A tight lower bound for simulating multiset model by set model in distributed computing

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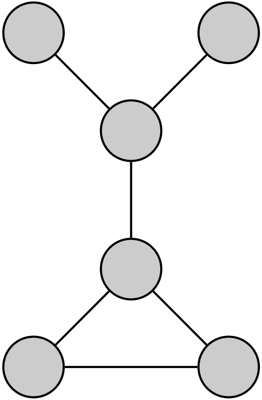
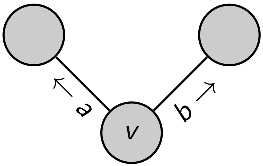
Distributed system



A graph, whose each node

- runs the same algorithm,
- can communicate with its neighbours,
- produces a local output.

Communication in synchronous rounds



In every round, each node v

1. sends messages to its neighbours,
2. receives messages from its neighbours,
3. updates its state.

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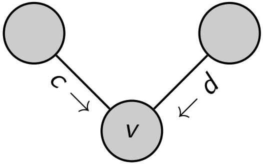

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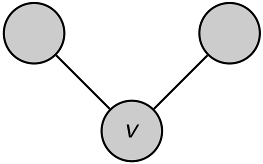

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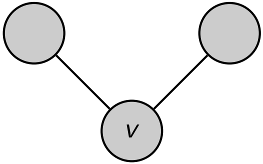

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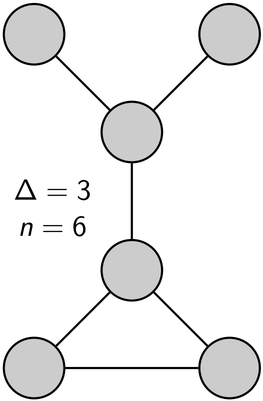
3. updates its state.

After the final round, each node announces its own output.

Focus on communication, not computation



The running time of an algorithm is the *number of communications rounds*.

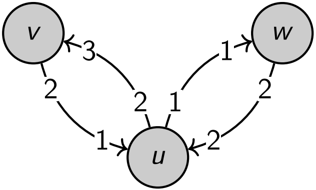
The running time may depend on

- the maximum degree of the graph, Δ,
- the number of nodes, n.

Port numbering

A *port* of a graph $G = (V, E)$ is a pair (v, i), where $v \in V$ and $i \in \{1, 2, \ldots, \deg(v)\}$. Let $P(G)$ be the set of all ports of G. A *port numbering* of G is a bijection $p\colon P(G) \to P(G)$ such that

$$p(v, i) = (u, j) \text{ for some } i \text{ and } j \quad \text{if and only if} \quad \{v, u\} \in E.$$



Intuitively, if $p(v, i) = (u, j)$, then (v, i) is an output port of node v that is connected to an input port (u, j) of node u.

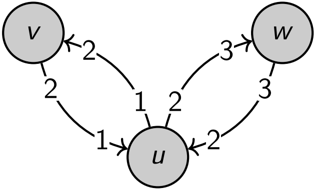
$$\begin{gathered} p(u, 2) = (v, 3), \\ p(v, 2) = (u, 1), \\ p(u, 1) = (w, 1), \\ \vdots \end{gathered}$$

Consistent port numbering

We say that a port numbering p is *consistent* if we have

$$p(p(v,i)) = (v,i) \text{ for all } (v,i) \in P(G),$$

or, in other words, if the input port and the output port connected to the same neighbour always have the same number.



$$\begin{aligned} p(u,1) &= (v,2), \\ p(v,2) &= (u,1), \\ p(u,2) &= (w,3), \\ p(w,3) &= (u,2), \\ &\vdots \end{aligned}$$

Bounded-degree graphs and local inputs

For each positive integer Δ, denote by $\mathcal{F}(\Delta)$ the class of all simple undirected graphs of maximum degree at most Δ.

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An *input* for a graph $G = (V, E)$ is a function $f \colon V \to X$, where $X \ni \emptyset$ is a finite set. For each $v \in V$, the value $f(v)$ is called the *local input* of v.

The symbol $\emptyset \in X$ is used to indicate "no input".

Algorithms as state machines

Let $\Delta \in \mathbb{N}_+$ and let X be a set of local inputs. A *distributed state machine* for $(\mathcal{F}(\Delta), X)$ is a tuple $\mathcal{A} = (Y, Z, \sigma_0, M, \mu, \sigma)$, where

- Y is a set of states,
- $Z \subseteq Y$ is a finite set of stopping states,
- $\sigma_0 \colon \{0, 1, \ldots, \Delta\} \times X \to Y$ is a function that defines the initial state,
- M is a set of messages such that $\epsilon \in M$,
- $\mu \colon Y \times [\Delta] \to M$ is a function that constructs the outgoing messages, such that $\mu(z, i) = \epsilon$ for all $z \in Z$ and $i \in [\Delta]$,
- $\sigma \colon Y \times M^\Delta \to Y$ is a function that defines the state transitions, such that $\sigma(z, \overline{m}) = z$ for all $z \in Z$ and $\overline{m} \in M^\Delta$.

The special symbol $\epsilon \in M$ indicates "no message".

Execution

Let $G = (V, E) \in \mathcal{F}(\Delta)$, let p be a port numbering of G, let $f\colon V \to X$, and let $\mathcal{A}$ be a distributed state machine for $(\mathcal{F}(\Delta), X)$.

The state of the system in round $r \in \mathbb{N}$ is a function $x_r\colon V \to Y$, where $x_r(v)$ is the *state* of node v in round r. To initialise the nodes, set

$$x_0(v) = \sigma_0(\deg(v), f(v)) \quad \text{for each } v \in V.$$

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Then, assume that x_r is defined for some $r \in \mathbb{N}$. Let $(u, j) \in P(G)$ and $(v, i) = p(u, j)$. Now, node v receives the message

$$a_{r+1}(v, i) = \mu(x_r(u), j)$$

from its port (v, i) in round $r + 1$. For each $v \in V$, we define

$$\overline{a}_{r+1}(v) = (a_{r+1}(v, 1), a_{r+1}(v, 2), \ldots, a_{r+1}(v, \deg(v)), \epsilon, \epsilon, \ldots, \epsilon) \in M^{\Delta}.$$

Now we can define the new state of each node $v \in V$ as follows:

$$x_{r+1}(v) = \sigma(x_r(v), \overline{a}_{r+1}(v)).$$

Running time

Let $t \in \mathbb{N}$. If $x_t(v) \in Z$ for all $v \in V$, we say that $\mathcal{A}$ *stops in time* t in (G, f, p).

The *running time* of $\mathcal{A}$ in (G, f, p) is the smallest t for which this holds.

