm that computes a 2-ruling set.

Can be implemented in:

- Linear-memory MPC: $O(1)$ rounds whp.
- Congested Clique: $O(1)$ rounds whp.
- Streaming: $O(1)$ passes whp, with $O(n)$ space (insertion-only).

Relaxation 2: Correlation Clustering

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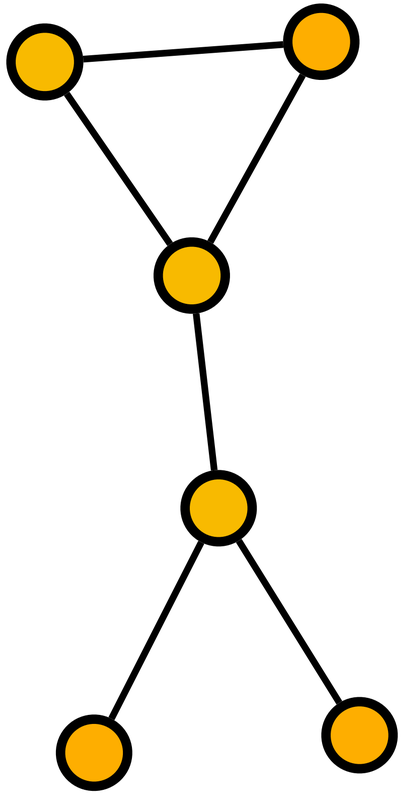

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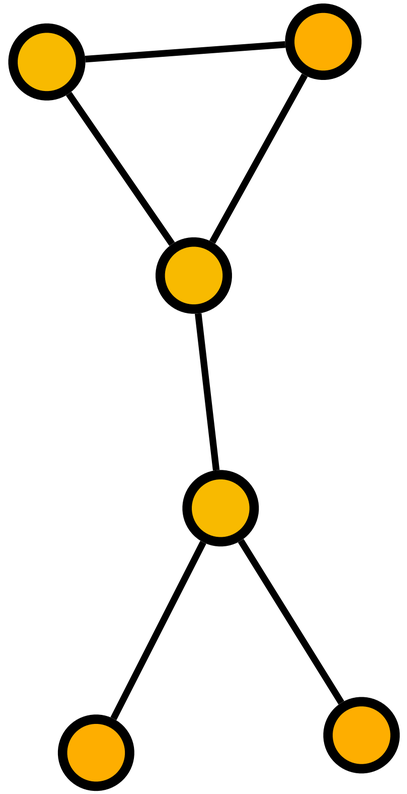

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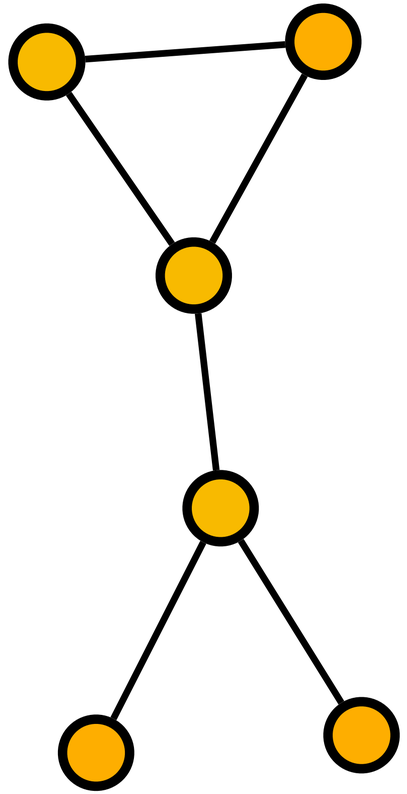

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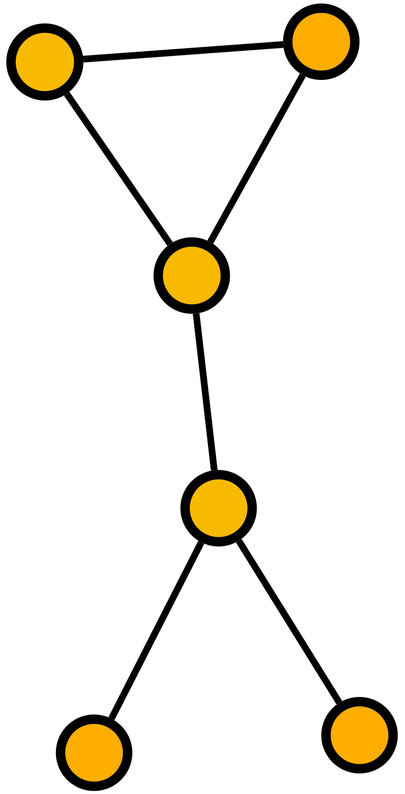

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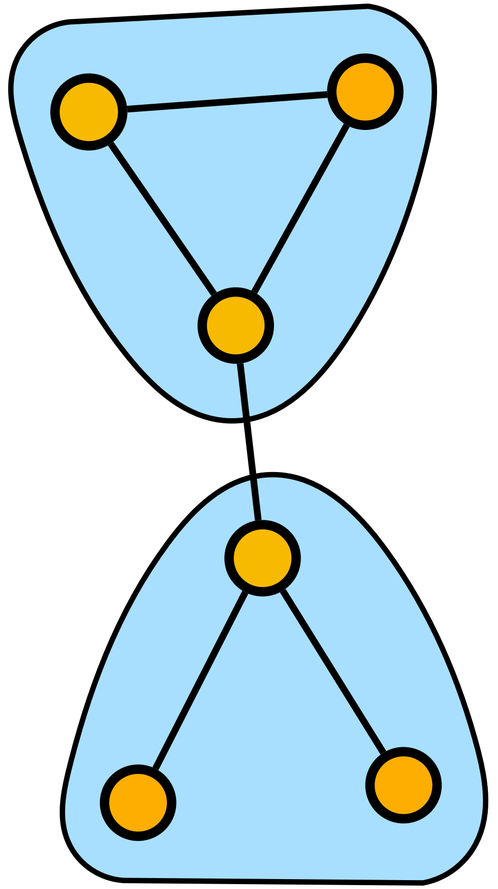

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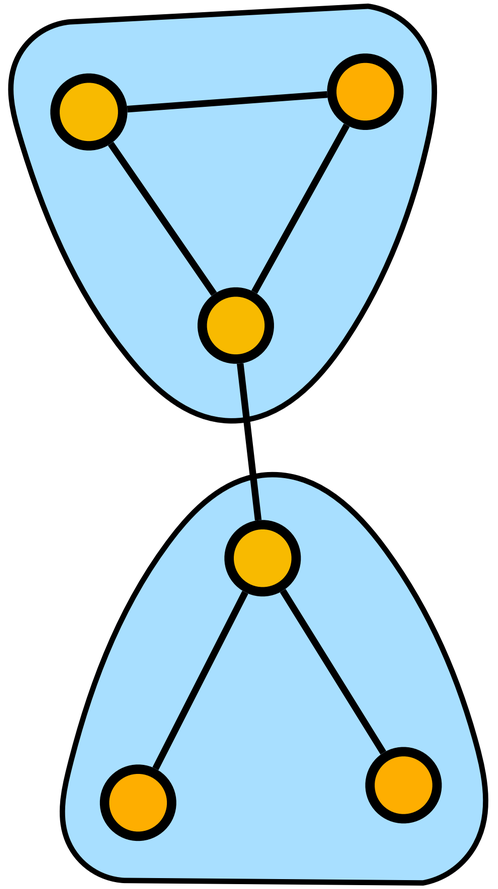

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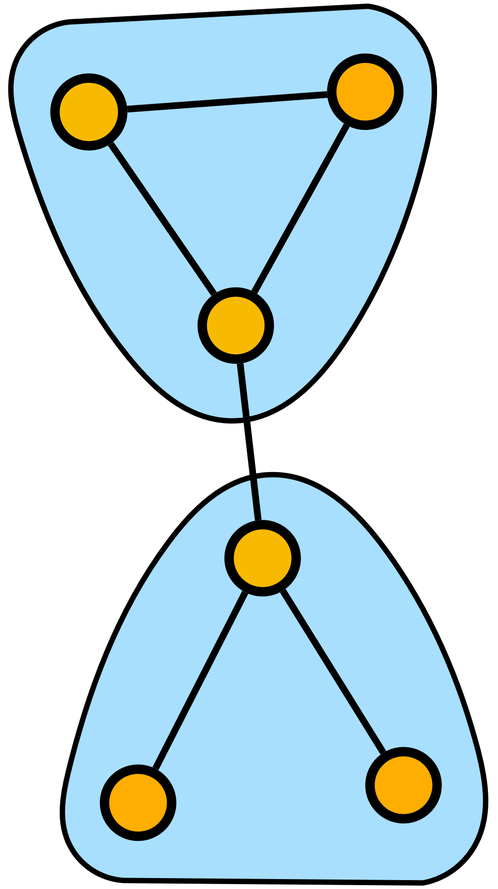

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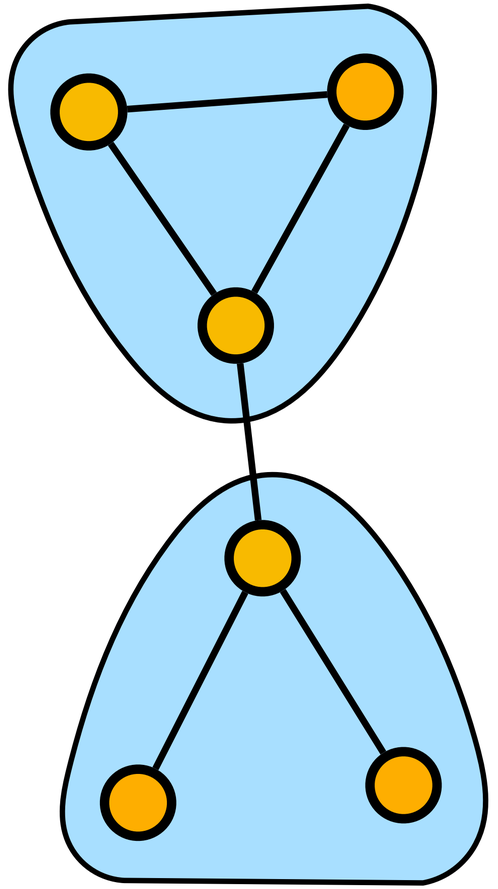

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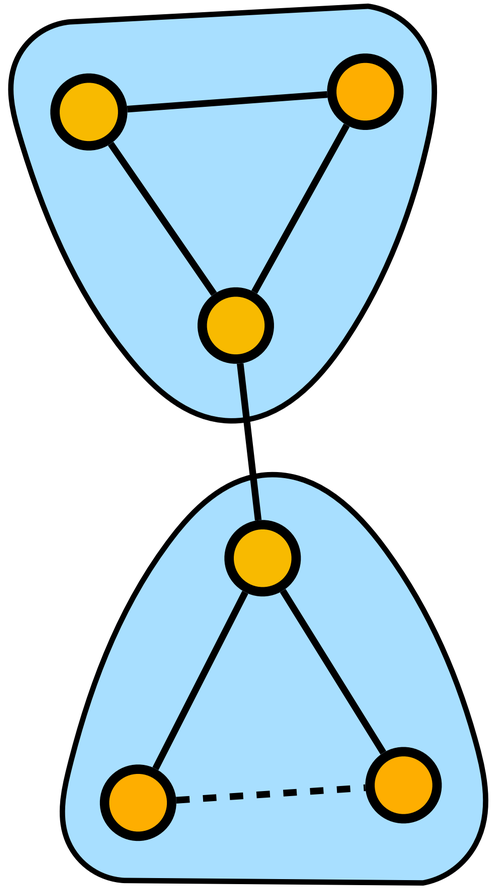

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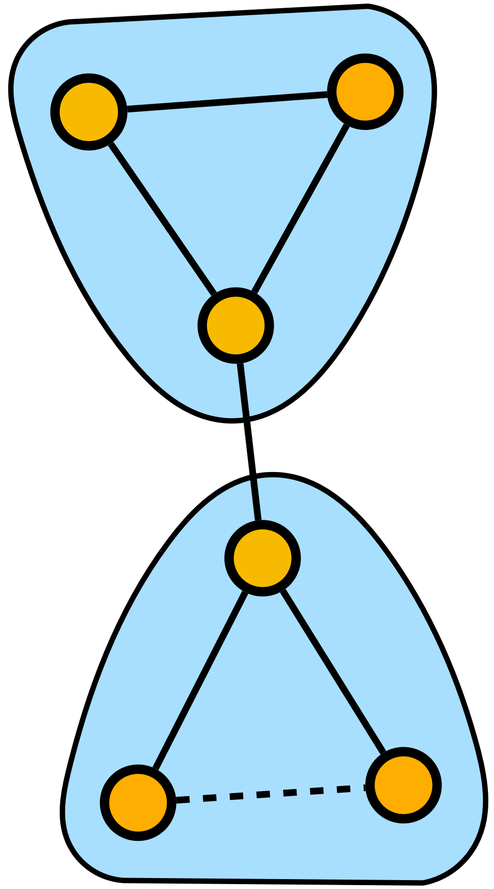

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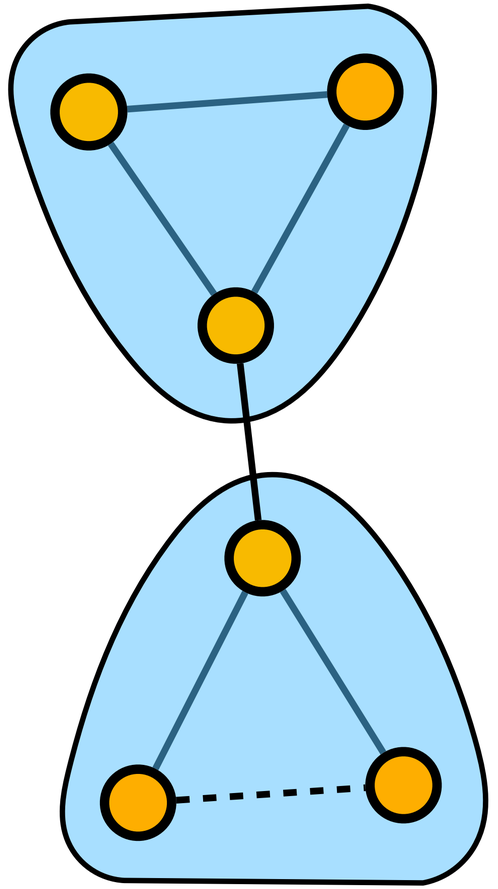

