VORTICES IN SUPERFLUID ^{3}He

Lectures for Kevo winter school
20.-26.4.2002

Erkki Thuneberg

Department of physical sciences
University of Oulu

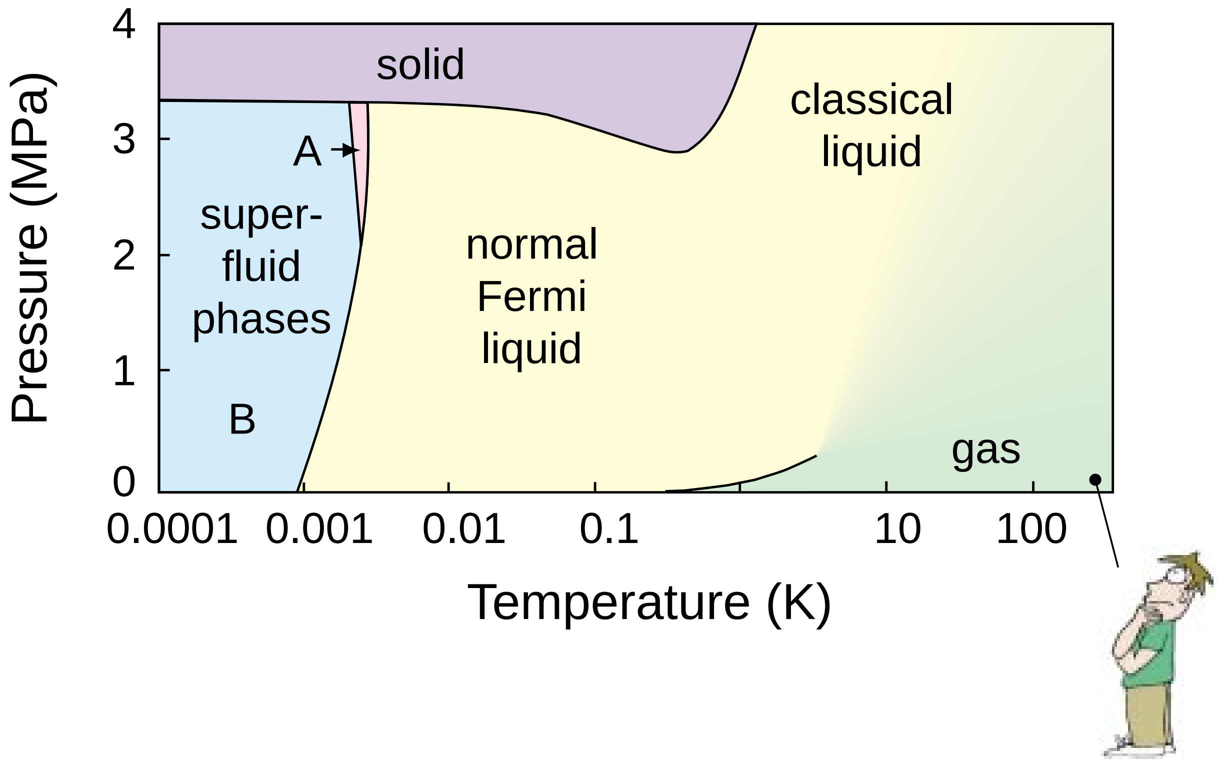
Thanks to the
Low temperature laboratory,
Helsinki university of technology
where most of this work was
done

