

# A Lower Bound for the Distributed Lovász Local Lemma

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25th November 2015

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# Outline

- 1 The Lovász local lemma
- 2 Our model of distributed computing
- 3 A lower bound for the distributed LLL

# The probabilistic method

- Primarily used in combinatorics to give existence proofs.
- Randomly choose objects from a certain class, and show that the probability that the object is of the desired kind is larger than zero.
- It follows that at least one such object has to exist.

# The probabilistic method

- Primarily used in combinatorics to give existence proofs.
- Randomly choose objects from a certain class, and show that the probability that the object is of the desired kind is larger than zero.
- It follows that at least one such object has to exist.
  
- Let  $\mathcal{E} = \{E_1, \dots, E_n\}$  be a set of *bad* events that make the object undesirable.
- If the events are mutually independent and  $\Pr(E_i) < 1$  for each  $i$ , we have trivially  $\Pr(\bigcap_{i=1}^n \overline{E}_i) > 0$ .
- What if there is some dependence between the events?

# The Lovász local lemma (LLL)

## Theorem (Erdős and Lovász, 1975)

*Let  $\mathcal{E} = \{E_1, \dots, E_n\}$  be a finite set of events such that each  $E_i$  depends on at most  $d$  other events. If  $\Pr(E_i) \leq p$  and  $4pd \leq 1$ , then there is a positive probability that none of the events occur.*

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## Theorem (Lovász, 1977)

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$e = 2.718\dots$  is the base of the natural logarithm.

### Proposition

*Any instance  $\phi$  of  $k$ -SAT where no variable appears in more than  $\frac{2^{k-2}}{k}$  clauses is satisfiable.*



# LLL: example

## Proposition

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## Proof.

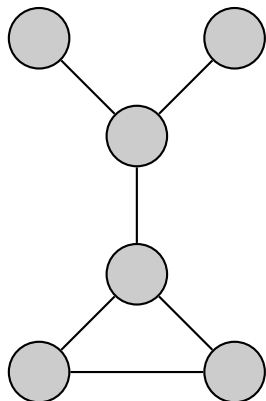
- 1 Pick a truth assignment uniformly at random.
- 2 Let  $E_i$  denote the event “clause  $i$  is not satisfied”.
- 3  $\Pr(E_i) = 2^{-k} =: p$ .
- 4  $E_i$  depends on at most  $d := k \frac{2^{k-2}}{k} = 2^{k-2}$  other events.
- 5 We have  $4pd = 4 \cdot 2^{-k} \cdot 2^{k-2} = 1$ .
- 6 Now LLL implies that  $\Pr(\bigcap \bar{E}_i) > 0$ .



# The algorithmic LLL

- LLL itself does not give a method for finding the object whose existence it proves.
- Beck showed in 1991 that there exist a deterministic polynomial-time algorithm for a weaker variant of LLL.
- This inspired a long line of research about algorithms for various versions of LLL.
- The breakthrough result of Moser and Tardos (2010) shows that there is a simple randomised resampling algorithm for a very general form of LLL.
- But we are interested in the *distributed* algorithmic LLL.

## Distributed computing: the LOCAL model

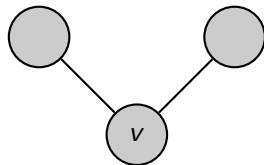


A simple connected undirected graph  $G = (V, E)$ , where each node  $v \in V$

- is given its own input,
- runs the same algorithm,
- communicates with its neighbours,
- produces its own output.

# Communication in synchronous rounds

Initially, each node  $v$  knows the total number of nodes  $n$ , the maximum degree of the graph  $\Delta$ , and a task-specific local input  $f(v)$ .

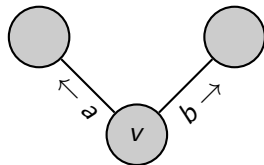


In every round, each node  $v \in V$

- 1 sends messages to its neighbours,
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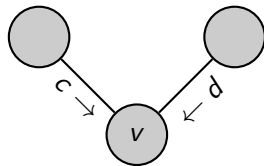


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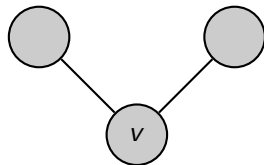


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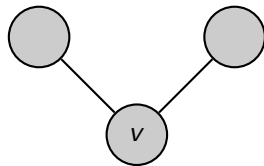


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In every round, each node  $v \in V$

- 1 sends messages to its neighbours,
- 2 receives messages from its neighbours,
- 3 updates its state.