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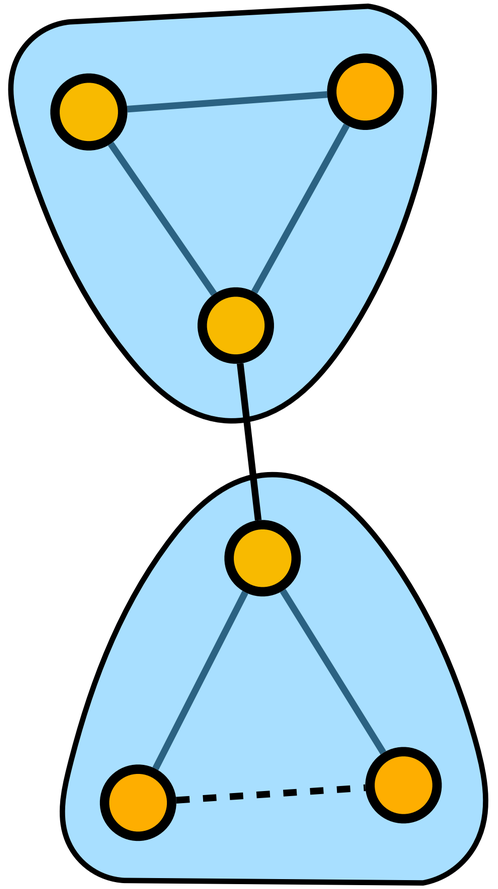

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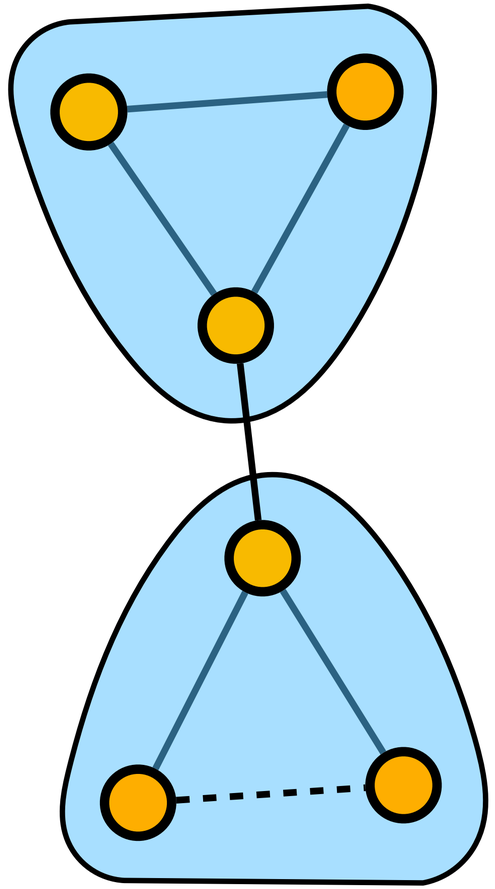

[CGW05] Charikar, Guruswami, Wirth Journal of Computer and System Sciences 2005

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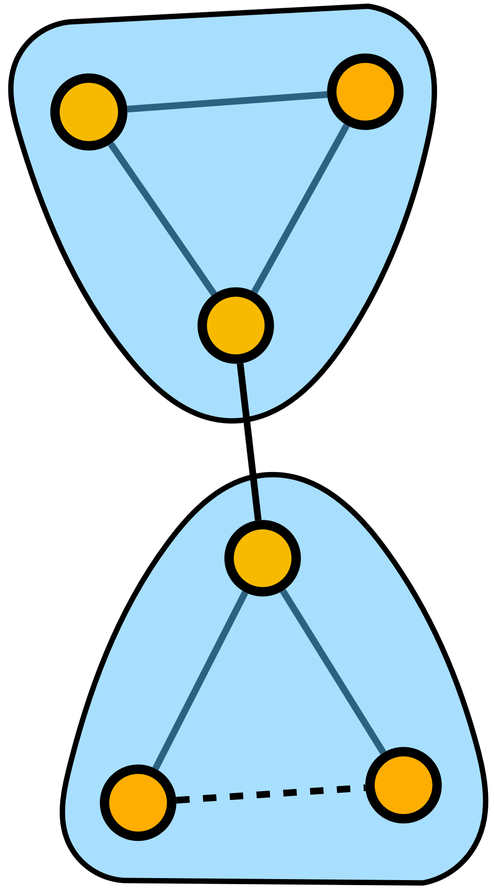

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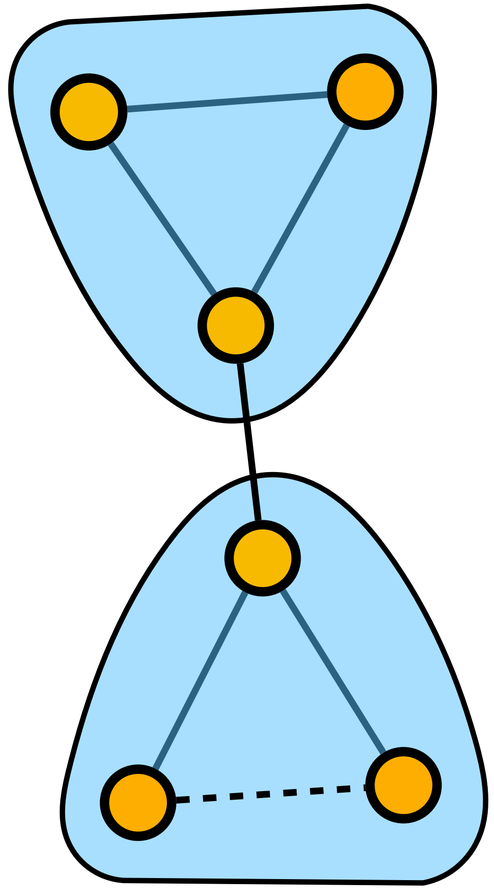

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[CLP+24] Cohen-Addad, Lolck, Pilipczuk, Thorup, Yan, and Zhang. STOC 2024

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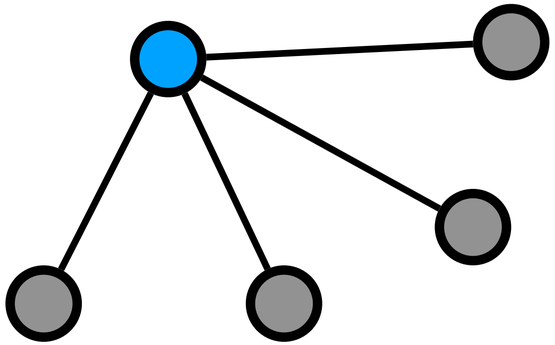

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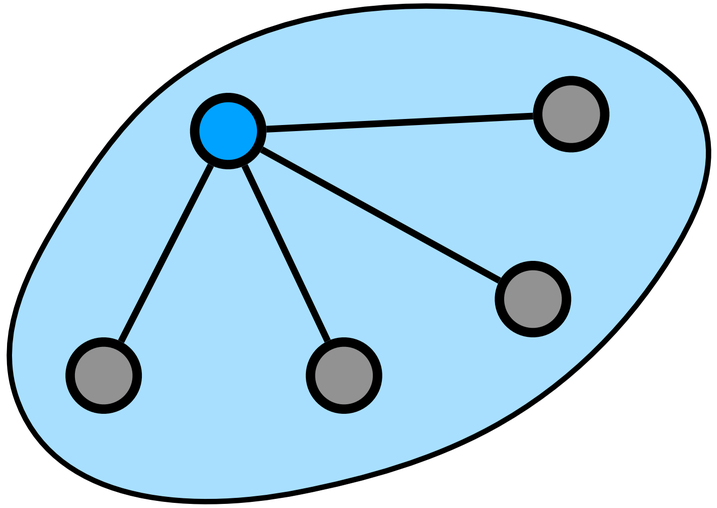

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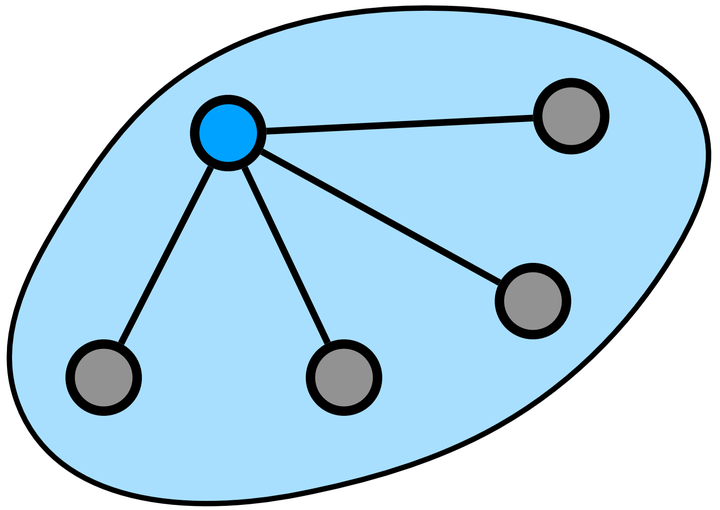

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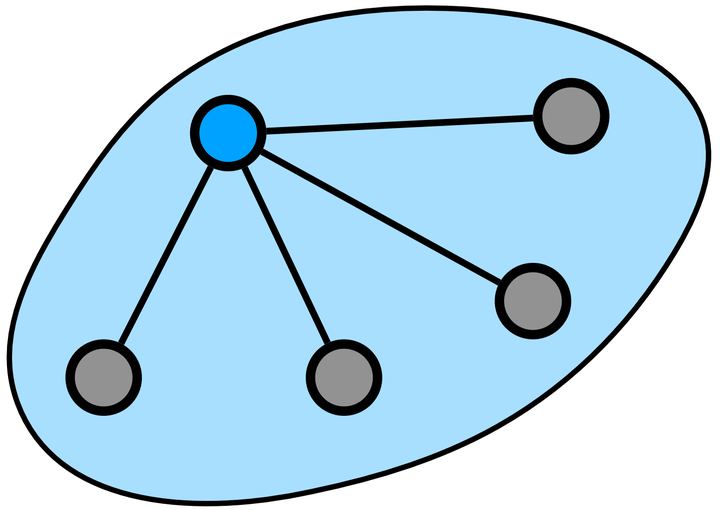

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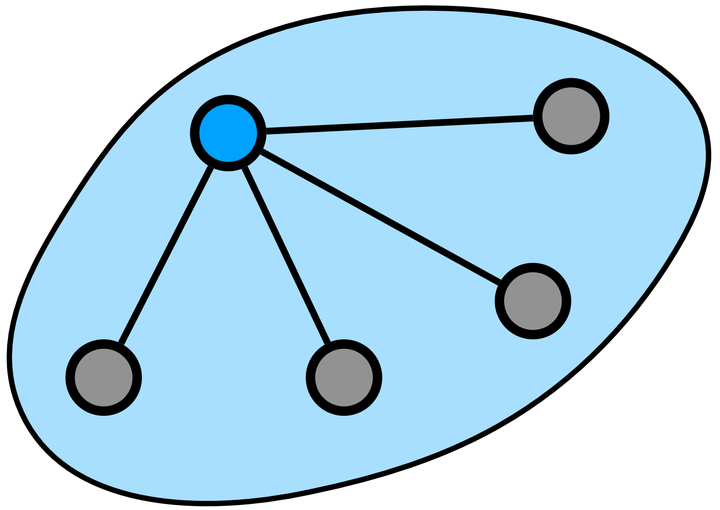

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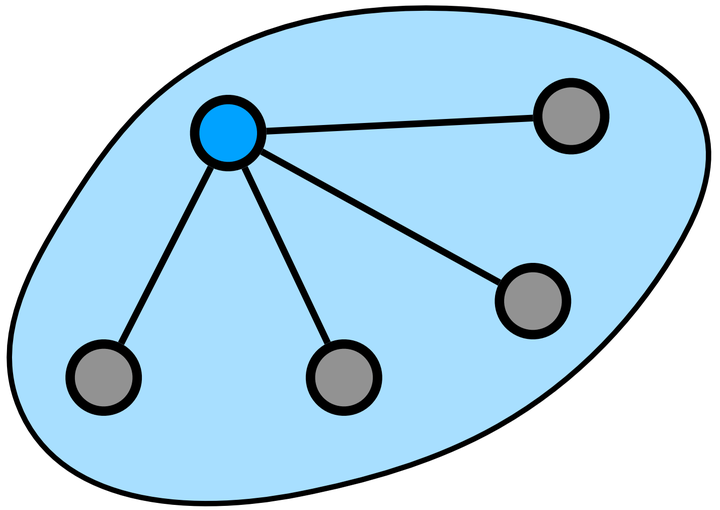

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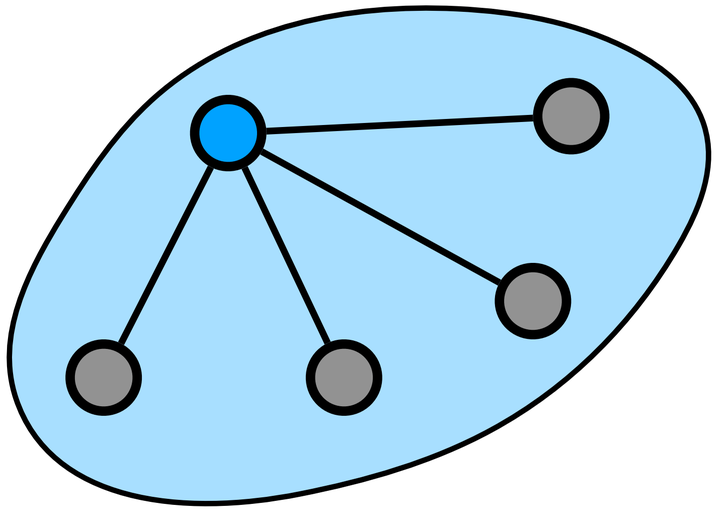