These lecture notes are available at
http://boojum.hut.fi/research/theory/

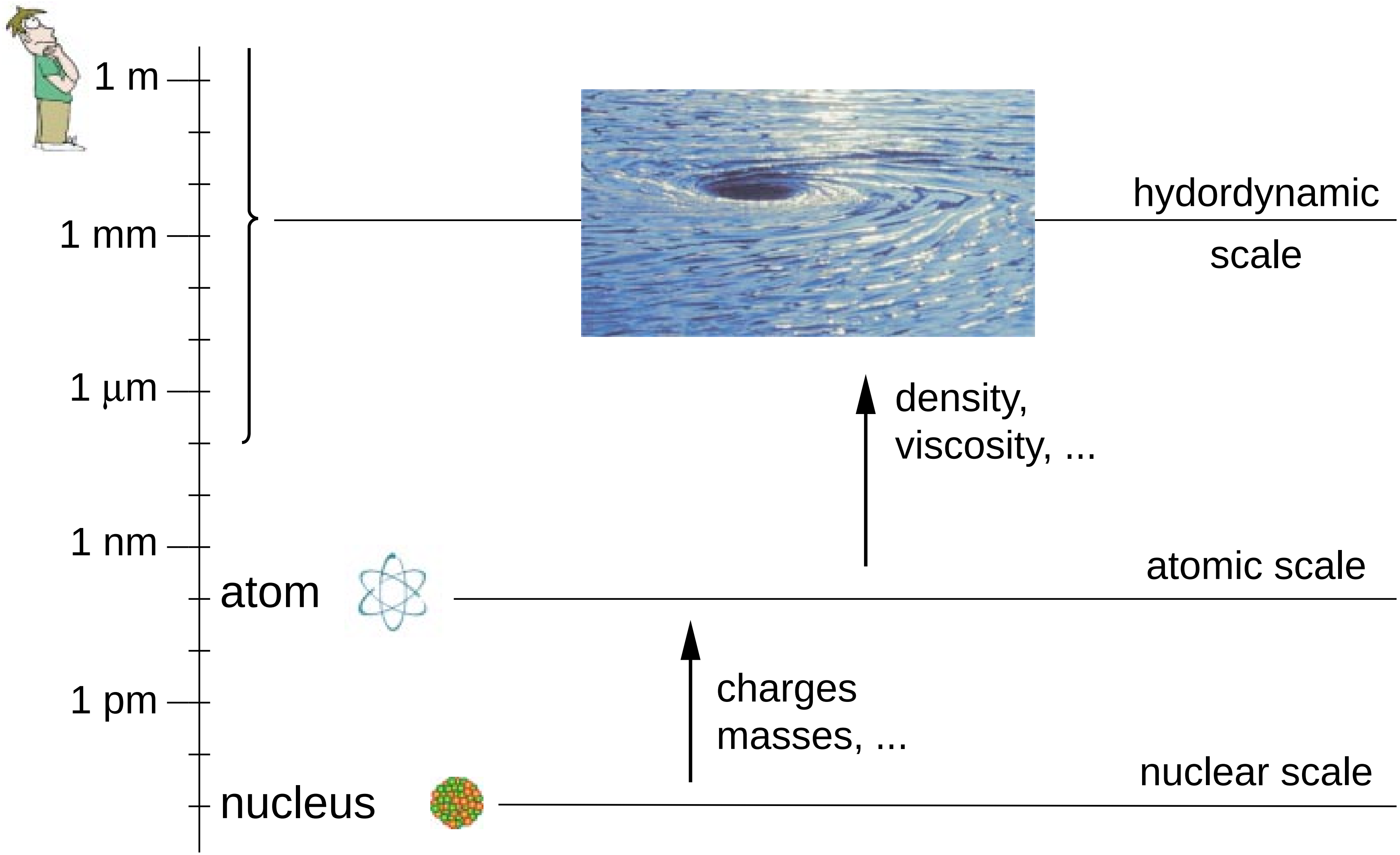
Content

Phase diagram of superfluid ^{3}He

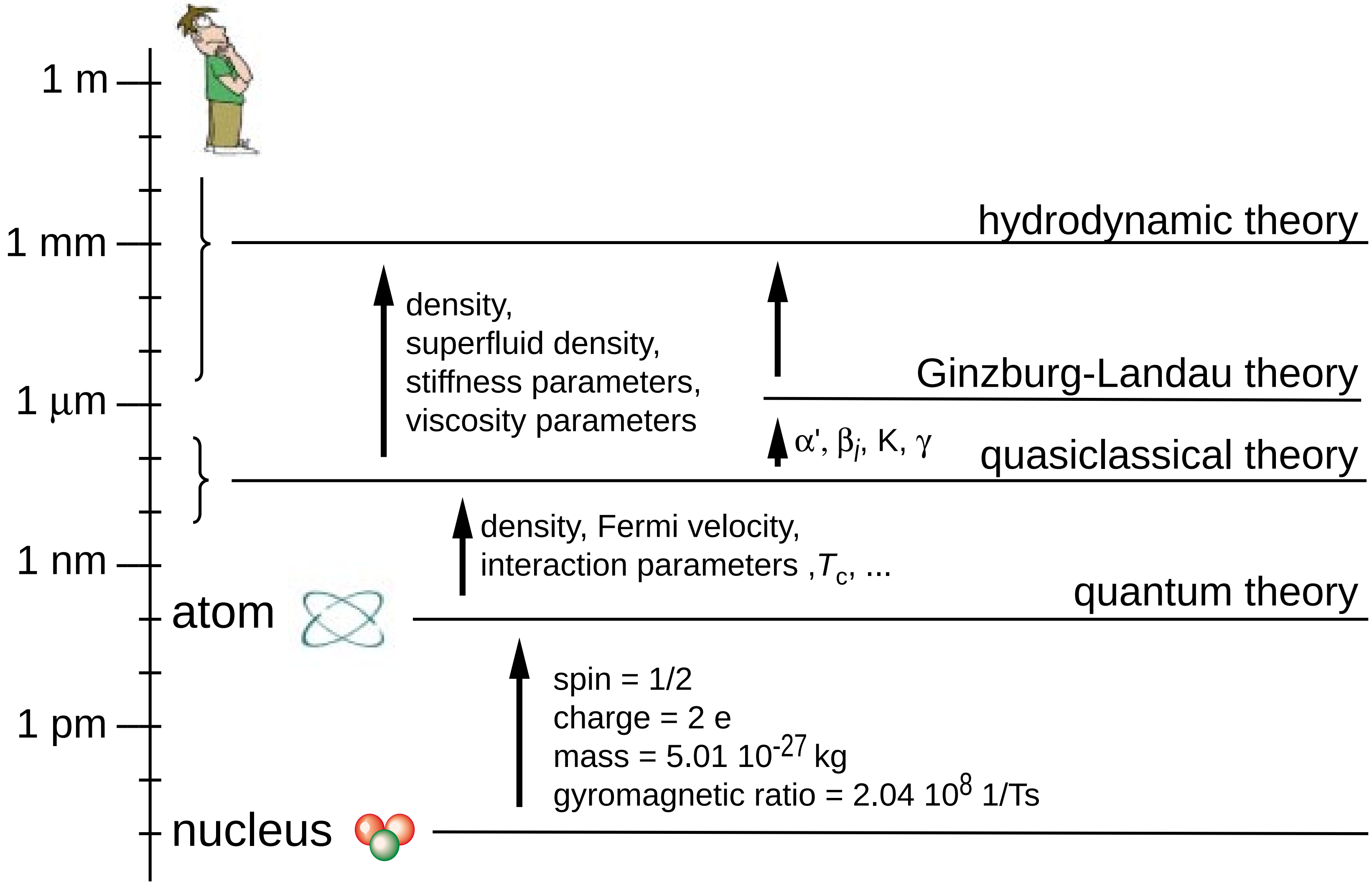
^{3}He is the less common isotope of helium. It is a fermion.



Length scales



Length scales in superfluid ^{3}He



Superfluidity

Superfluids can flow without friction.

Superfluidity is a quantum phenomenon on a macroscopic scale.

Macroscopic number of particles is in the same quantum state.

$\Rightarrow$ a macroscopic wave function (order parameter)

Bosons

A macroscopic number of bosons is in the same quantum state.

$\Rightarrow$ superfluidity of liquid ^{4}He and gases of alkali atoms