If $\mathcal{A}$ stops in time t in (G, f, p), the *output* of $\mathcal{A}$ in (G, f, p) is $x_t \colon V \to Y$.

For each $v \in V$, the *local output* of v is $x_t(v)$.

Graph problems

We study *graph problems* where

- problem instance is the communication graph (and the possible local inputs),
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Let X and Y be finite nonempty sets.

A *graph problem* is a function $\Pi_{X,Y}$ that maps each undirected simple graph $G = (V, E)$ and each input $f\colon V \to X$ to a set $\Pi_{X,Y}(G, f)$ of solutions.

Each *solution* $S \in \Pi_{X,Y}(G, f)$ is a function $S\colon V \to Y$.

Solving a graph problem

Let $\Pi_{X,Y}$ be a graph problem, $T\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ such that each $\mathcal{A}_\Delta$ is a distributed state machine for $(\mathcal{F}(\Delta), X)$. Algorithm $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in time* T if the following holds for all $\Delta \in \mathbb{N}$, all finite graphs $G = (V, E) \in \mathcal{F}(\Delta)$, all inputs $f\colon V \to X$ and all port numberings p of G:

1. $\mathcal{A}_\Delta$ stops in time $T(\Delta, |V|)$ in (G, f, p).
2. The output of $\mathcal{A}_\Delta$ in (G, f, p) is in $\Pi_{X,Y}(G, f)$.

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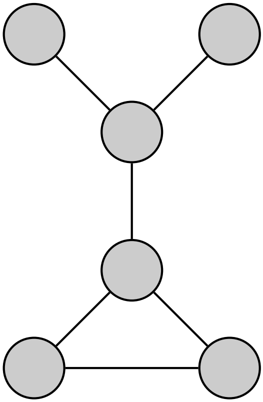
We say that $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in time* T *assuming consistency* if the above holds for all consistent port numberings p of G.

If $T(\Delta, n)$ does not depend on n, we say that $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in constant time* or that $\mathbf{A}$ *is a local algorithm for* $\Pi_{X,Y}$.

Graph problems



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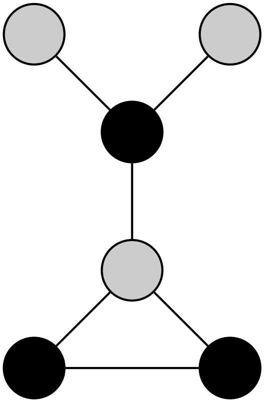

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Example problems:

- minimum vertex cover,

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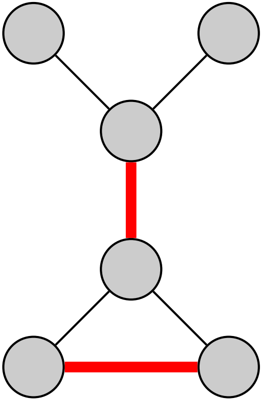

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Example problems:

- minimum vertex cover,
- maximal matching.

Variants of the model of computation

$\mathcal{VV}$ is the class of all distributed state machines (send a vector, receive a vector).

We can place different restrictions on the algorithms:

- $\mathcal{VB}$: broadcast the same message to all neighbours:

$$\mu(y, i) = \mu(y, j) \text{ for all } i, j \in \{1, 2, \ldots, \Delta\} \text{ and } y \in Y,$$

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- $\mathcal{MV}$: receive a multiset of messages:

$$\text{multiset}(\overline{a}) = \text{multiset}(\overline{b}) \Rightarrow \sigma(y, \overline{a}) = \sigma(y, \overline{b}) \text{ for all } y \in Y,$$

- $\mathcal{SV}$: receive a set of messages:

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- $\mathcal{MB} = \mathcal{MV} \cap \mathcal{VB}$ and $\mathcal{SB} = \mathcal{SV} \cap \mathcal{VB}$.

Variants of the model of computation

$$\begin{aligned}
\mathbf{VV} &= \{(\mathcal{A}_1, \mathcal{A}_2, \dots) : \mathcal{A}_\Delta \in \mathcal{VV} \text{ for all } \Delta\},\\
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\end{aligned}$$

Complexity classes

Let P be the class of all graph problems.

$$
\begin{aligned}
\mathsf{VV_c} &= \{\Pi \in \mathsf{P} : \text{there is } \mathbf{A} \in \mathbf{VV} \text{ that solves } \Pi \text{ assuming consistency}\},\\
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$$

Containment relations between the classes

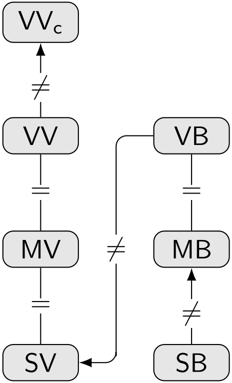
$$
\begin{tikzcd}
\mathsf{VV_c} & \\
\mathsf{VV} \arrow[u] & \mathsf{VB} \arrow[l] \\
\mathsf{MV} \arrow[u] & \mathsf{MB} \arrow[l] \arrow[u] \\
\mathsf{SV} \arrow[u] & \mathsf{SB} \arrow[l] \arrow[u]
\end{tikzcd}
$$

Trivial relations:

- $\mathsf{SV} \subseteq \mathsf{MV} \subseteq \mathsf{VV} \subseteq \mathsf{VV_c}$,
- $\mathsf{SB} \subseteq \mathsf{MB} \subseteq \mathsf{VB}$,
- $\mathsf{VB} \subseteq \mathsf{VV}$,
- $\mathsf{MB} \subseteq \mathsf{MV}$,
- $\mathsf{SB} \subseteq \mathsf{SV}$.

Non-trivial: $\mathsf{SV} \subseteq \mathsf{VB}$? $\mathsf{VB} \subseteq \mathsf{SV}$?

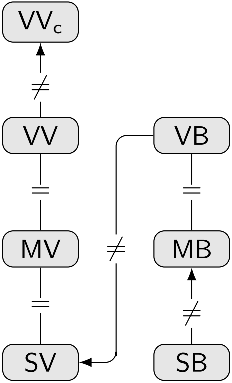
Containment relations between the classes



Hella, Järvisalo, Kuusisto, Laurinharju, L., Luosto, Suomela, Virtema (PODC 2012):

$$\mathsf{SB} \subsetneq \mathsf{MB} = \mathsf{VB} \subsetneq \mathsf{SV} = \mathsf{MV} = \mathsf{VV} \subsetneq \mathsf{VV_c}.$$

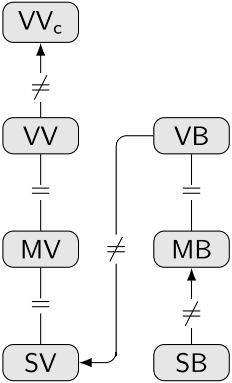
Simulation results



Equalities are proved by showing that seemingly more powerful algorithms can be simulated by seemingly weaker algorithms.