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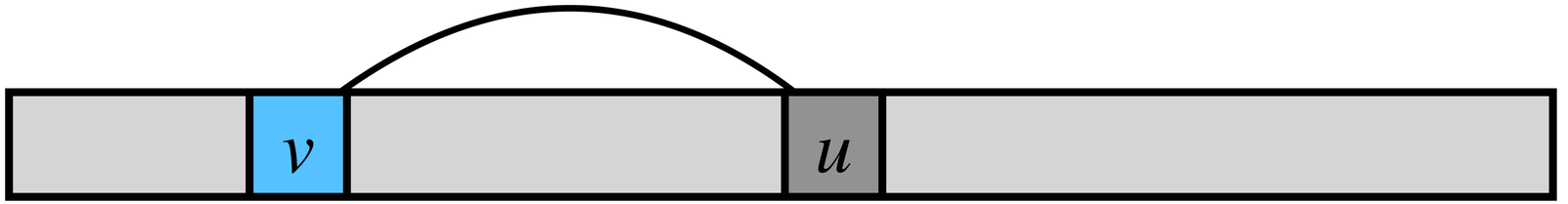

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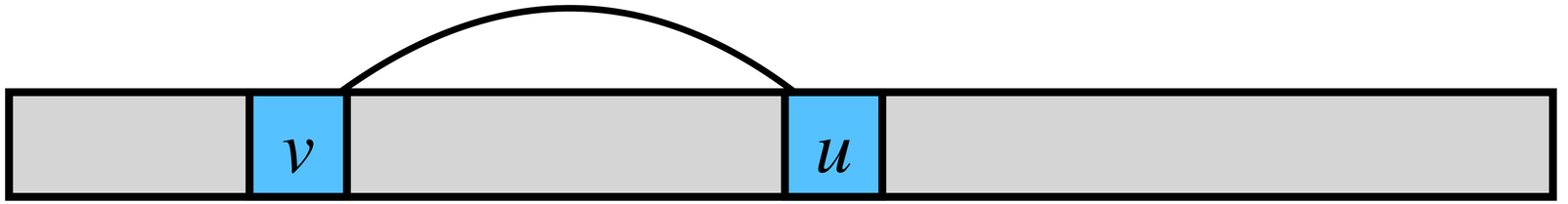

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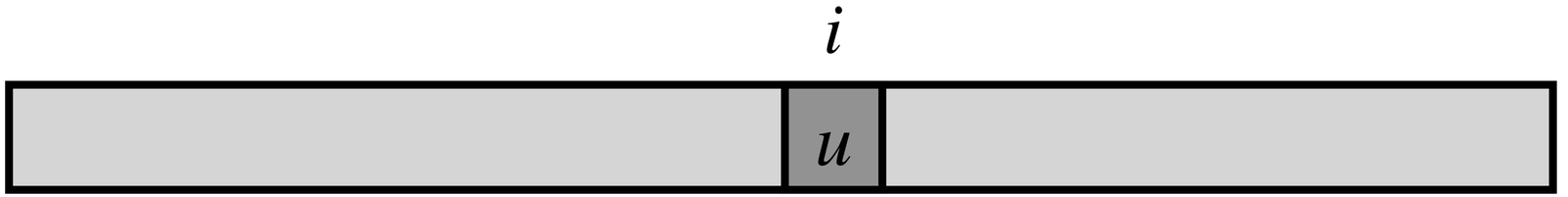

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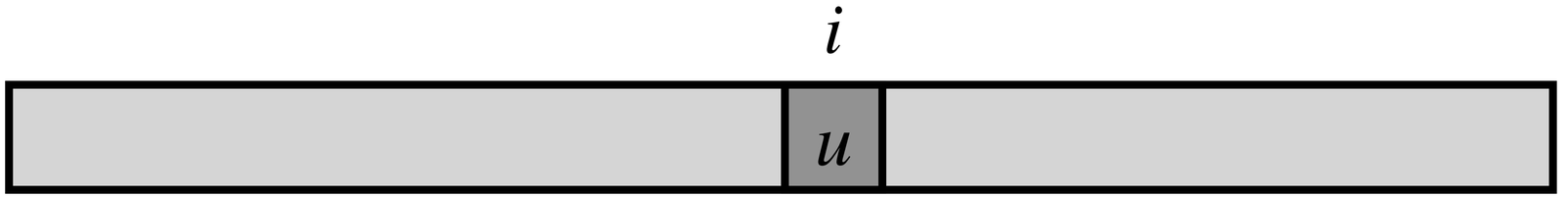

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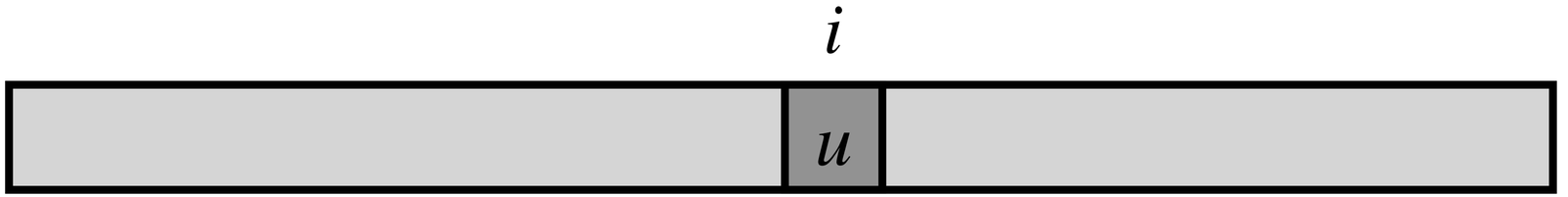

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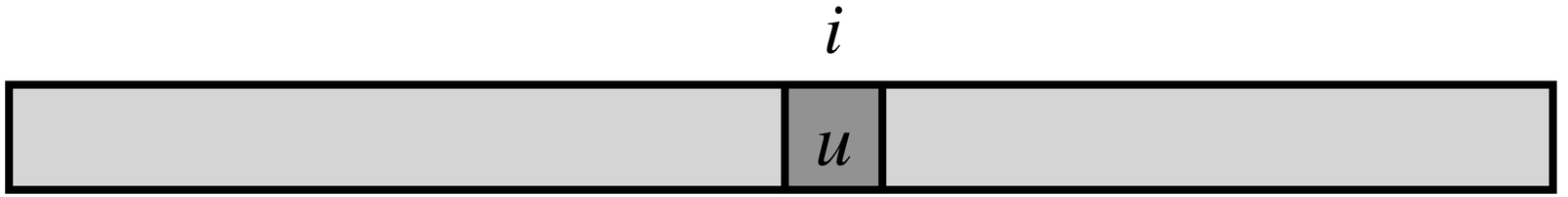

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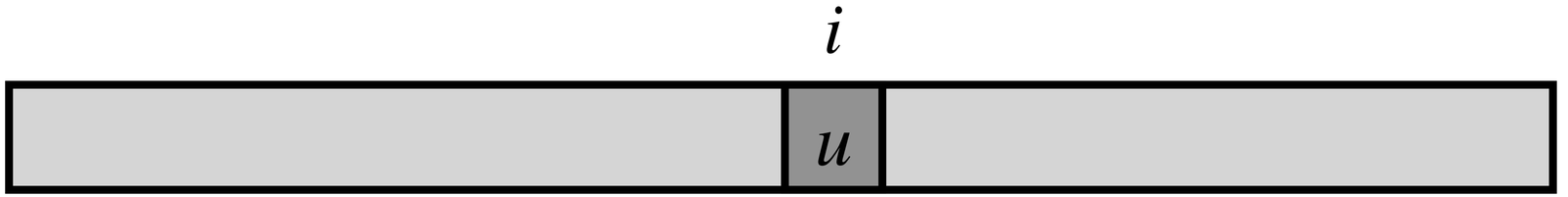

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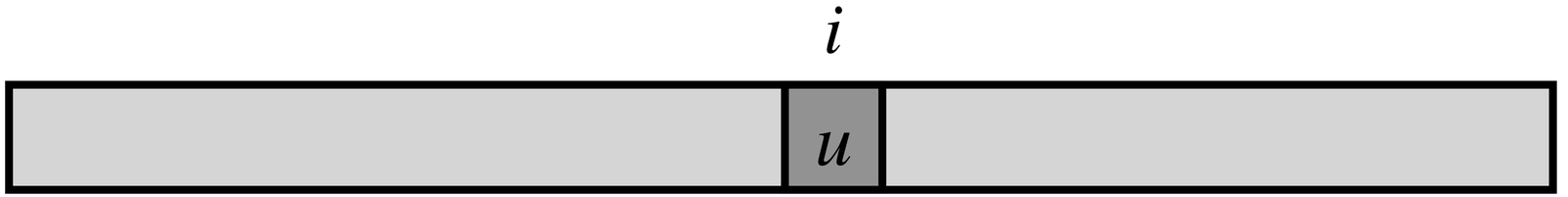

[CM23] improves this to $O(n/\varepsilon)$.

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[CM23] Chakrabarty, Makarychev. NeurIPS 2023

Approximation Analysis

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“$+\varepsilon$”-approximation

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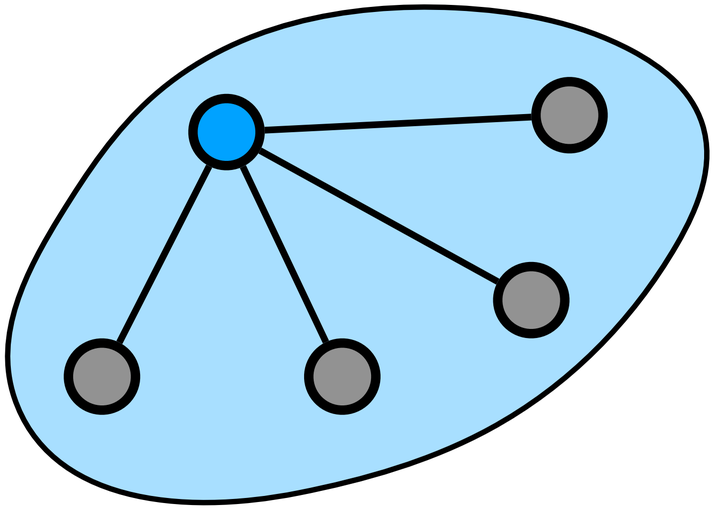

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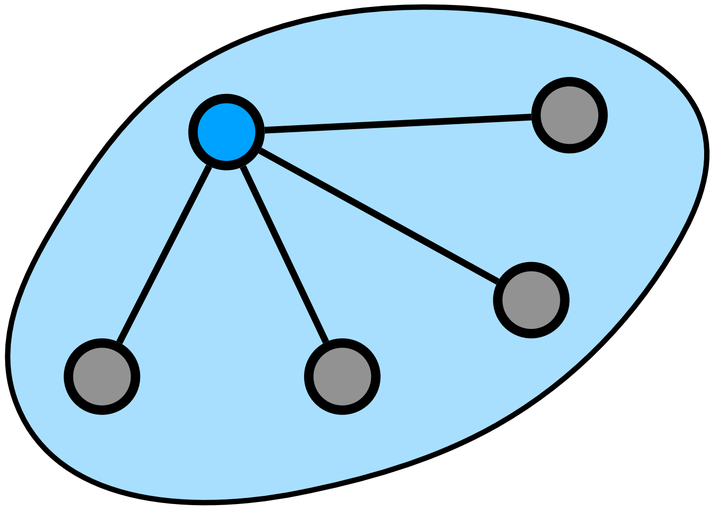

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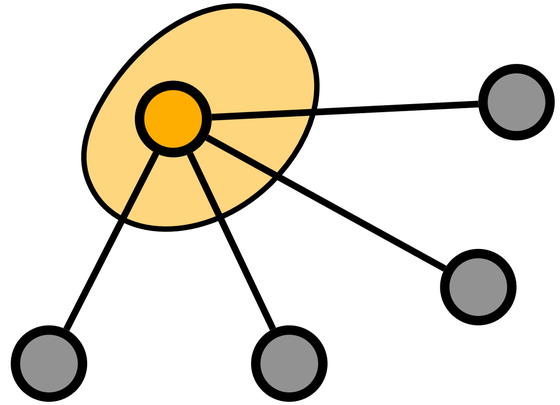

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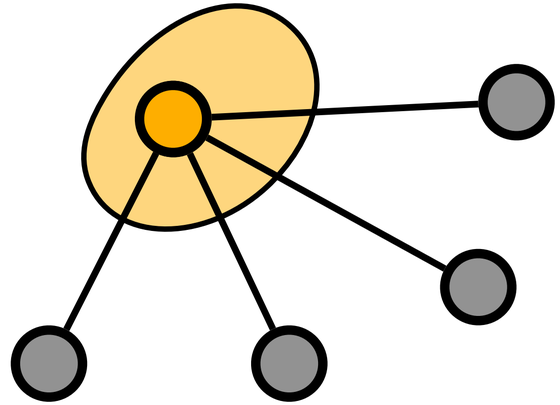

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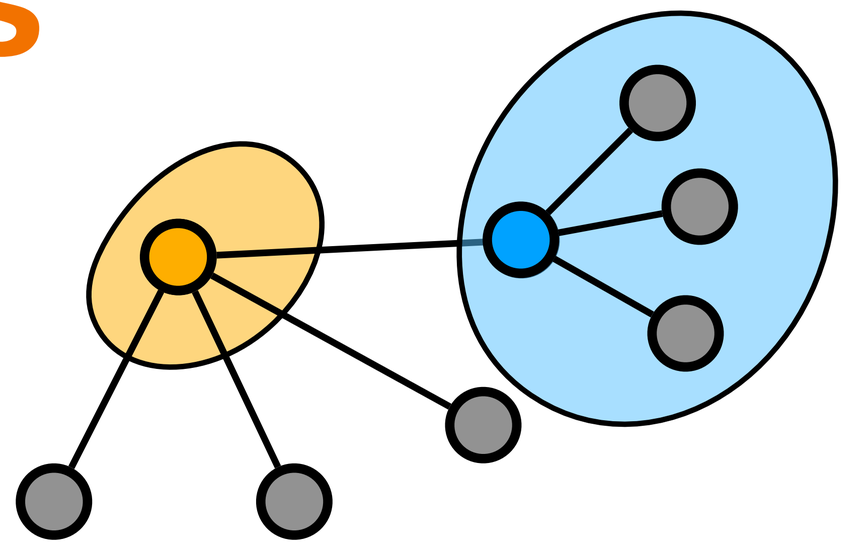