After the final round, each node announces its output.

The running time of an algorithm is the *number of communications rounds* until all nodes have stopped, as a function of  $n$ .



# Input and output

- The same graph  $G = (V, E)$  serves both as the communication network and as the *problem instance*.
- A *graph problem* is defined by a function  $\Pi$  that maps each graph  $G$  and each labelling  $f: V \rightarrow X$  to a set  $\Pi(G, f)$  of *solutions*  $S: V \rightarrow Y$ .
- The *output* of algorithm  $A$  in  $(G, f)$  is the function  $g: V \rightarrow Y$  such that  $g(v)$  is the local output of  $v$  for each node  $v$ .
- Algorithm  $A$  *solves* problem  $\Pi$  if for each graph  $G$  and labelling  $f$  the output  $g$  of  $A$  in  $(G, f)$  is in  $\Pi(G, f)$ .

# Randomised algorithms

- We assume that each node can toss a countably infinite number of random coins.
- Equivalently, each node  $v$  is given a real number  $x(v)$  taken uniformly at random from  $[0, 1]$ .
- With probability 1, the values  $x(v)$  are globally unique and can thus be used as identifiers.

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- With probability 1, the values  $x(v)$  are globally unique and can thus be used as identifiers.
- Monte Carlo algorithms:
  - Running time is deterministic.
  - Output is a valid solution with high probability (with probability at least  $1 - 1/n^c$  for an arbitrarily large constant  $c$ ).

# The distributed Lovász local lemma

- Let  $\mathcal{X} = \{X_1, \dots, X_m\}$  be a set of mutually independent random variables and let  $\mathcal{E} = \{E_1, \dots, E_n\}$  be a set of events.
- Denote by  $\text{vbl}(E_i) \subseteq \mathcal{X}$  the subset of variables that  $E_i$  depends on.
- Define a *dependency graph*  $G_{\mathcal{E}} = (\mathcal{E}, \mathcal{D})$ , where  $\mathcal{D} = \{\{E_i, E_j\} : \text{vbl}(E_i) \cap \text{vbl}(E_j) \neq \emptyset\}$ .

## Problem

Let the communication network be isomorphic to  $G_{\mathcal{E}} = (\mathcal{E}, \mathcal{D})$ ; each node  $v$  corresponds to an event  $E_v \in \mathcal{E}$  and knows the set  $\text{vbl}(E_v)$ . The task is to have each node output an assignment  $a_v$  of the variables  $\text{vbl}(E_v)$  such that

- 1 for any  $\{E_u, E_v\} \in \mathcal{D}$  and  $X \in \text{vbl}(E_u) \cap \text{vbl}(E_v)$  it holds that  $a_u(X) = a_v(X)$ ,
- 2 the event  $E_v$  does not occur under assignment  $a_v$ .

## Existing algorithms and lower bounds

- The algorithm of Moser and Tardos (2010) can be adapted to the distributed setting; the running time is  $O(\log^2 n)$  rounds.
- Chung et al. (2014) gave a distributed algorithm running in  $O(\log n)$  rounds in bounded-degree graphs.
- LLL can be used to properly colour a cycle graph using a constant number of colours. This is known to require  $\Omega(\log^* n)$  rounds.

## Our lower bound

### Theorem

*Let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be such that  $f(4) \leq 16$ . Let  $A$  be a Monte Carlo distributed algorithm for LLL that finds an assignment avoiding all the bad events under the LLL criteria  $pf(d) \leq 1$  with high probability. Then the running time of  $A$  is  $\Omega(\log \log n)$  rounds.*

Note that we can plug in, for example, either of the LLL criteria  $ep(d+1) \leq 1$  or  $4pd \leq 1$ .