Fermions

Only one fermion can occupy one quantum state.

Fermions form pairs (Cooper pairs).

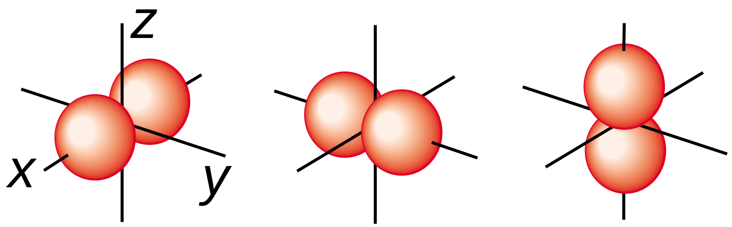
$\Rightarrow$ superconductivity of metals, superfluidity of liquid ^{3}He

Order parameter of superfluid ^{3}He

Cooper pairs form in p-wave state

$\Rightarrow$ the spin state is triplet

Orbital wave functions (L=1)



Spin wave functions (S=1)

$S_x=0$: $(-\uparrow\uparrow+\downarrow\downarrow)$

$S_y=0$: $i(\uparrow\uparrow+\downarrow\downarrow)$

$S_z=0$: $(\uparrow\downarrow+\downarrow\uparrow)$

$$\begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

order parameter

$A_{ij}(\mathbf{r})$ is the wave function for the center of mass of a Cooper pair.