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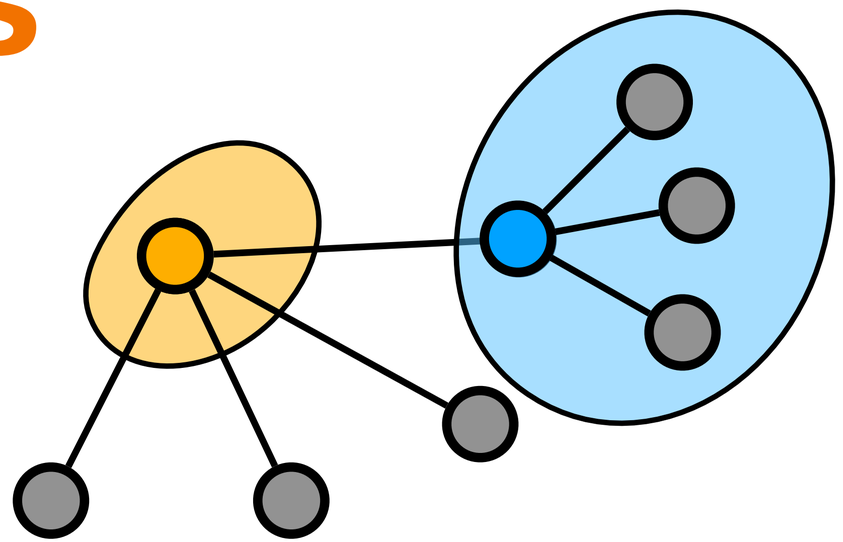

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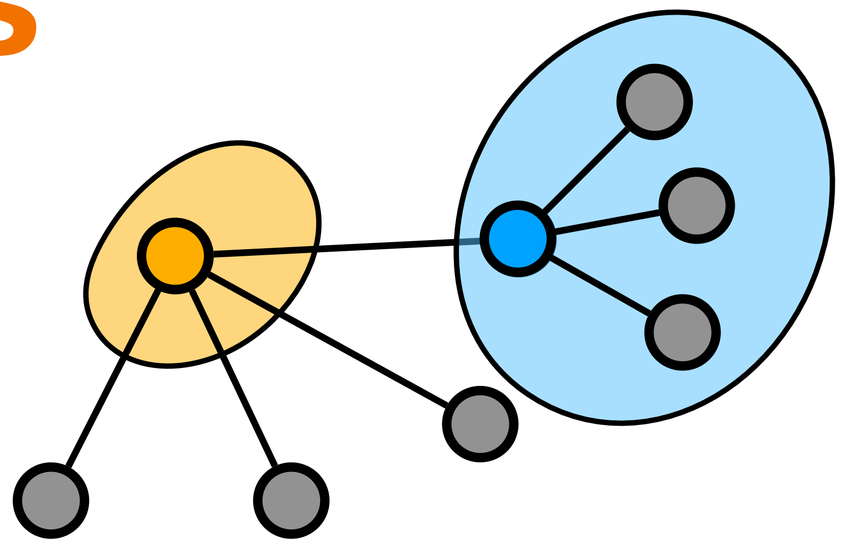

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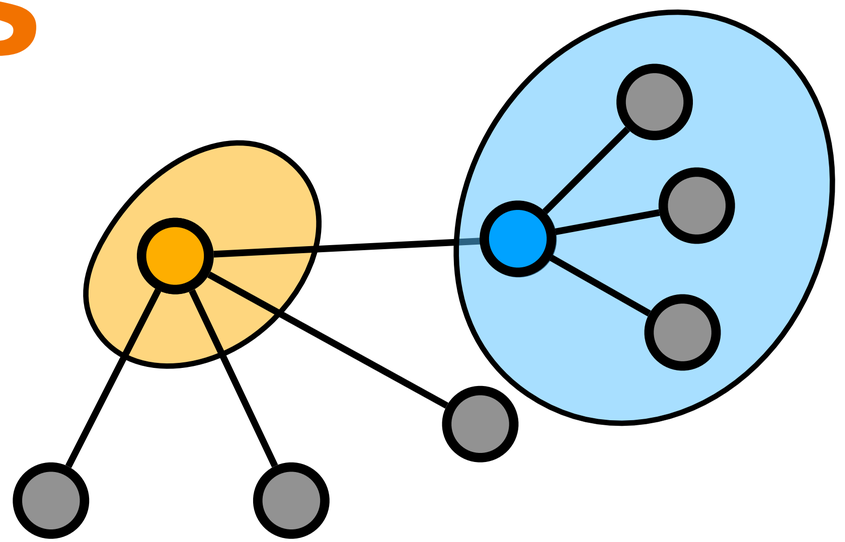

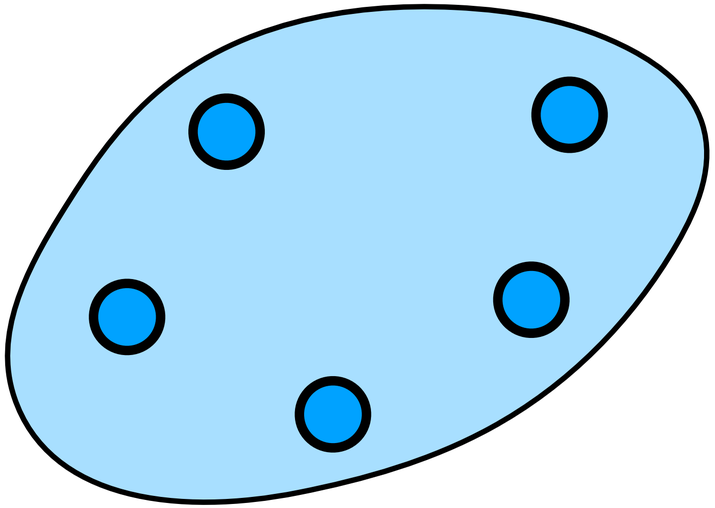

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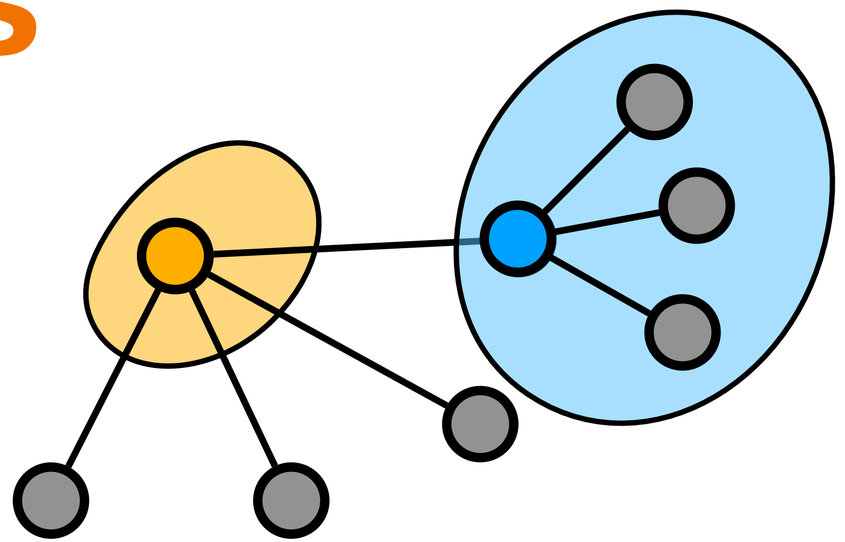

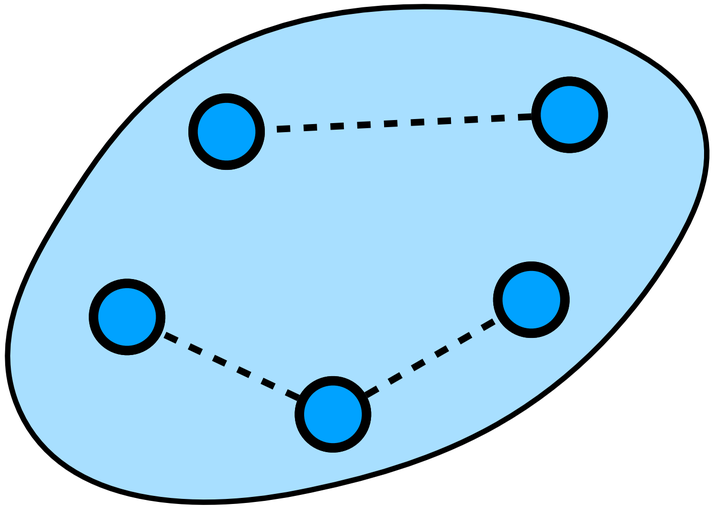

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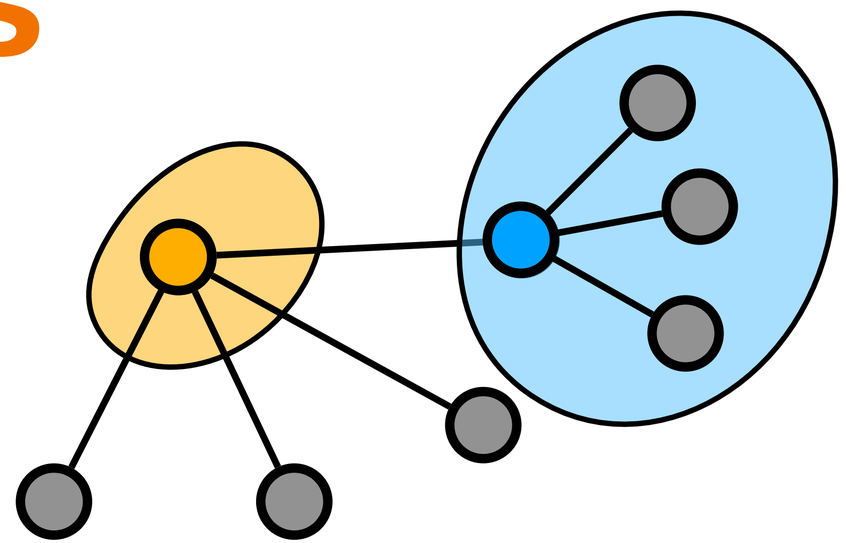

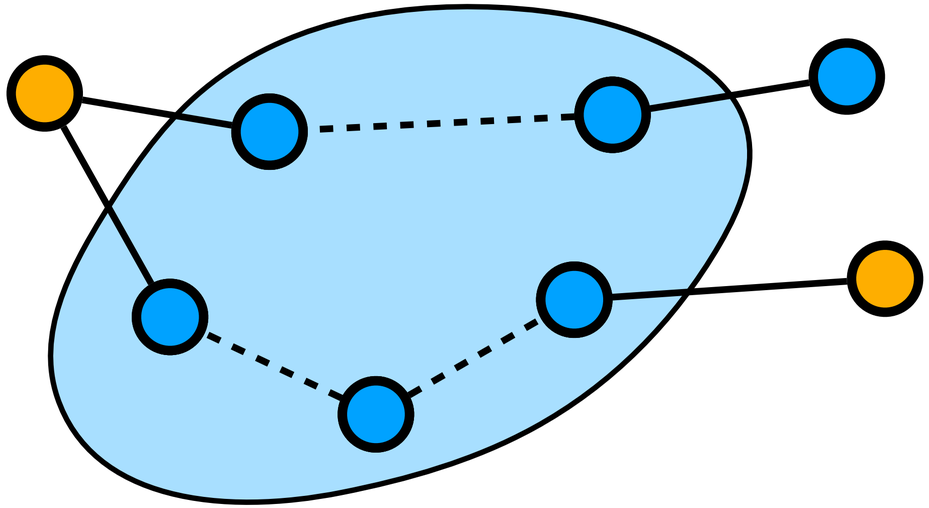

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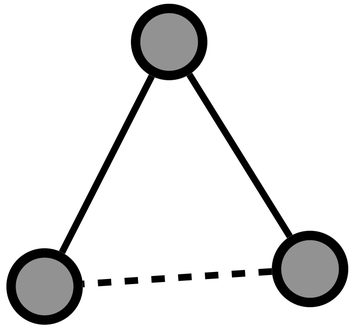
Bad Triangles



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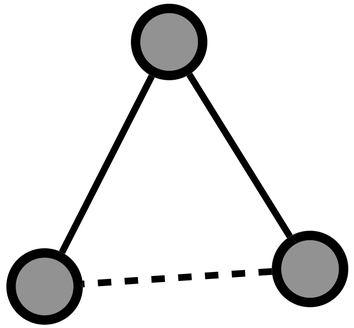

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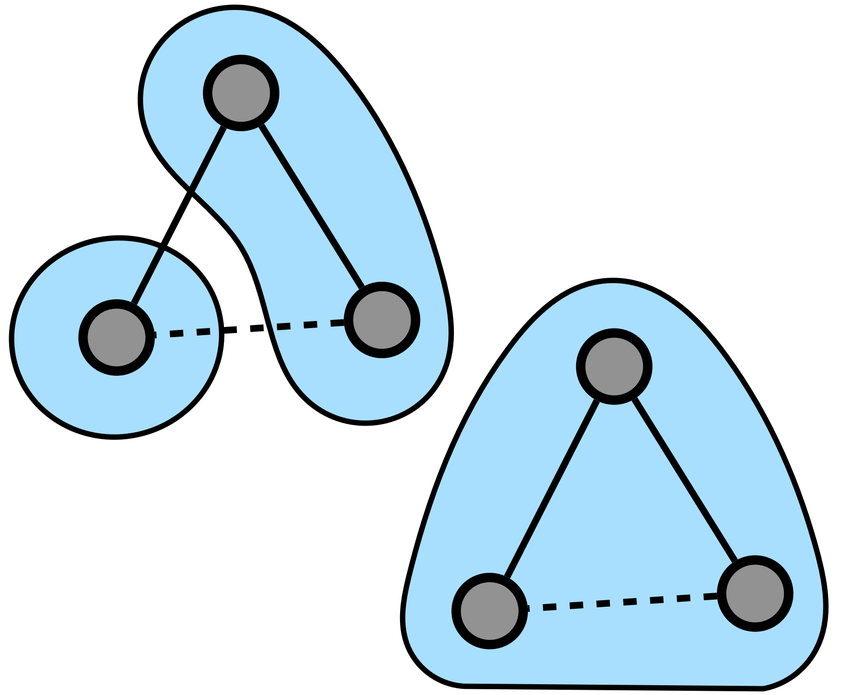