- MV = VV and MB = VB: the simulation does not increase running time.
- SV = MV: the running time increases by $2\Delta - 2$ rounds.

New contributions



SV $=$ MV: the running time increases by $2\Delta - 2$ rounds.

Question: Is the overhead of $2\Delta - 2$ rounds optimal?

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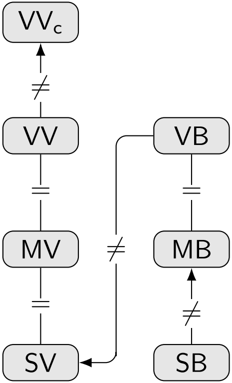


SV = MV: the running time increases by $2\Delta - 2$ rounds.

Question: Is the overhead of $2\Delta - 2$ rounds optimal?

Answer: Yes

- A so-called simulation problem requires exactly $2\Delta - 2$ rounds.
- In the case of graph problems, a linear-in-Δ overhead is necessary.

The relationship of MV and SV

Trivially SV $\subseteq$ MV.

Hella, Järvisalo, Kuusisto, Laurinharju, L., Luosto, Suomela, Virtema (PODC 2012):

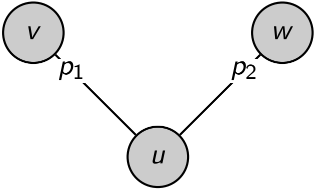
Theorem

Let Π be a graph problem and let $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Assume that there is an algorithm $\mathbf{A} \in \mathbf{MV}$ that solves Π in time T. Then there is an algorithm $\mathbf{B} \in \mathbf{SV}$ that solves Π in time T', where $T'(n, \Delta) = T(n, \Delta) + 2\Delta - 2$.

It follows that SV $=$ MV.

Idea behind the simulation theorem

First, solve the following *simulation problem* by an $\mathcal{SV}$-algorithm:



If $p_1 = p_2$, then output$(v) \neq$ output(w).

Now the pair

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is distinct for each neighbour.

This takes $2\Delta - 2$ communication rounds.

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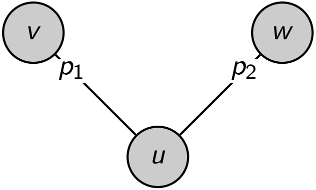

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This takes $2\Delta - 2$ communication rounds.

Then, simulate the $\mathcal{MV}$-algorithm by attaching the above pair to each message. That way we can reconstruct the message multiplicities.

The new results: lower bounds for the simulation

Is the overhead of $2\Delta - 2$ rounds really needed to reconstruct the message multiplicities by an $\mathcal{SV}$-algorithm?

Theorem

For each $\Delta \geq 2$ there is a graph $G = (V, E) \in \mathcal{F}(\Delta)$, a port numbering p of G and nodes $v, u, w \in V$ such that when executing any algorithm $\mathcal{A} \in \mathcal{SV}$ in (G, p), node v receives identical messages from its neighbours u and w in rounds $1, 2, \ldots, 2\Delta - 2$.

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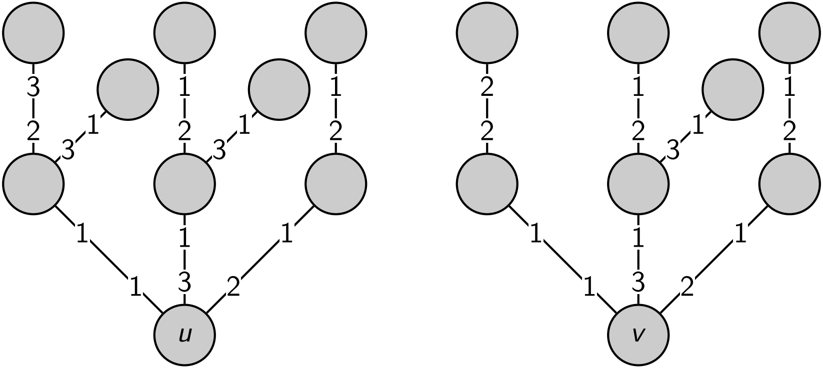
Theorem

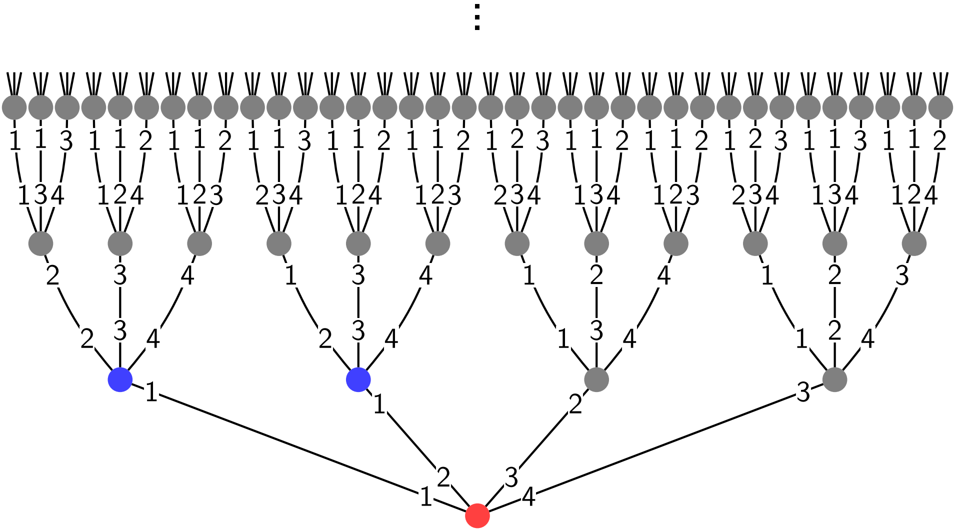
There is a graph problem Π that can be solved in one round by an algorithm in **MV** *but that requires at least time T, where $T(n, \Delta) \geq \Delta$ for all $\Delta \geq 2$, when solved by an algorithm in* **SV**.

Example: a problem instance separating $\mathcal{SV}$ and $\mathcal{MV}$



Output 1 if there is an even number of neighbours of even degree, 0 otherwise.

Lower-bound construction for the simulation problem



Proof idea

1. Investigate walks that start from the blue nodes and follow an identical sequence of port numbers.
    1. In which cases we cannot extend the walks in a consistent manner?
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    1. In which cases we cannot extend the walks in a consistent manner?
    2. What is the length of such maximal walks?
2. Prove a lower bound for the length of the walks.
3. Show that the lower bound on walks implies bisimilarity of the blue nodes up to a certain distance.
4. Bisimilarity entails a lower bound for the running time of any distributed algorithm that is able to distinguish the nodes.