# Outline of the proof

- Two new graph problems: *sinkless orientation* and *sinkless colouring*.
- LLL can be used to solve the sinkless orientation in 3-regular graphs.
- A *mutual speedup lemma*:
  - If we can find a sinkless colouring in  $t$  rounds, we can find a sinkless orientation in  $t$  rounds.
  - If we can find a sinkless orientation in  $t$  rounds, we can find a sinkless colouring in  $t - 1$  rounds.
- By iterating the lemma, we obtain an algorithm that finds a sinkless orientation in 0 rounds, which leads to a contradiction.

# Orientations

- An *orientation*  $\sigma$  of a graph  $G = (V, E)$  assigns a direction  $\sigma(\{u, v\}) \in \{u \rightarrow v, u \leftarrow v\}$  for each edge  $\{u, v\} \in E$ .
- For all  $v \in V$  define  $\text{in-deg}(v, \sigma) = |\{u : (u, v) \in \sigma(E)\}|$ ,  $\text{out-deg}(v, \sigma) = |\{u : (v, u) \in \sigma(E)\}|$  and  $\text{deg}(v) = \text{in-deg}(v, \sigma) + \text{out-deg}(v, \sigma)$ .
- A node  $v$  with  $\text{in-deg}(v, \sigma) = \text{deg}(v)$  is called a *sink*. We call an orientation  $\sigma$  *sinkless* if no node is a sink, that is, every node  $v$  has  $\text{out-deg}(v, \sigma) > 0$ .



# Colourings

- We write  $[k] = \{0, 1, \dots, k - 1\}$ .
- $\psi: E \rightarrow [\chi]$  is a *proper edge  $\chi$ -colouring* if any two adjacent edges have a different colour.
- Given a properly edge  $\chi$ -coloured graph  $G = (V, E, \psi)$ , we call  $\varphi: V \rightarrow [\chi]$  a *sinkless colouring* of  $G$  if for all edges  $e = \{u, v\} \in E$  it holds that  $\varphi(u) = \psi(e) \Rightarrow \varphi(v) \neq \psi(e)$ .

## Graph problem definitions

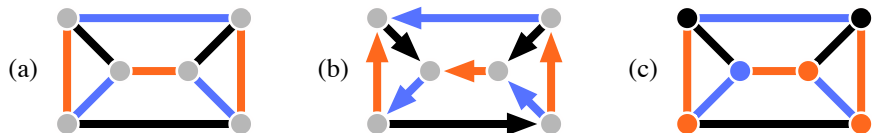
### Problem (Sinkless colouring)

*Given an edge  $d$ -coloured  $d$ -regular graph  $G = (V, E, \psi)$ , find a sinkless colouring  $\varphi$ . That is, compute a colouring  $\varphi$  such that for no edge  $e = \{u, v\} \in E$  we have  $\varphi(u) = \varphi(v) = \psi(e)$ .*

### Problem (Sinkless orientation)

*Given an edge  $d$ -coloured  $d$ -regular graph  $G = (V, E, \psi)$ , find a sinkless orientation. That is, compute an orientation  $\sigma$  such that  $\text{out-deg}(v, \sigma) > 0$  for all  $v \in V$ .*

# Graph problem definitions: example



**Figure:** (a) A 3-regular edge 3-coloured graph. (b) A sinkless orientation. (c) A sinkless colouring.

## From LLL to Sinkless Orientation

### Theorem

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- We start with 4-regular graphs  $G = (V, E)$ .
- Set  $\text{vbl}(E_v) = \{X_e : v \in e\}$  for each  $v \in V$
- For each  $e = \{u, v\} \in E$ , the variable  $X_e$  ranges over  $\{u \rightarrow v, u \leftarrow v\}$
- The bad event  $E_v$  occurs exactly when for all neighbours  $u$  of  $v$  the variable  $X_{\{v,u\}}$  takes the value  $u \rightarrow v$

## From LLL to Sinkless Orientation

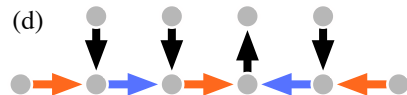
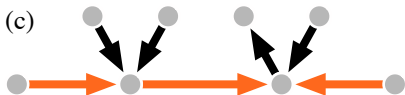
- If the variables  $X_e$  are sampled uniformly at random, we have  $\Pr(E_v) = 1/2^4 = 1/16$  for each  $v \in V$ .
- Let  $p = 1/16$  and  $d = 4$ . Now  $\Pr(E_v) \leq p$  and  $E_v$  depends on  $d$  other events for each  $v \in V$ , and the condition  $pf(d) \leq 1$  holds, given  $f(4) \leq 16$ .
- Run the algorithm  $A$  and define an orientation  $\sigma$  of  $G$  by setting  $\sigma(e) = a_v(X_e)$ , where  $v \in e$ , for each  $e \in E$ .
- Now  $\sigma$  is a sinkless orientation of  $G$ .