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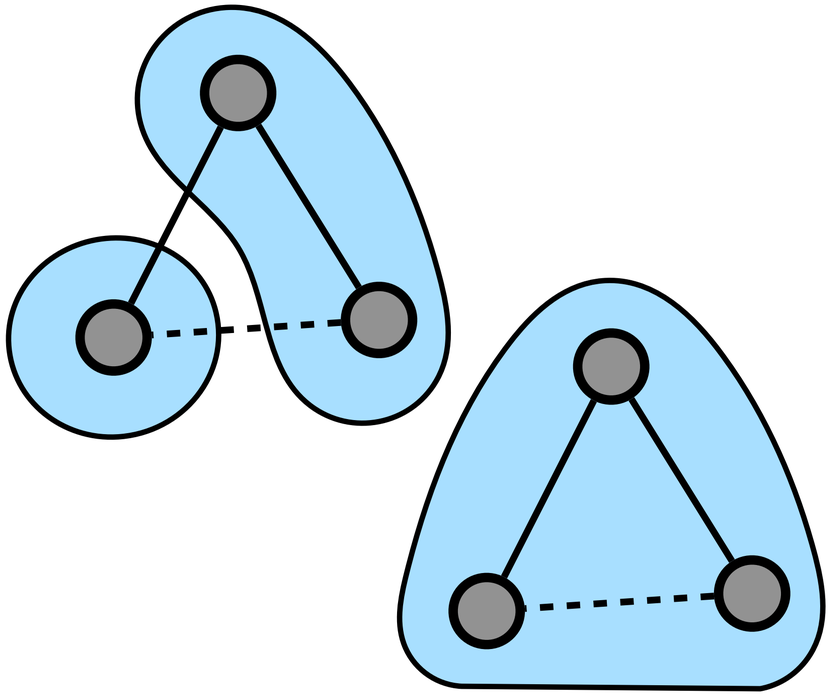

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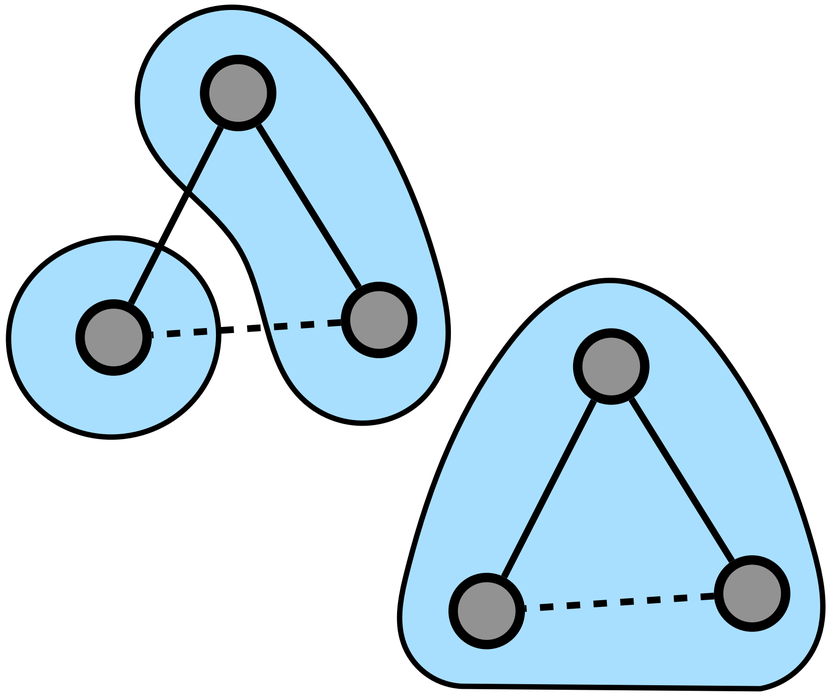

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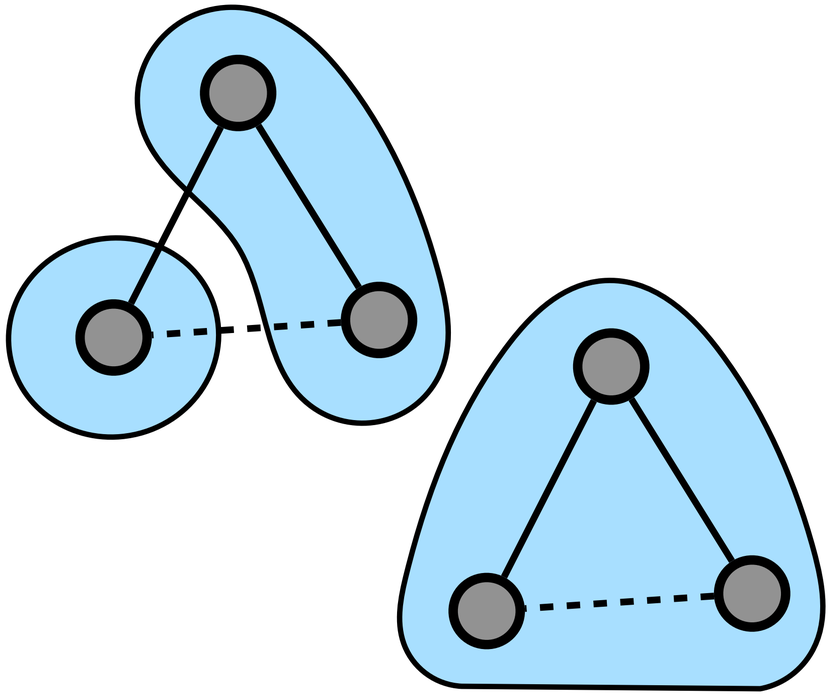

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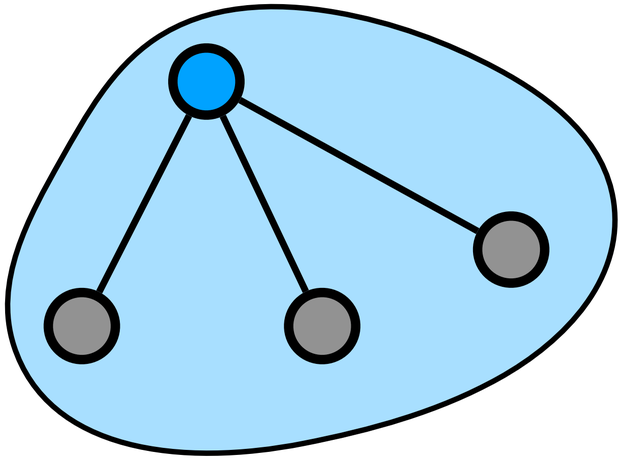

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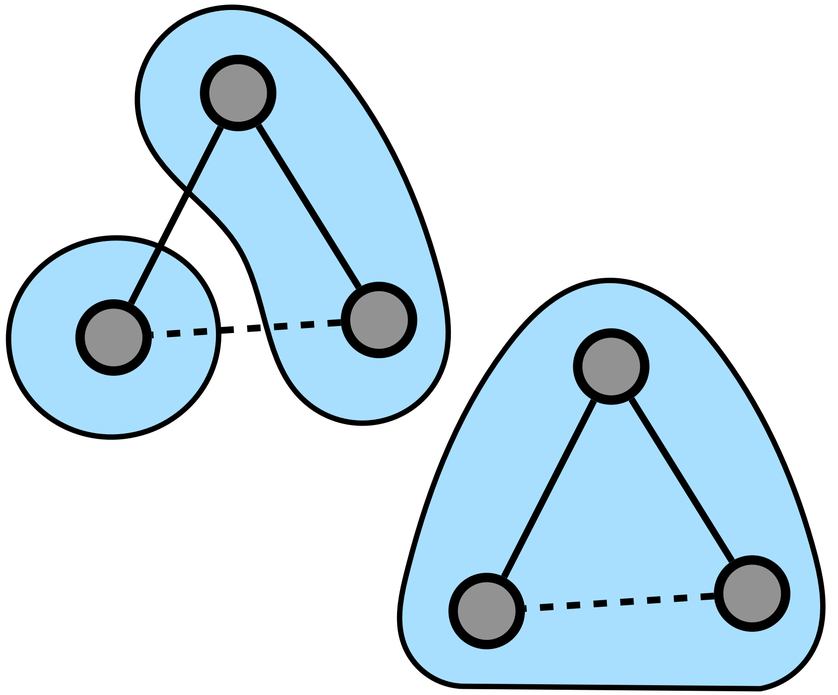

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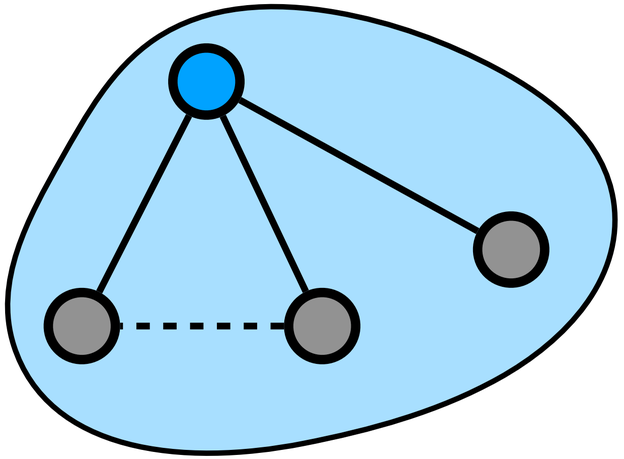

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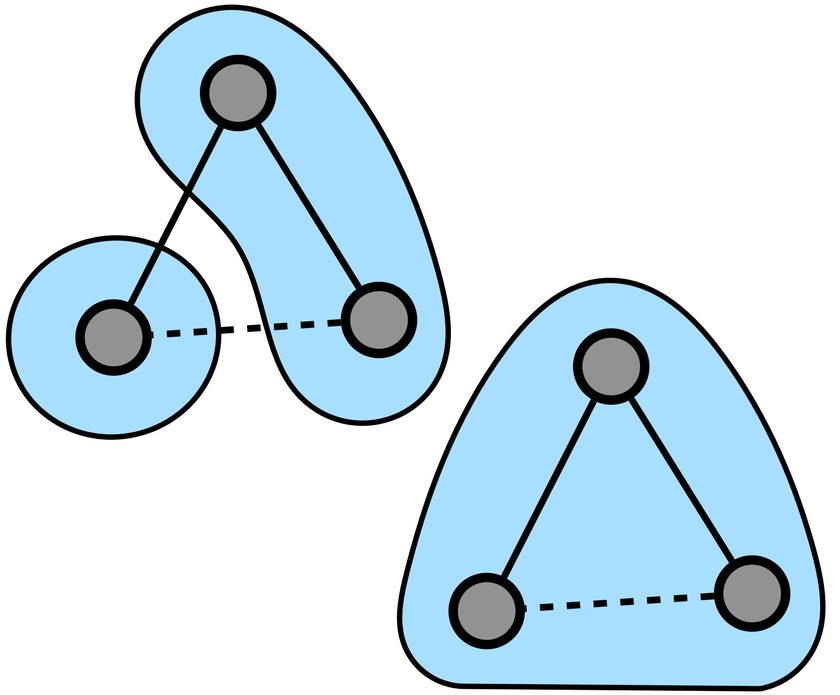

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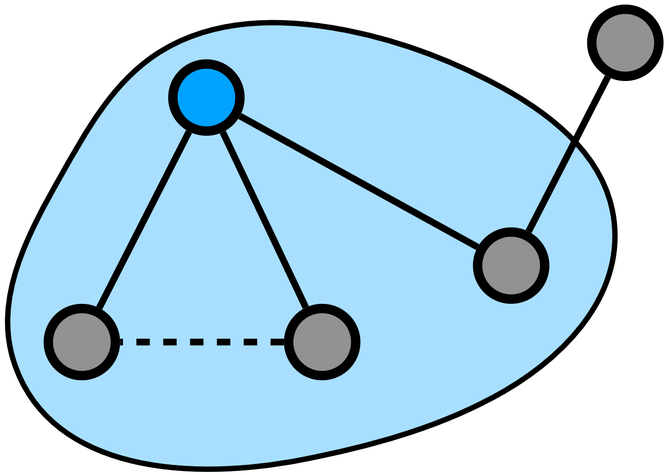

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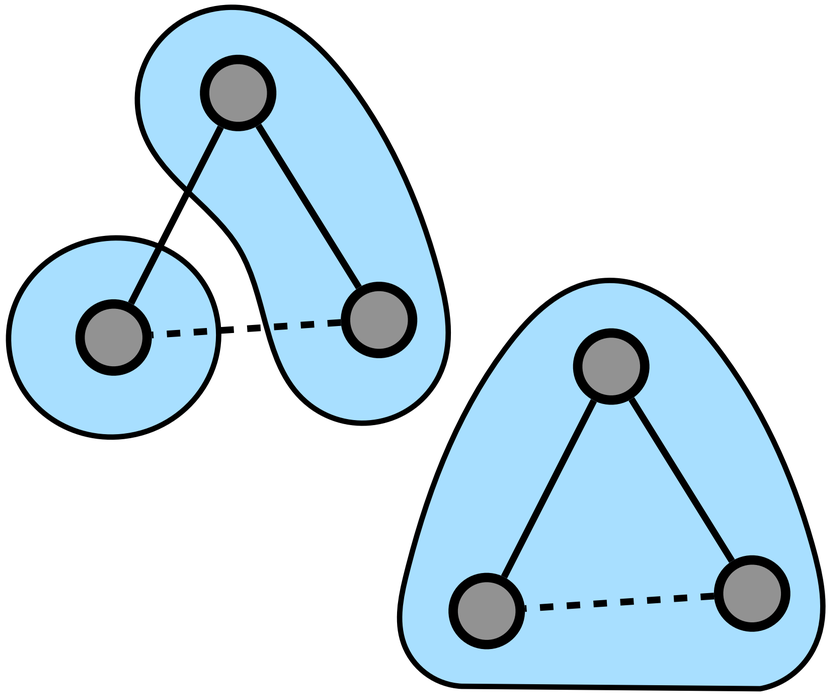
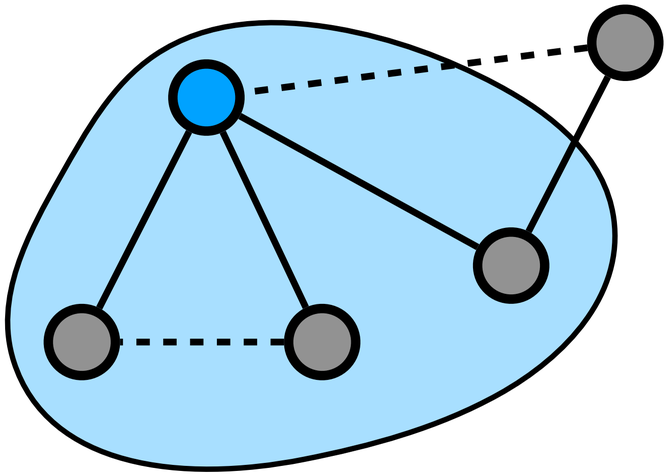


- A bad triangle is a triple containing two edges and one non-edge.
    - No matter how you cluster a bad triangle, you will make **at least one mistake**.
    - If the graph has t edge-disjoint bad triangles, we will make at least t mistakes.
- Each mistake made by creating a pivot cluster can be charged to a unique bad triangle.