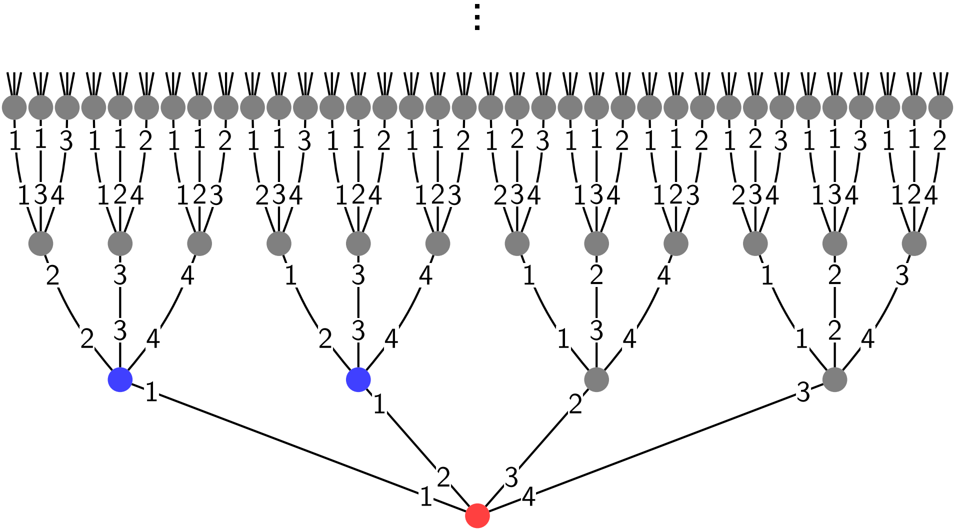
Notation for outgoing port numbers

If $p(v, i) = (u, j)$, we write $\pi(v, u) = i$. That is, $\pi(v, u)$ is the number of the output port of v that is connected to u.

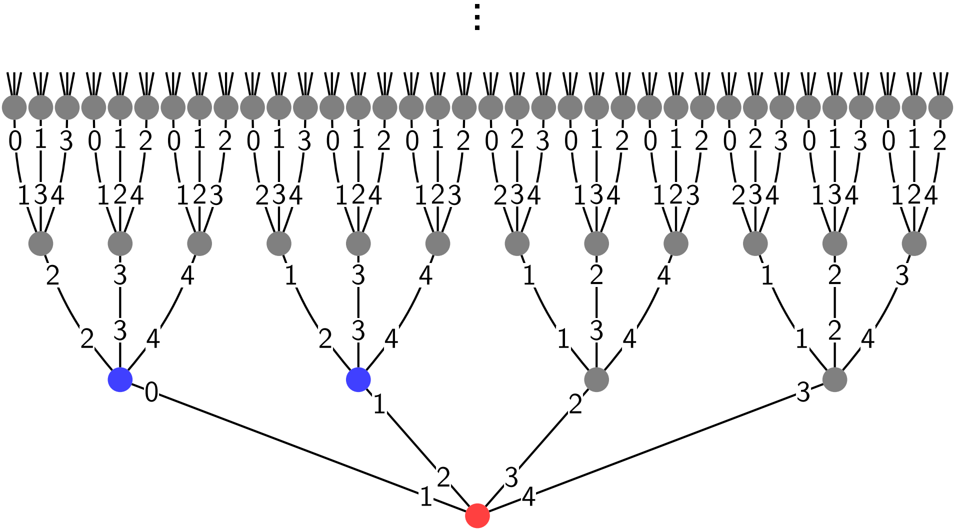
Definition of the graph G_d

1. $\emptyset \in V_d$.
2. $((1,0)),((2,1)),((3,2)),((4,3)),\ldots,((d,d-1)) \in V_d$.
3. If $(a_1,a_2,\ldots,a_i) \in V_d$, where i is odd and $i < 2d$, then $(a_1,a_2,\ldots,a_{i+1}^j) \in V_d$ for all $j = 1,2,\ldots,d-1$, where $a_{i+1}^j = (c_1^j, c_2^j)$ is defined as follows. Let $(b_1,b_2) = a_i$ and $b_2^+ = 1$ if $b_2 = 0$, $b_2^+ = b_2$ otherwise. Define
$$c_1^j = \min(\{1,2,\ldots,d\} \setminus \{b_2^+, c_1^1, c_1^2, \ldots, c_1^{j-1}\}),$$
$$c_2^j = \min(\{1,2,\ldots,d\} \setminus \{b_1, c_2^1, c_2^2, \ldots, c_2^{j-1}\}).$$
4. If $(a_1,a_2,\ldots,a_i) \in V_d$, where i is even and $0 < i < 2d$, then $(a_1,a_2,\ldots,a_{i+1}^j) \in V_d$ for all $j = 1,2,\ldots,d-1$, where $a_{i+1}^j = (c_1^j, c_2^j)$ is defined as follows. Let $(b_1,b_2) = a_i$. Define
$$c_1^j = \min(\{1,2,\ldots,d\} \setminus \{b_2, c_1^1, c_1^2, \ldots, c_1^{j-1}\}),$$
$$c_2^j = \min(\{0,1,\ldots,d-1\} \setminus \{b_1, c_2^1, c_2^2, \ldots, c_2^{j-1}\}).$$

Graph G_d for $d = 4$



Graph G_d for $d = 4$



Definition of the graph G_d

The set E_d of edges consists of all pairs $\{v, u\}$, where $v = (a_1, a_2, \ldots, a_i) \in V_d$ and $u = (a_1, a_2, \ldots, a_i, a_{i+1}) \in V_d$ for some $i \in \{0, 1, \ldots\}$.

If $v = (a_1, a_2, \ldots, a_i)$ and $u = (a_1, a_2, \ldots, a_{i+1})$, where $a_{i+1} = (b_1, b_2)$, the outgoing port number from v to u is $\pi_d(v, u) = b_1$ and the outgoing port number from u to v is $\pi_d(u, v) = b_2$.