## From 3-regular to 4-regular graphs

- LLL is not directly applicable: the probability of bad events would be  $p = 1/2^3 = 1/8$  and thus  $ep(d + 1) > 1$ .
- *Contract* edges of one colour class to obtain a 4-regular graph.
- Simulate the algorithm for the 4-regular case in the 3-regular graph.

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# The Mutual Speedup Lemma

## Lemma

*Suppose  $B$  is a sinkless colouring algorithm that runs in  $t$  rounds such that for any edge  $e = \{u, v\}$  the probability of outputting a forbidden configuration  $B(u) = \psi(e) = B(v)$  is at most  $p$ . Then there exists a sinkless orientation algorithm  $B'$  that runs in  $t$  rounds such that for any node  $u$  the probability of being a sink is at most  $6p^{1/3}$ .*

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## Lemma

*Suppose  $B'$  is a sinkless orientation algorithm that runs in time  $t$  such that the probability that any node  $u$  is a sink is at most  $\ell$ . Then there exists a sinkless colouring algorithm  $B''$  that runs in time  $t - 1$  such that the probability for any edge  $e = \{u, v\}$  having a forbidden configuration  $B''(u) = \psi(e) = B''(v)$  is less than  $4\ell^{1/4}$ .*

## From colouring to orientation: the proof idea

- Given a randomised sinkless *colouring* algorithm  $B$  running in  $t$  rounds, construct a randomised sinkless *orientation* algorithm  $B'$  that also runs in  $t$  rounds.
- We write  $B(u)$  for the colour that  $u$  outputs according to  $B$  and  $B'(e)$  for the orientation  $B'$  outputs for edge  $e$ .
- We denote the radius- $t$  neighbourhood of a node  $u$  by  $N^t(u) = \{v \in V : \text{dist}(u, v) \leq t\}$ , where  $\text{dist}(u, v)$  is the length of the shortest path between  $u$  and  $v$ .
- The radius- $t$  neighbourhood of an *edge*  $\{u, v\}$  is  $N^t(\{u, v\}) = N^t(u) \cap N^t(v)$ .

# From colouring to orientation: the proof idea

Consider any node  $u \in V$ . Algorithm  $B'$  consists of three steps:

- 1 Node  $u$  gathers its radius- $t$  neighbourhood  $N^t(u)$  in  $t$  rounds.
- 2 Node  $u$  computes the set  $C(u)$  of *candidate colours*:

$$C(u) = \{\psi(e) : \Pr[B(u) = \psi(e) \mid N^t(e)] \geq p^{1/3} \text{ and } e = \{u, v\}\},$$

In addition, for each  $e = \{u, v\}$  node  $u$  calculates the probability of outputting the colour  $\psi(e)$  when executing  $B$ . Thus  $u$  can determine whether  $\psi(e) \in C(v)$ .

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- 3 If  $\psi(e) \in C(u) \cap C(v)$  or  $\psi(e) \notin C(u) \cup C(v)$ , choose the orientation  $B'(e)$  of edge  $e$  arbitrarily. Otherwise, edge  $e$  is oriented according to the following rule:

$$B'(e) = \begin{cases} u \rightarrow v & \text{if } \psi(e) \in C(u) \text{ and } \psi(e) \notin C(v), \\ u \leftarrow v & \text{if } \psi(e) \notin C(u) \text{ and } \psi(e) \in C(v). \end{cases}$$

# Conclusion

- There is a randomised distributed algorithm for LLL that runs in  $O(\log n)$  rounds in bounded-degree graphs.
- The best previously known lower bound was  $\Omega(\log^* n)$  rounds.
- We show that any randomised Monte Carlo algorithm for LLL that finds a satisfying assignment with high probability requires  $\Omega(\log \log n)$  rounds.