Charging Bad Triangles

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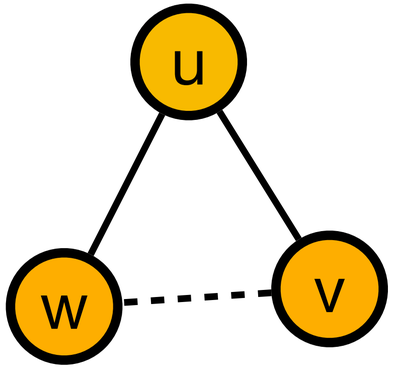

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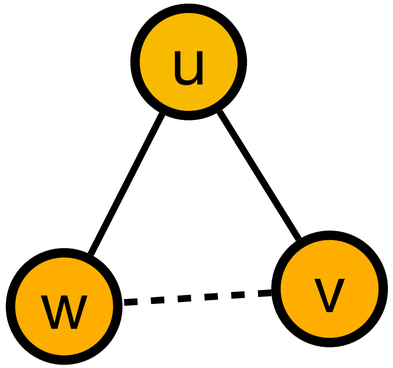

A_t : **all** of u, v, w are **active** and
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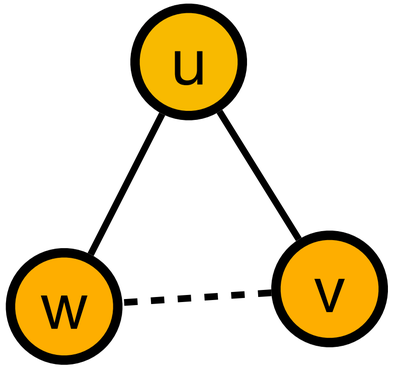

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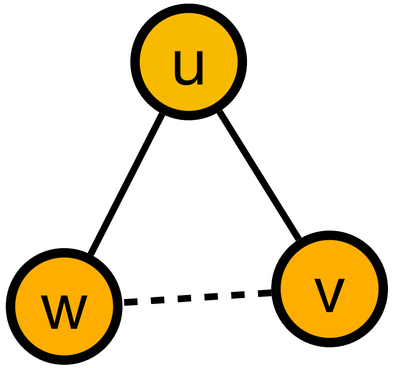

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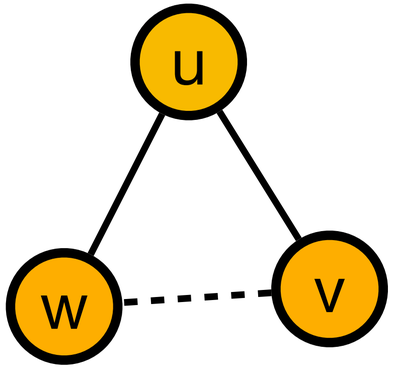

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How do we relate the p_t values to OPT?

Linear Program Relaxation

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Dual LP:

$$\max \sum_{t \in BT} y_t$$

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Dual Feasible Solution

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- Why does $y_t = p_t/3$ satisfy all the dual constraints?

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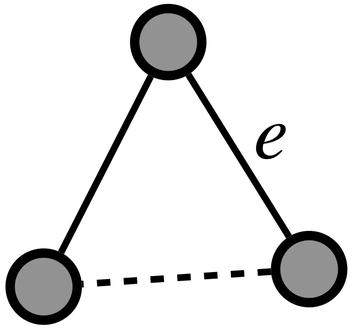

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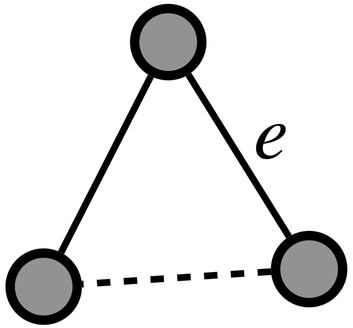

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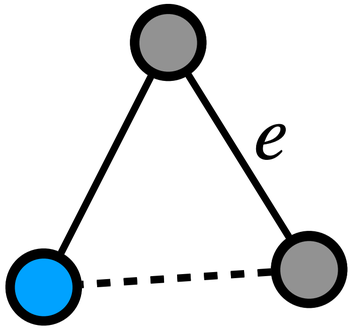

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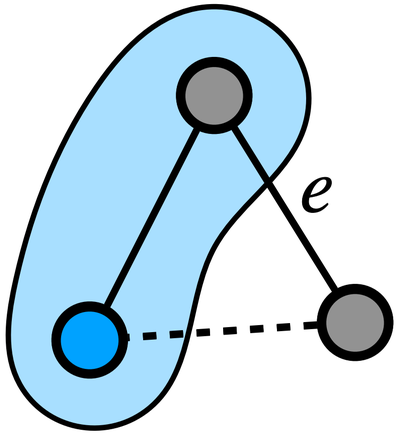

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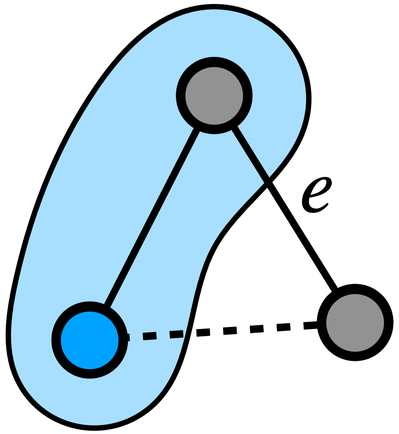

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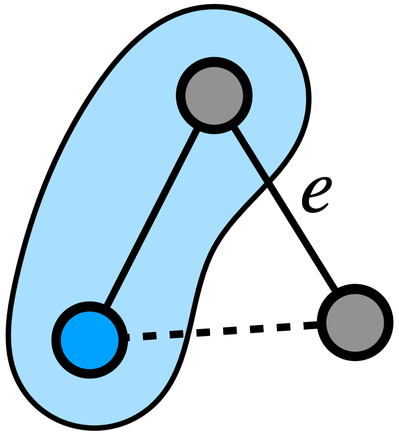

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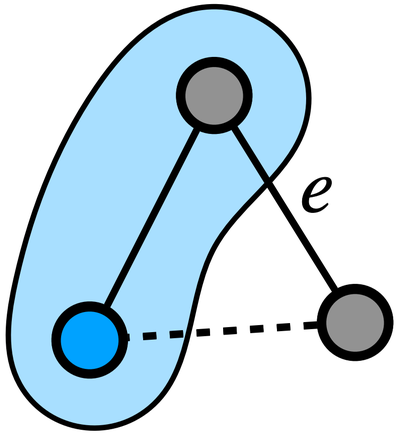

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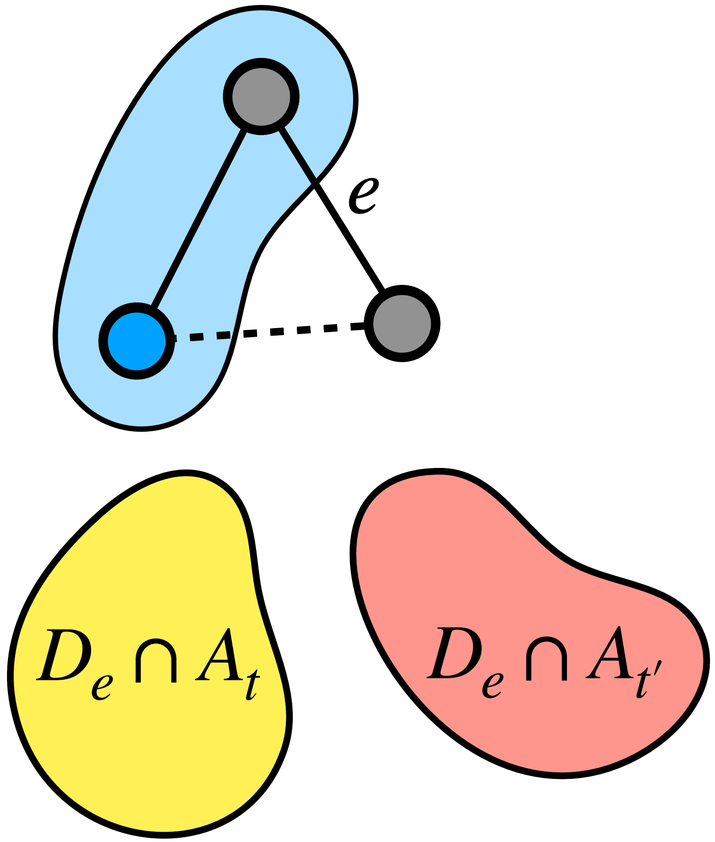

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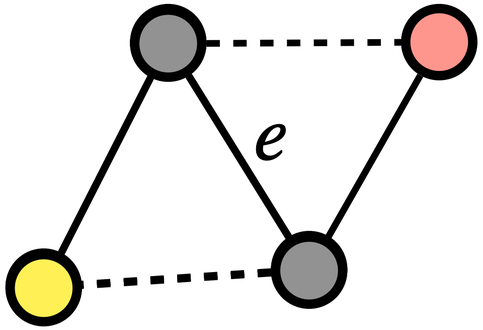

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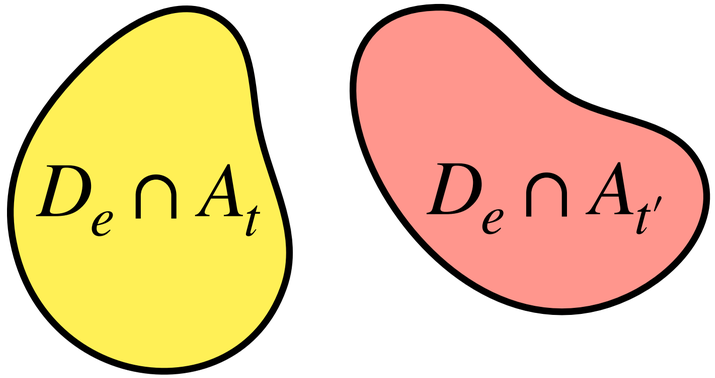

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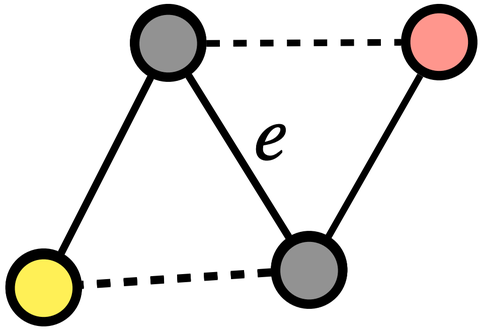

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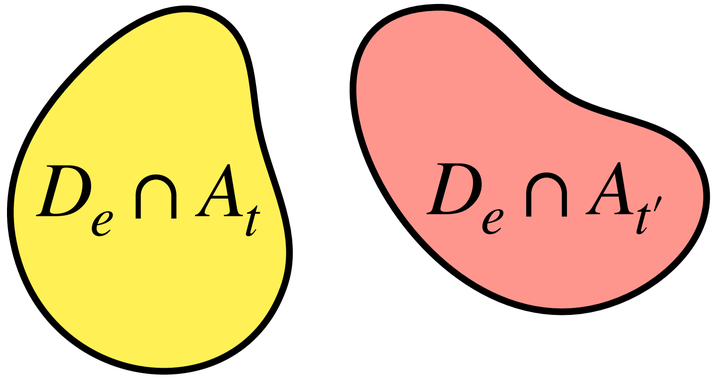