Pairs of separating walks (PSWs)

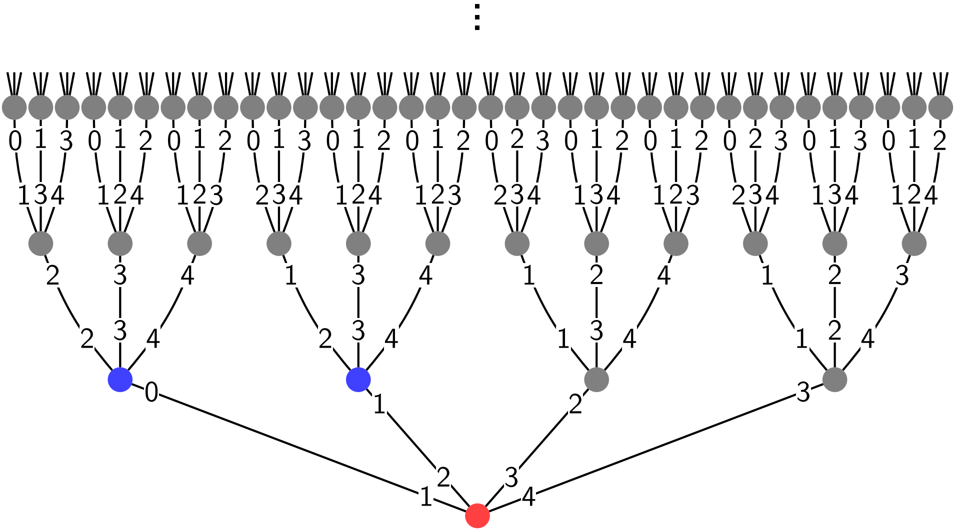
A *walk* is a sequence $\overline{v} = (v_0, v_1, \ldots, v_k)$ of nodes such that $\{v_i, v_{i+1}\} \in E_d$ for all $i = 0, 1, \ldots, k-1$.

A pair $(\overline{v}_1, \overline{v}_2)$ of walks, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for all $i = 1, 2$, is called a *pair of separating walks (PSW) of length k in G_d* if the following conditions hold:

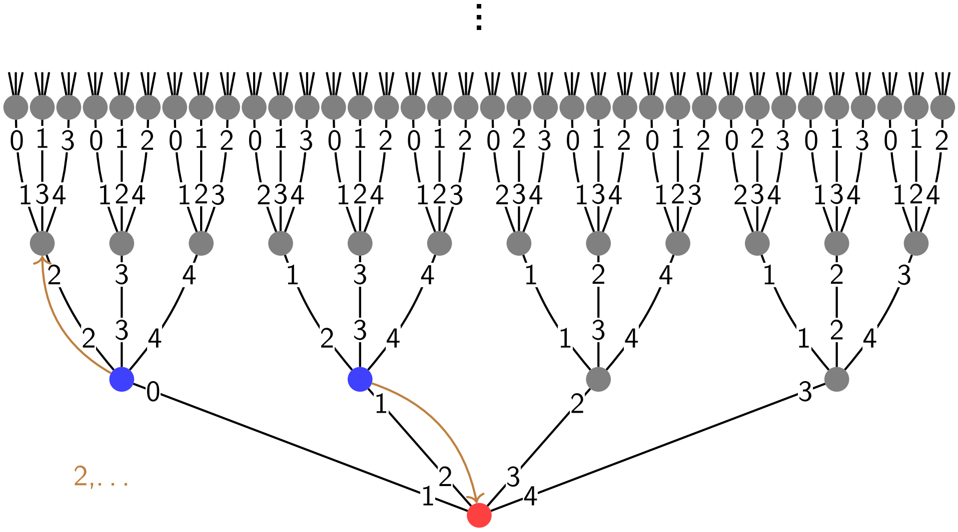
1. $v_0^1 = ((1, 0))$ and $v_0^2 = ((2, 1))$.
2. $\pi_d(v_j^1, v_{j-1}^1) = \pi_d(v_j^2, v_{j-1}^2)$ for all $j = 1, 2, \ldots, k$.
3. There is $v_{k+1}^1 \in V_d$ with $\{v_k^1, v_{k+1}^1\} \in E_d$ such that there is no $v_{k+1}^2 \in V_d$ for which $\{v_k^2, v_{k+1}^2\} \in E_d$ and $\pi_d(v_{k+1}^1, v_k^1) = \pi_d(v_{k+1}^2, v_k^2)$.

We say that a pair of separating walks of length k in G_d is *critical* if there does not exist a pair of separating walks of length k' in G_d for any $k' < k$.

A pair of separating walks in G_4



A pair of separating walks in G_4



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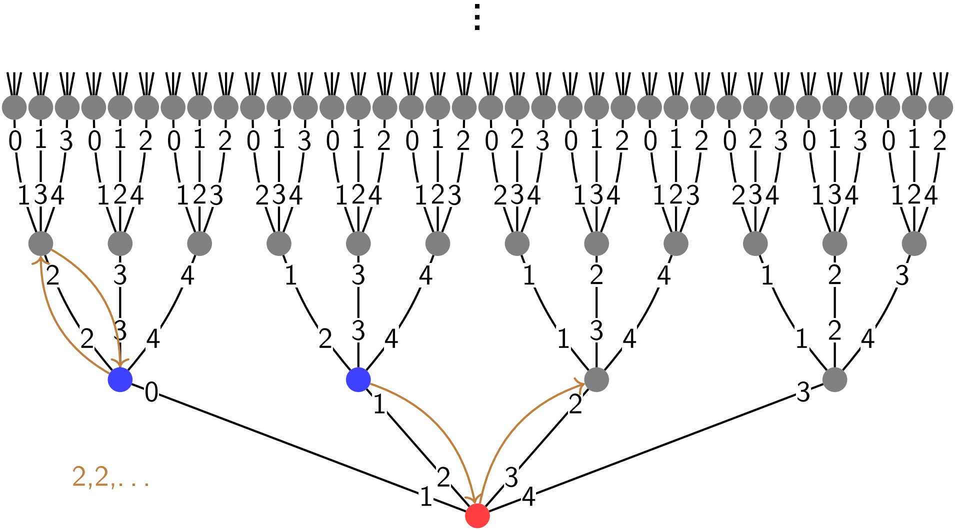

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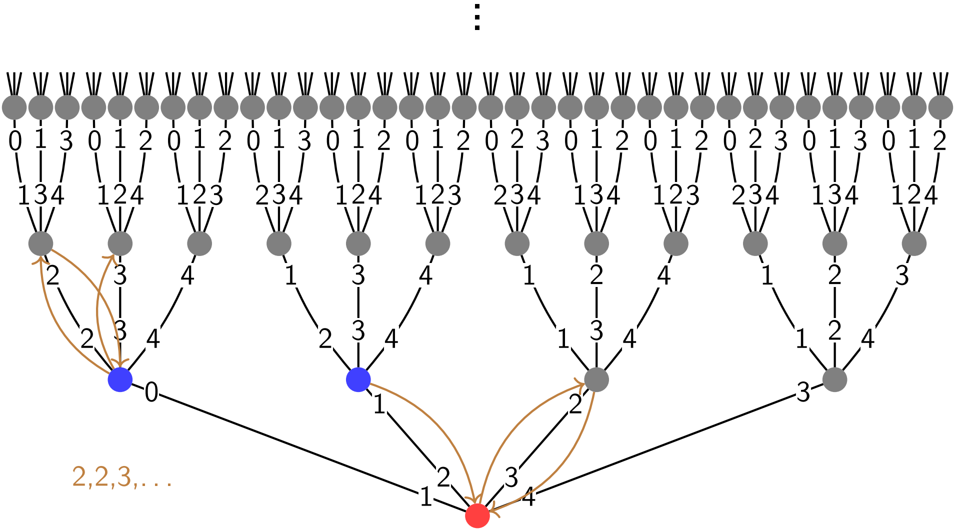

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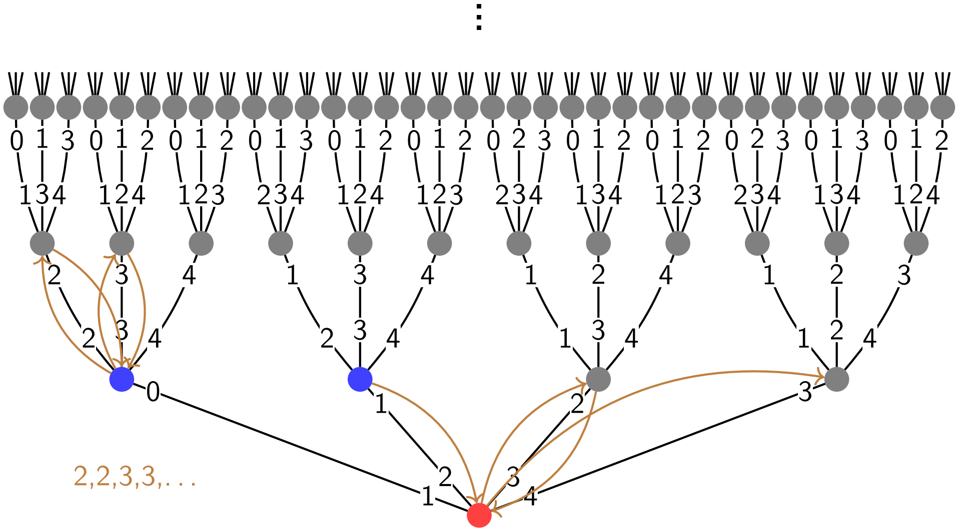

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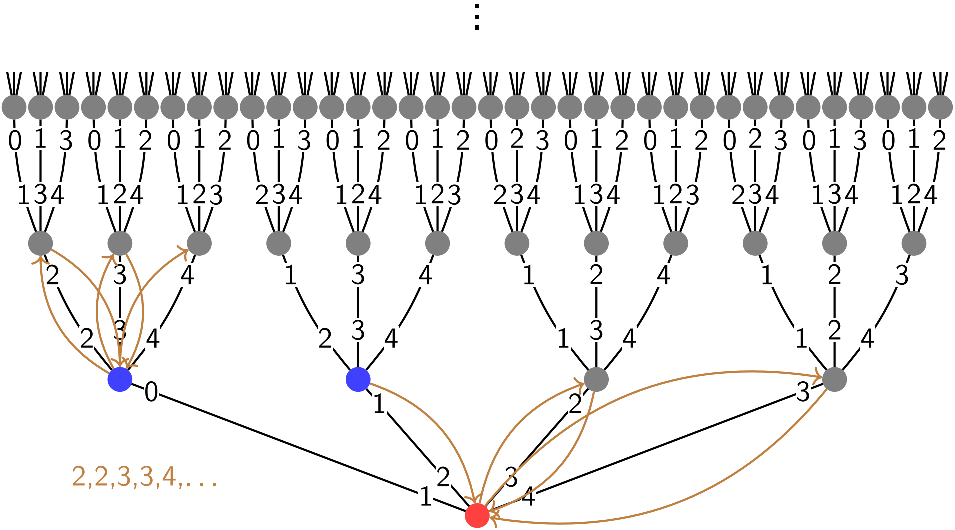
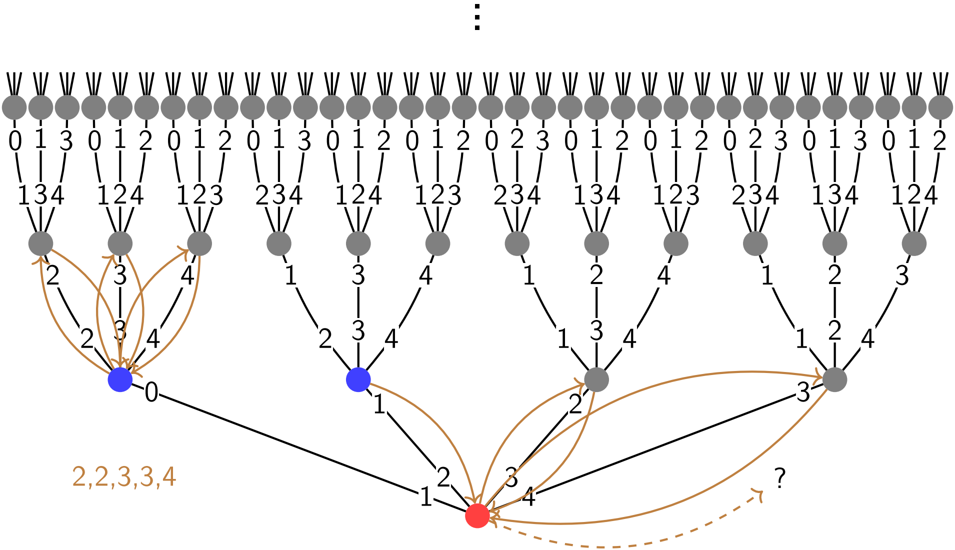


A pair of separating walks in G_4



How to prove that two nodes stay in the same state?

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, and let p and p' be port numberings of G and G', respectively. An *r-$\mathcal{SV}$-bisimulation* between nodes $v \in V$ and $v' \in V'$ is a sequence of binary relations $B_r \subseteq B_{r-1} \subseteq \cdots \subseteq B_0 \subseteq V \times V'$ such that the following conditions hold for $1 \leq i \leq r$:

1. $(v, v') \in B_r$.
2. If $(u, u') \in B_0$, then $\deg_G(u) = \deg_{G'}(u')$ and $f(u) = f'(u')$.
3. If $(u, u') \in B_i$ and $\{u, w\} \in E$, then there is $w' \in V'$ such that $\{u', w'\} \in E'$, $(w, w') \in B_{i-1}$ and $\pi(w, u) = \pi'(w', u')$.
4. If $(u, u') \in B_i$ and $\{u', w'\} \in E'$, then there is $w \in V$ such that $\{u, w\} \in E$, $(w, w') \in B_{i-1}$ and $\pi(w, u) = \pi'(w', u')$.