Approximation Analysis

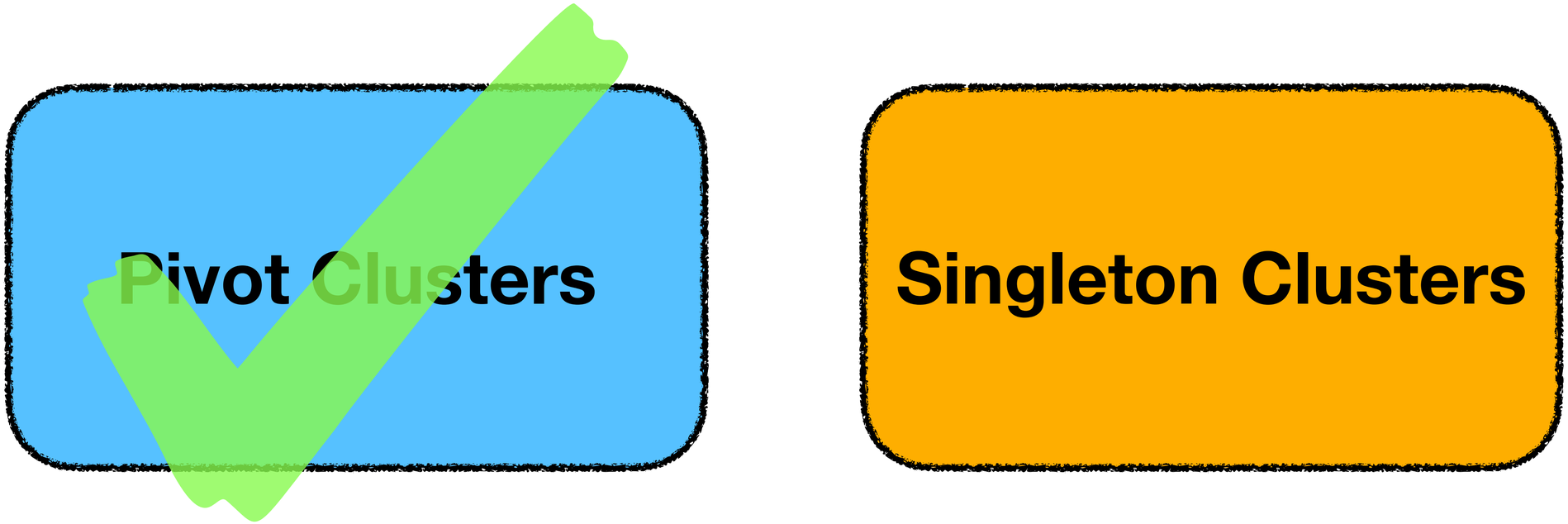
Pivot Clusters

3-approximation

Singleton Clusters

“$+\varepsilon$”-approximation

Approximation Analysis



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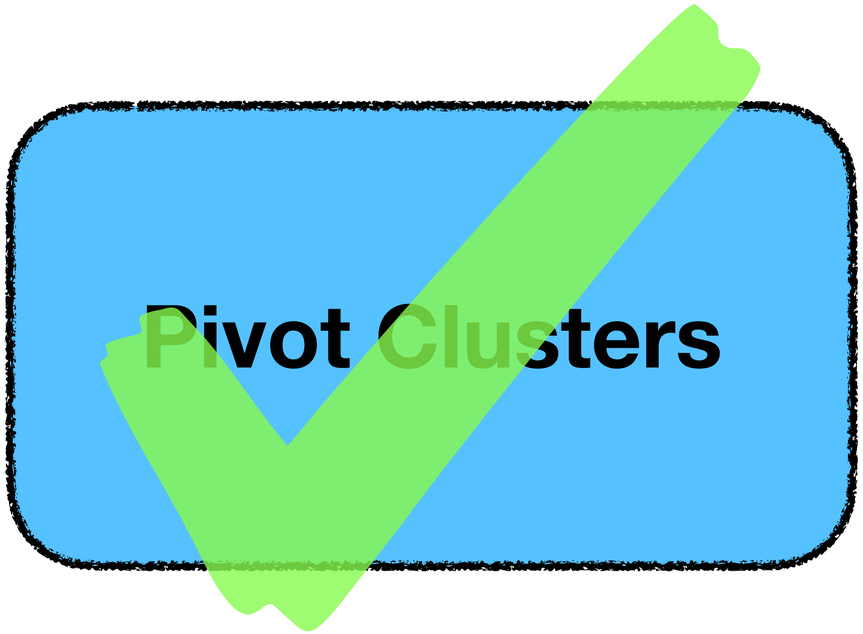

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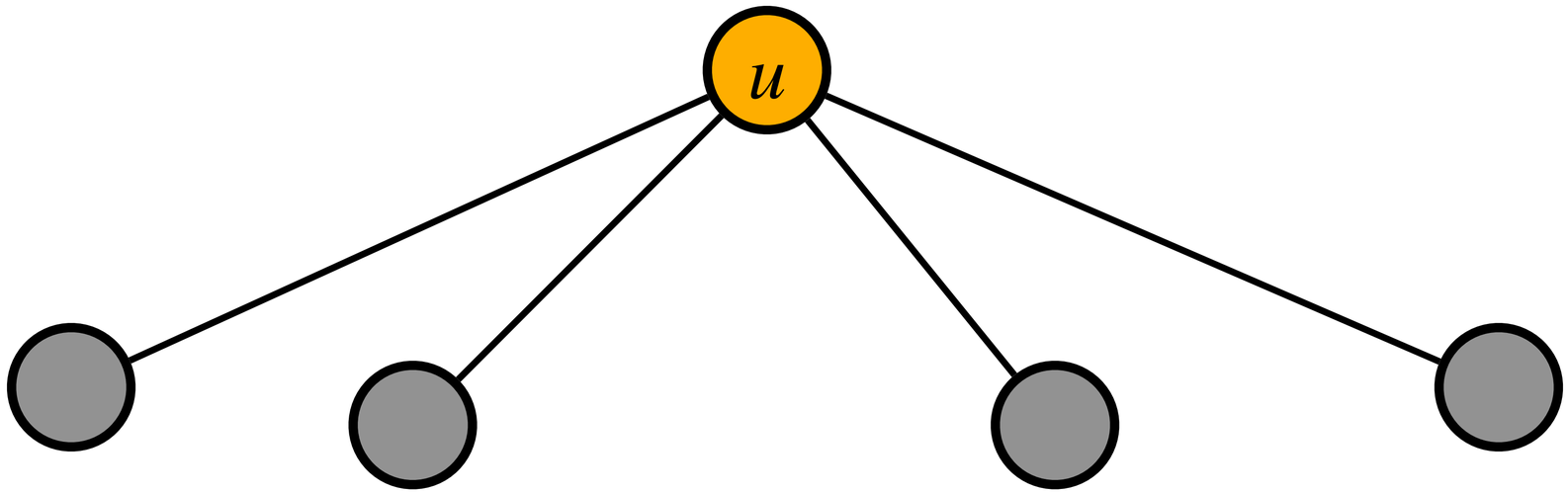

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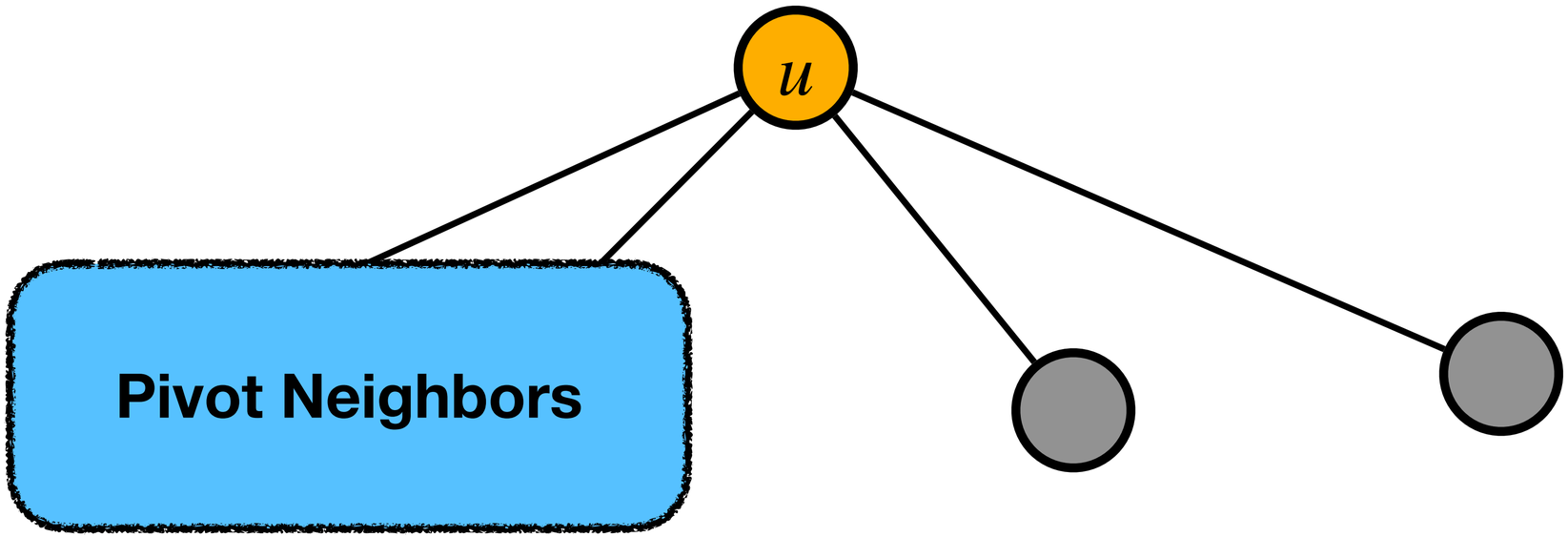

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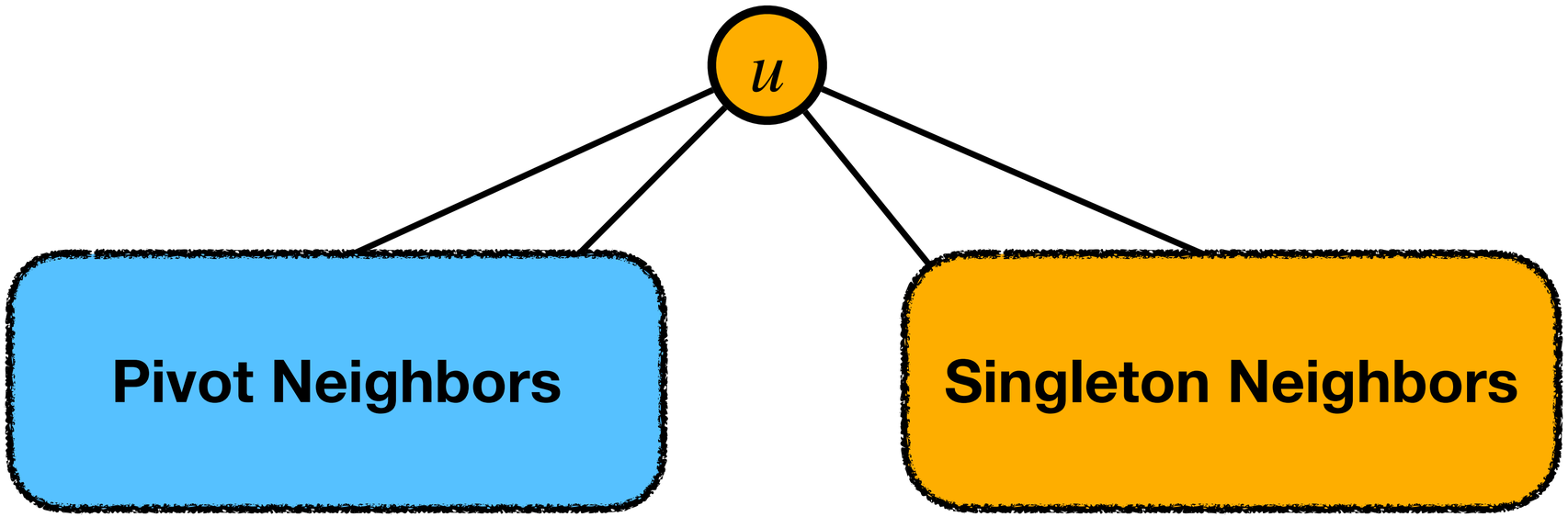

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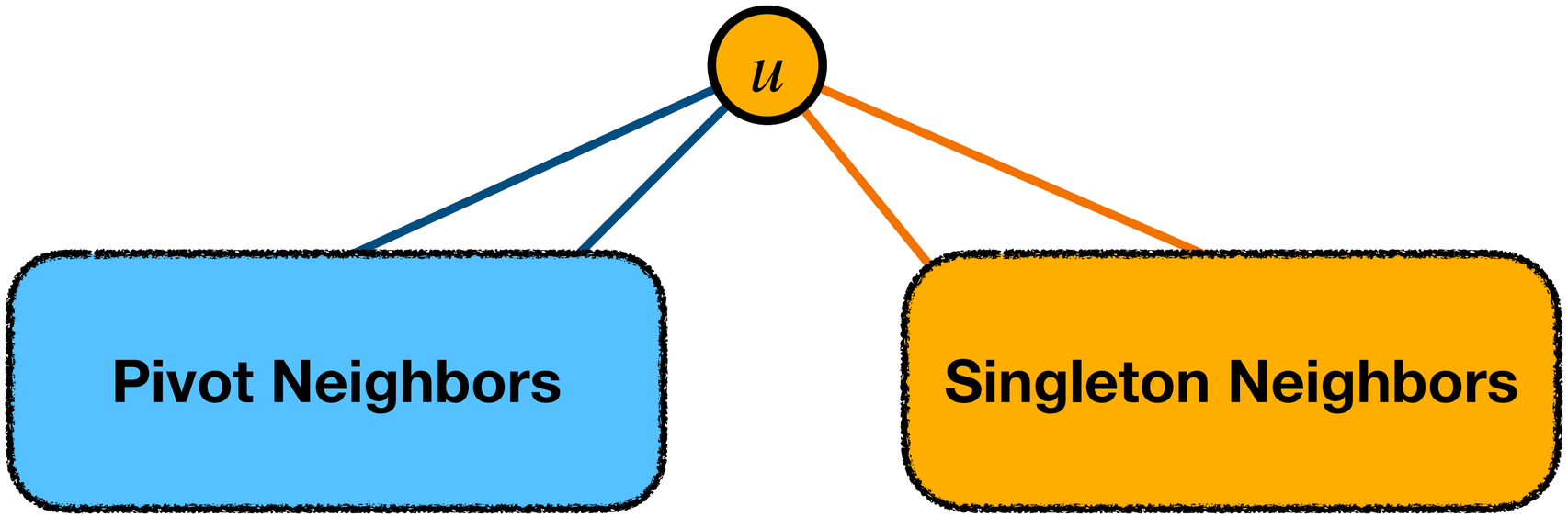

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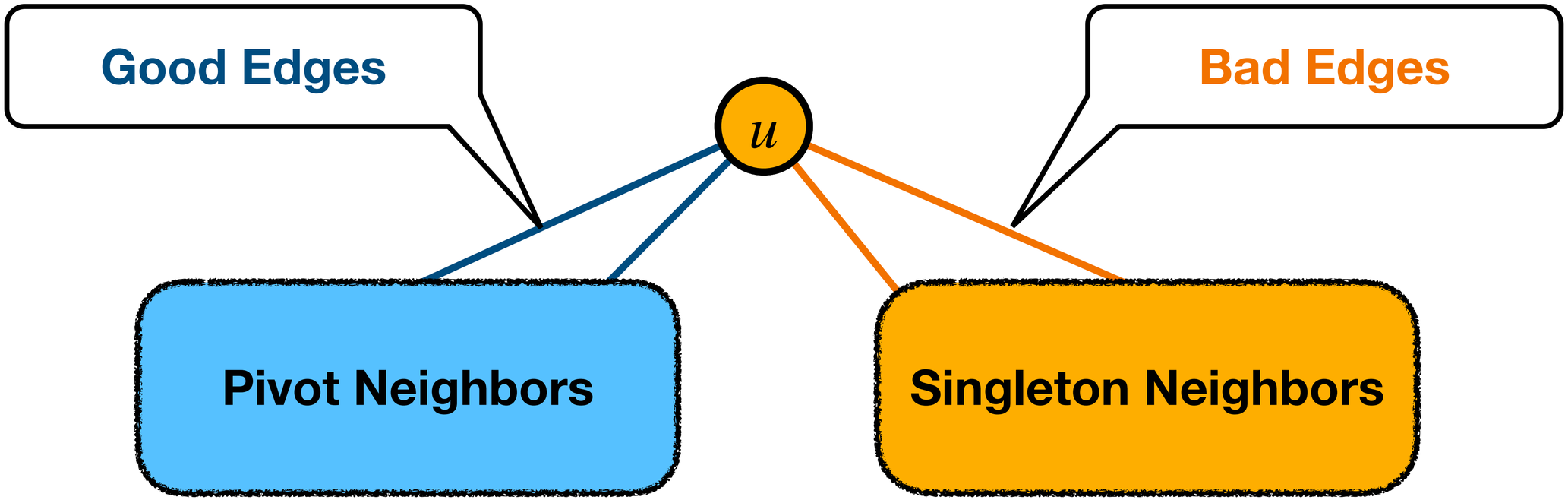

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- This is not true for each node as all edges can be bad.