How to prove that two nodes stay in the same state?

We say that $v \in V$ and $v' \in V'$ are *r-$\mathcal{SV}$-bisimilar* and write $(G, f, v, p) \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} (G', f', v', p')$ (or simply $v \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} v'$) if there exists an *$r$-$\mathcal{SV}$-bisimulation* between them.

Lemma

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, and let p and p' be port numberings of G and G', respectively. If $(G, f, v, p) \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} (G', f', v', p')$ for some $r \in \mathbb{N}$, $v \in V$ and $v' \in V'$, then for all algorithms $\mathcal{A} \in \mathcal{SV}$ we have $x_t(v) = x'_t(v')$ for all $t = 0, 1, \ldots, r$, that is, the state of v and v' is identical in rounds $0, 1, \ldots, r$.

Bisimilarity

Lemma

The r-$\mathcal{SV}$-bisimilarity relation $\underline{\leftrightarrow}_r^{\mathcal{SV}}$ is an equivalence relation in the class of quadruples (G, f, v, p), where $G = (V, E)$ is a graph, f is an input for G, p is a port numbering of G and $v \in V$.

Lemma

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, let p and p' be port numberings of G and G', respectively, and let $v \in V$, $v' \in V'$. Then $(G, f, v, p) \underline{\leftrightarrow}_r^{\mathcal{SV}} (G', f', v', p')$ iff the following conditions hold:

1. $(G, f, v, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', v', p')$.
2. *If $\{v, w\} \in E$, then there is $w' \in V'$ such that $\{v', w'\} \in E'$, $(G, f, w, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', w', p')$ and $\pi(w, v) = \pi'(w', v')$.*
3. *If $\{v', w'\} \in E'$, then there is $w \in V$ such that $\{v, w\} \in E$, $(G, f, w, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', w', p')$ and $\pi(w, v) = \pi'(w', v')$.*

Definitions

If $v = (a_1, a_2, \ldots, a_i)$ and $u = (a_1, a_2, \ldots, a_{i+1})$, we say that node v is the *parent* of node u and that u is a *child* of v.

We say that the node v is *even* if i is even and *odd* if i is odd.

If $a_i = (b_1, b_2)$, we call (b_1, b_2) the *type* of node v.

Easy observations

Lemma

For each d, we have $\deg(v) \in \{1, d\}$ for all $v \in V_d$, and thus $G_d \in \mathcal{F}(d)$. Additionally, G_d is a subgraph of G_{d+1}.

Easy observations

Lemma

For each d, we have $\deg(v) \in \{1, d\}$ for all $v \in V_d$, and thus $G_d \in \mathcal{F}(d)$. Additionally, G_d is a subgraph of G_{d+1}.

Lemma

Let $v \in V_d$ and $a \in \{0, 1, \ldots, d\}$. Then there is at most one node $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$.

Easy observations

Lemma

Let $v = (a_1, a_2, \ldots, a_i) \in V_d$*, where* $i < 2d$*. If* v *is odd, then for all* $a \in \{1, 2, \ldots, d\}$ *there exists* $u \in V_d$ *such that* $\{v, u\} \in E_d$ *and* $\pi_d(u, v) = a$*. If* v *is even, then either for all* $a \in \{0, 1, \ldots, d-1\}$ *or for all* $a \in \{0, 1, \ldots, d-2, d\}$ *there exists* $u \in V_d$ *such that* $\{v, u\} \in E_d$ *and* $\pi_d(u, v) = a$*. In the case of even* v *and* $a = d$*, node* u *is the parent of node* v*.*

Easy observations

Lemma

Let $v = (a_1, a_2, \ldots, a_i) \in V_d$, where $i < 2d$. If v is odd, then for all $a \in \{1, 2, \ldots, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. If v is even, then either for all $a \in \{0, 1, \ldots, d-1\}$ or for all $a \in \{0, 1, \ldots, d-2, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. In the case of even v and $a = d$, node u is the parent of node v.

Lemma

Let $\{v, u\} \in E_{d+1} \setminus E_d$ be such that $v \in V_d$. Then u is a child of v. If v is odd, then $\pi_{d+1}(v, u) = \pi_{d+1}(u, v) = d + 1$. If v is even, then $\pi_{d+1}(v, u) = d + 1$ and $\pi_{d+1}(u, v) \in \{d - 1, d\}$.

Walks in isomorphic subtrees

Lemma

Let $(\overline{v}_1, \overline{v}_2)$, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for some $k \leq 2d-3$ and all $i = 1, 2$, be a PSW in G_d. If for some $\ell \in \{0, 1, \ldots, k-1\}$ the node $v_{\ell+1}^i$ is a child of node v_ℓ^i for all $i = 1, 2$, and we have $\pi_d(v_\ell^1, v_{\ell+1}^1) = \pi_d(v_\ell^2, v_{\ell+1}^2)$, then $(\overline{v}_1, \overline{v}_2)$ is not a critical PSW in G_d.

Extending a PSW

Lemma

Let $(\overline{v}_1, \overline{v}_2)$ be a PSW of length $k \leq 2d - 3$ in G_d. Then there is a PSW of length $k + 2$ in G_{d+1}.

Second-to-last node is in $V_d \setminus V_{d-1}$

Lemma

Let $(\overline{v}_1, \overline{v}_2)$, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for some $k \leq 2d-3$ and all $i = 1, 2$, be a critical PSW in G_d. Then we have $v_{k-1}^i \in V_d \setminus V_{d-1}$ for some $i \in \{1, 2\}$.

Pair of walks that is not a PSW

Lemma

Let $(\overline{v}_1, \overline{v}_2)$*, where* $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ *for some* $k \leq 2d - 3$ *and all* $i = 1, 2$*, be a pair of walks in* G_d *such that conditions (1) and (2) hold. If* $(\overline{v}_1, \overline{v}_2)$ *is not a PSW in* G_d*, then for each neighbour* $v_{k+1}^1 \in V_d$ *of* v_k^1 *there is a neighbour* $v_{k+1}^2 \in V_d$ *of* v_k^2 *such that* $\pi_d(v_{k+1}^1, v_k^1) = \pi_d(v_{k+1}^2, v_k^2)$*, and vice versa.*

The main lemma

Lemma

Let $(\overline{v}_1, \overline{v}_2)$, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for some $k \leq 2d-3$ and all $i = 1, 2$, be a critical PSW in G_d. Then $(\overline{v}'_1, \overline{v}'_2)$, where $\overline{v}'_i = (v_0^i, v_1^i, \ldots, v_{k-2}^i)$ for all $i = 1, 2$, is a PSW in G_{d-1}.