Counting Bad Edges

Random Permutation π :

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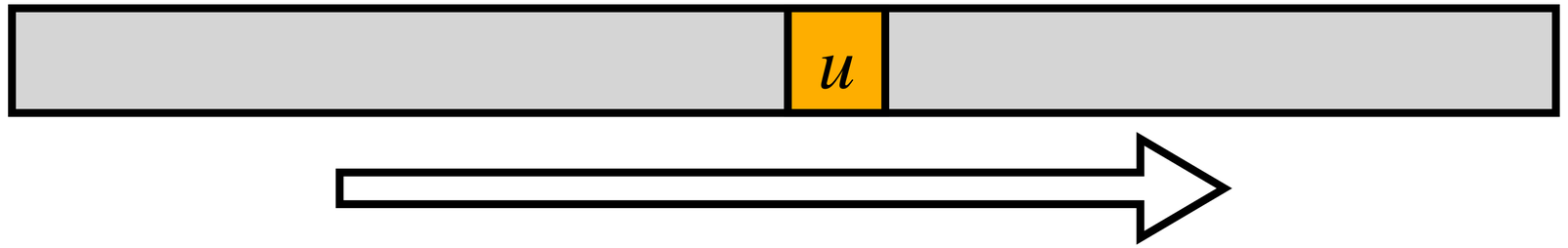

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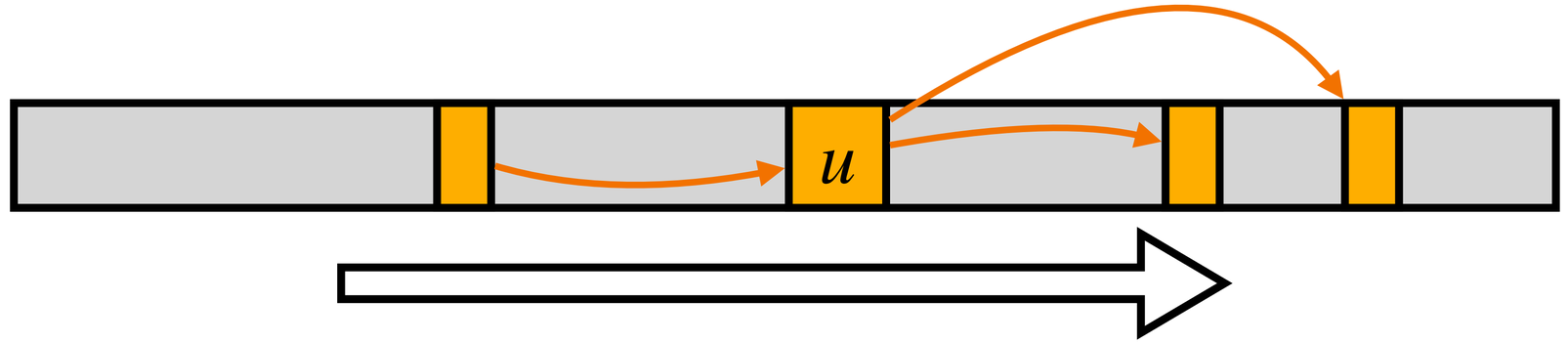

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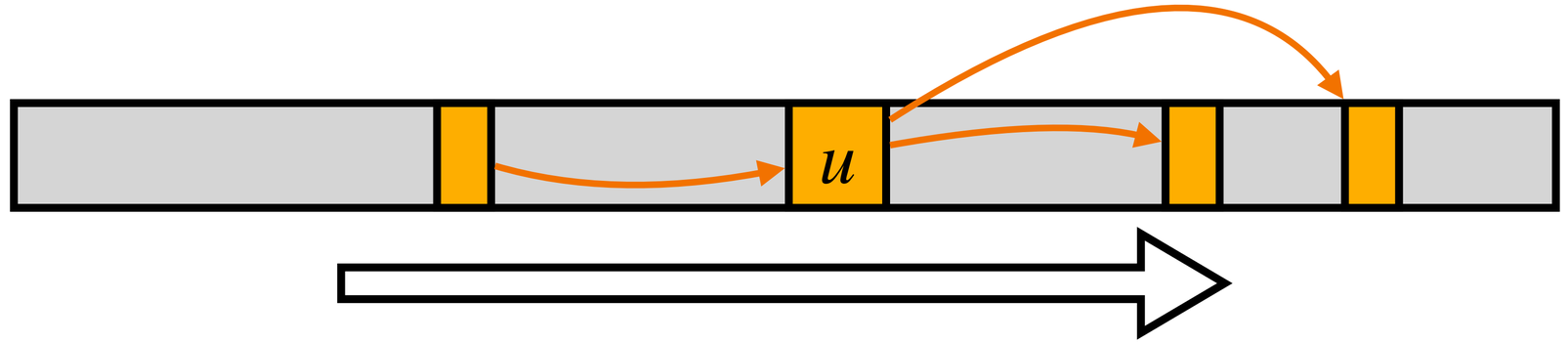

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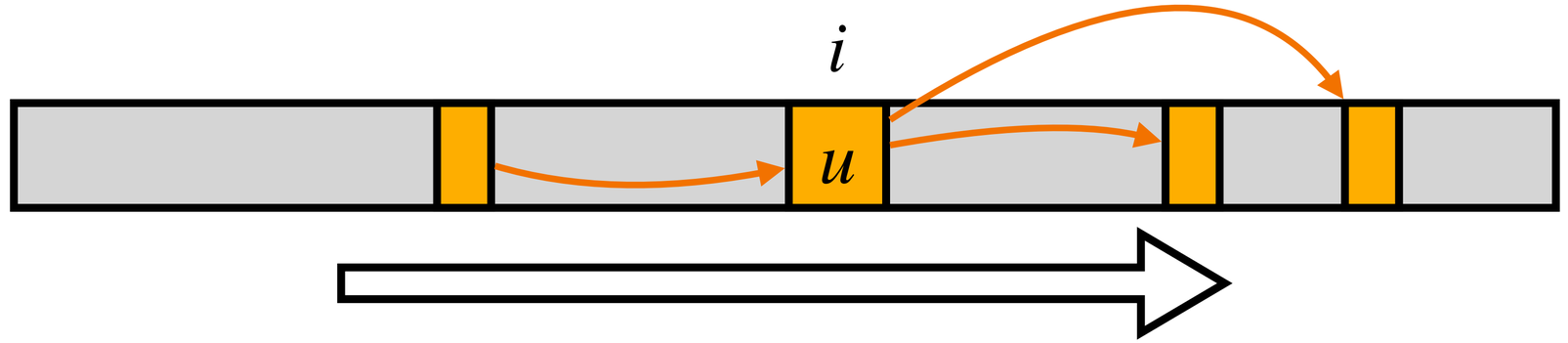

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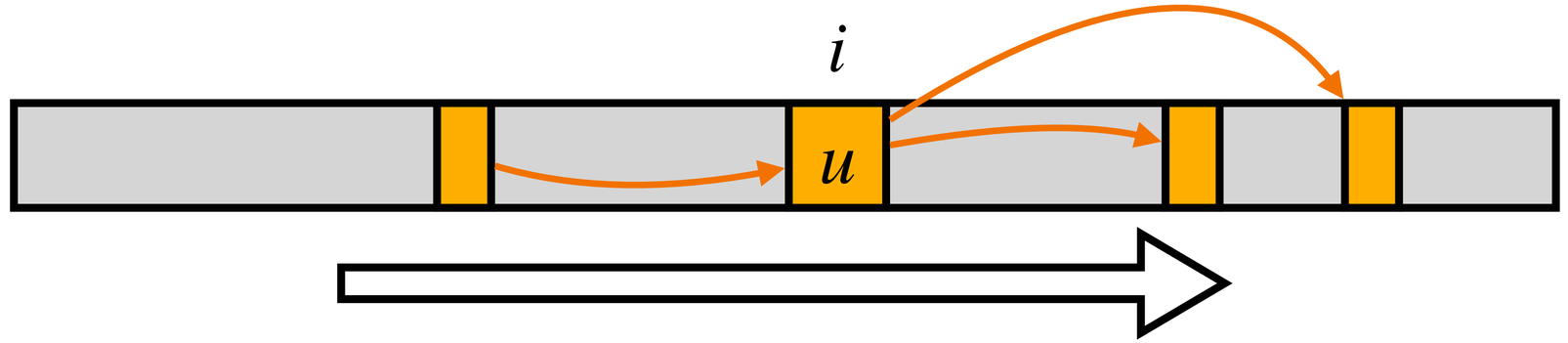

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It's unlikely that u is so far in the permutation and did not join a pivot cluster.

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- Therefore, $|\text{Bad Edges}| \leq 2\varepsilon \cdot |\text{Singleton Edges}|$

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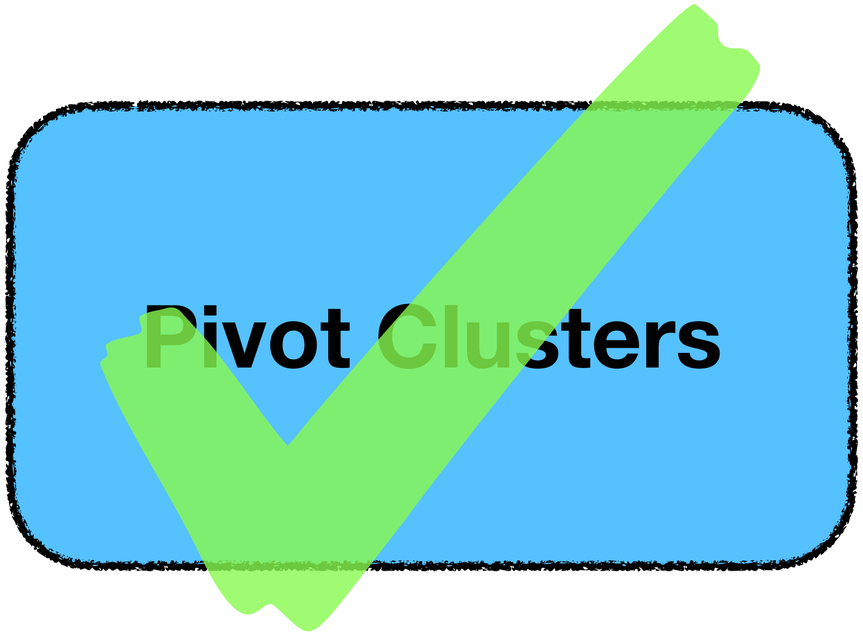
$$|\text{Bad Edges}| \leq 2\varepsilon \cdot |\text{Singleton Edges}| \leq \frac{2\varepsilon}{1 - 2\varepsilon} \cdot |\text{Good Edges}|$$

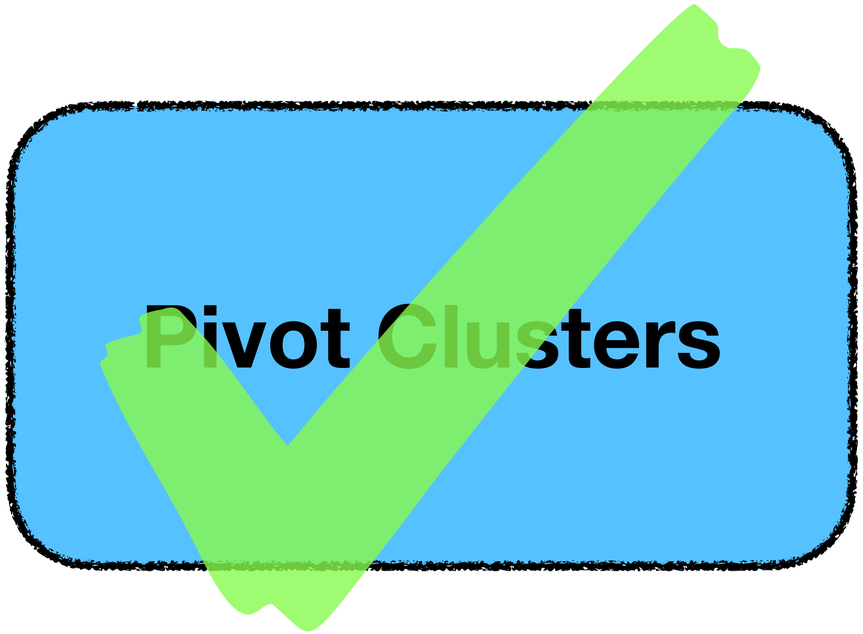
Approximation Analysis



3-approximation

“$+\varepsilon$”-approximation

Approximation Analysis



3-approximation

“$+\varepsilon$”-approximation

Correlation Clustering Summary

A $(3+\varepsilon)$-approximation algorithm using near-linear time and space.

Can be implemented in:

- Massively Parallel Computing: $O(1)$ rounds with $\tilde{O}(n/\varepsilon)$ local space.
- Dynamic Streaming: single pass with $\tilde{O}(n/\varepsilon)$ space.

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