Minimum length of a PSW and bisimilarity

Lemma

Let $(\overline{v}_1, \overline{v}_2)$ *be a PSW of length* $k \leq 2d - 3$ *in* G_d*. Then* $k \geq 2d - 3$*.*

Minimum length of a PSW and bisimilarity

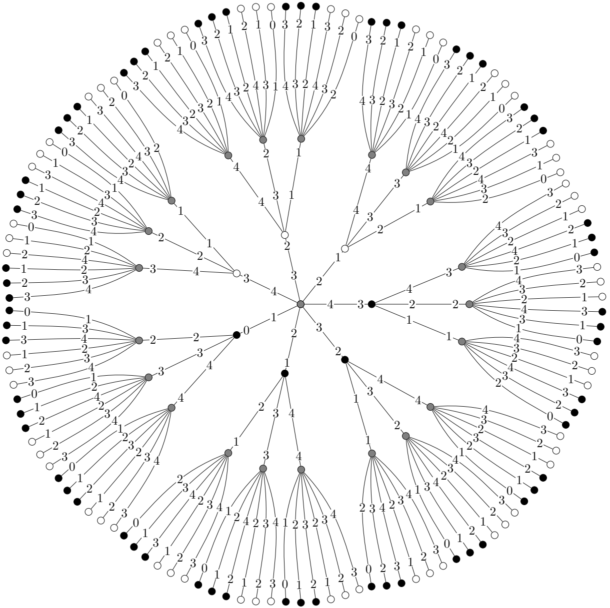
Lemma

Let $(\overline{v}_1, \overline{v}_2)$ *be a PSW of length* $k \leq 2d-3$ *in* G_d*. Then* $k \geq 2d-3$.

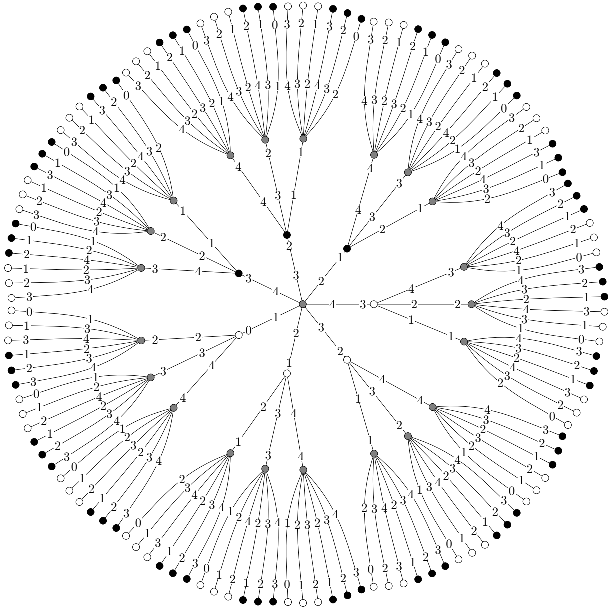
Lemma

We have $((1,0)) \underline{\leftrightarrow}^{\mathcal{SV}}_{2d-3} ((2,1))$*, that is, the nodes* $((1,0))$ *and* $((2,1))$ *of* G_d *are* $(2d-3)$*-*$\mathcal{SV}$*-bisimilar.*

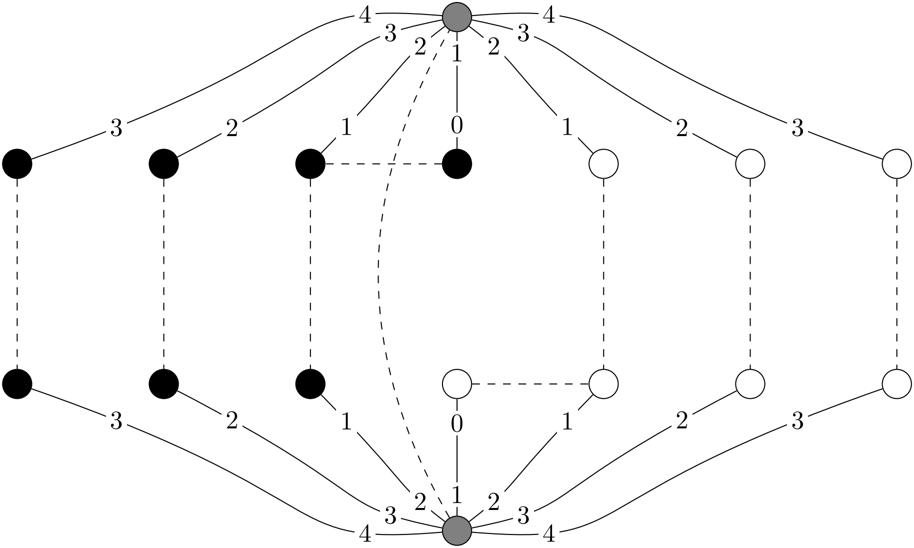
Separation by a graph problem



Separation by a graph problem



Separation by a graph problem



Connections to modal logic

Hella et al. (PODC 2012):

- Logical characterisations for constant-time variants of the problem classes
- In a certain class of structures, SV corresponds to *multimodal logic*
- . . . and MV corresponds to *graded multimodal logic*.

Connections to modal logic

Hella et al. (PODC 2012):

- Logical characterisations for constant-time variants of the problem classes
- In a certain class of structures, SV corresponds to *multimodal logic*
- ... and MV corresponds to *graded multimodal logic*.
- Our result: When given a formula ϕ of graded multimodal logic, we can find an equivalent formula ψ of multimodal logic, but in general, the modal depth $\mathrm{md}(\psi)$ of ψ has to be at least $\mathrm{md}(\phi) + \Delta - 1$.

Conclusion

$\mathcal{MV}$: Send a vector, receive a multiset.

$\mathcal{SV}$: Send a vector, receive a set.

Previously:

- It is possible to simulate $\mathcal{MV}$ in $\mathcal{SV}$ by using $2\Delta - 2$ extra rounds.

This work:

- $2\Delta - 2$ rounds are necessary to solve the simulation problem.
- There is a graph problem for which the difference in running time between $\mathcal{MV}$ and $\mathcal{SV}$ is $\Delta - 1$ rounds.

The thesis *A Classification of Weak Models of Distributed Computing* is available at `http://hdl.handle.net/10138/144214`.