Ginzburg-Landau theory of ^{3}He

The order parameter

$$\mathsf{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix}$$

is determined by a minimum of the free-energy functional

$$\begin{aligned} F &= \int d^3r \Big[\alpha A^*_{\mu i} A_{\mu i} \\ &+\beta_1 |A_{\mu i} A_{\mu i}|^2 + \beta_2 (A_{\mu i} A^*_{\mu i})^2 + \beta_3 A^*_{\mu i} A^*_{\nu i} A_{\nu j} A_{\mu j} \\ &+\beta_4 A^*_{\mu i} A_{\nu i} A^*_{\nu j} A_{\mu j} + \beta_5 A^*_{\mu i} A_{\nu i} A_{\nu j} A^*_{\mu j} \\ &+K(\gamma - 1) \partial_i A^*_{\mu i} \partial_j A_{\mu j} + K \partial_i A^*_{\mu j} \partial_i A_{\mu j} \Big]. \end{aligned} \quad (1)$$

where $\alpha = \alpha'(\frac{T}{T_c} - 1)$. Input parameters α', β_1, β_2, β_3, β_4, β_5, K and γ.

Expansion 1) in the amplitude of the order parameter and 2) in gradients.

Exercises

B-phase

$$\mathsf{A} = \Delta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Substitute in the free energy (1). Minimize with respect to Δ to obtain

$$\Delta^2 = \frac{|\alpha|}{6\beta_{12} + 2\beta_{345}}, \qquad (2)$$

where $\beta_{ij\ldots} = \beta_i + \beta_j + \ldots$. This gives the free energy density

$$f = \frac{F}{\text{volume}} = -\frac{3\alpha^2}{12\beta_{12} + 4\beta_{345}}. \qquad (3)$$

A-phase

$$\mathsf{A} = \Delta \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Substitution and minimization give

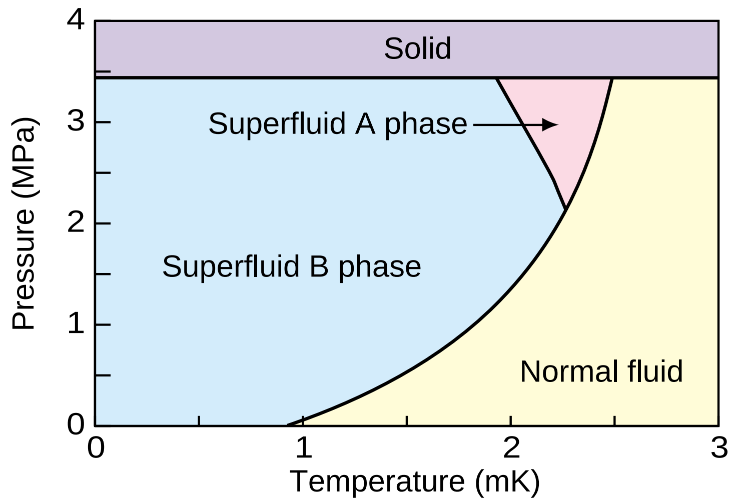
$$f = -\frac{\alpha^2}{4\beta_{245}}, \quad \Delta^2 = \frac{|\alpha|}{4\beta_{245}}. \qquad (4)$$

The same energies can be obtained by rotating the spin and orbital coordinates into arbitrary orientation. Also an arbitrary phase factor $\exp(i\chi)$ can be included. Thus for the B-phase

$$\mathsf{A} = \Delta \exp(i\chi) \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix}, \qquad (5)$$

where R is a rotation matrix. The simplest representation for the A phase uses an orthonormal triad $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{l}}$ and a unit vector $\hat{\mathbf{d}}$:

$$A_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j). \qquad (6)$$

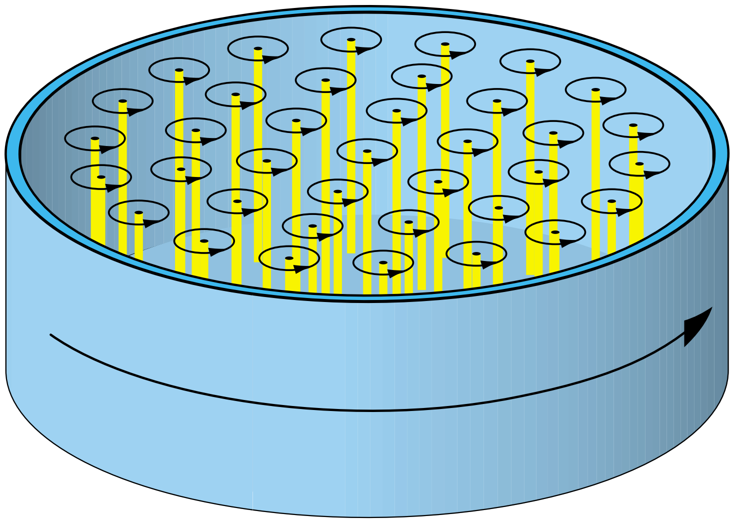


Problem of a rotating superfluid

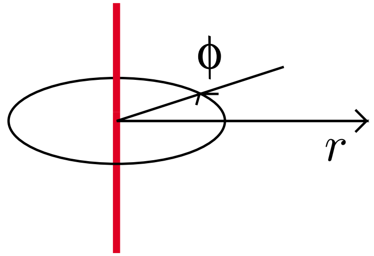
An uncharged superfluid cannot rotate homogeneously:

$$A(\mathbf{r}) = A_0 \exp[i\chi(\mathbf{r})]$$
$$\Rightarrow \mathbf{v} = \frac{\mathbf{p}}{2m} = \frac{-i\hbar\boldsymbol{\nabla}}{2m} = \frac{\hbar}{2m}\boldsymbol{\nabla}\chi$$
$$\Rightarrow \boldsymbol{\nabla} \times \mathbf{v} = 0.$$

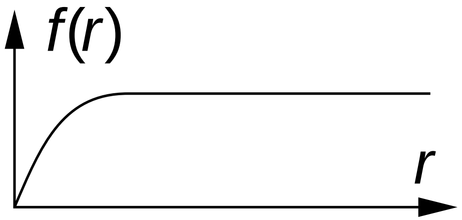
Rotation takes place via vortex lines



Simple model for a vortex



$$A(r,\phi) = A_0 e^{i\phi} f(r)$$



It follows that

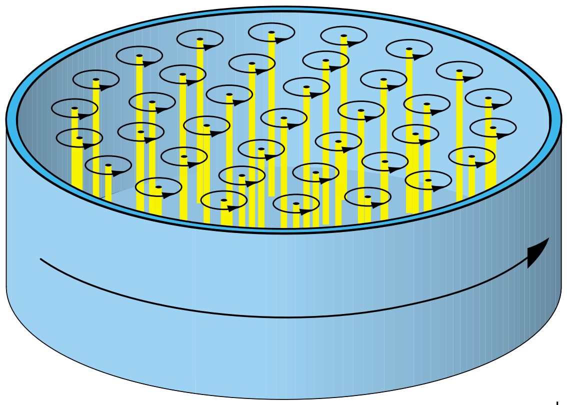
$$\mathbf{v} = \frac{\hbar}{2m}\boldsymbol{\nabla}\phi = \frac{\hbar}{2m}\frac{\hat{\phi}}{r}.$$

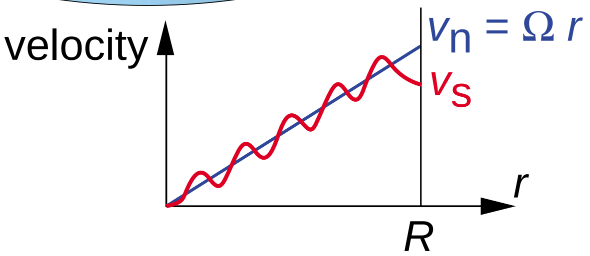
Circulation around a vortex line

$$\kappa = \oint d\mathbf{l}\cdot\mathbf{v} = \frac{\hbar}{2m}\oint d\mathbf{l}\cdot\boldsymbol{\nabla}\phi = \frac{h}{2m}.$$

Density of vortex lines n

Angular velocity Ω.



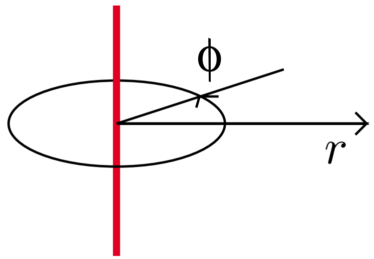


On the average, the superfluid rotates as a normal fluid. The circulation around the container equals

$$2\pi R \Omega R = n \kappa \pi R^2$$

$$\Rightarrow n = \frac{2\Omega}{\kappa}. \qquad (7)$$

The vortex core

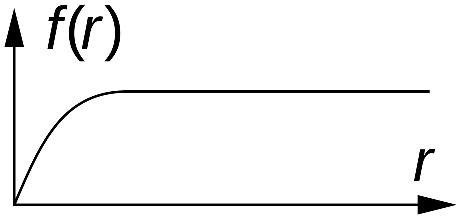


Boundary condition (B phase)

$$\mathsf{A}(r \to \infty, \phi) = e^{\mathrm{i}\phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (8)$$

Simple ansatz:

$$\mathsf{A}(r, \phi) = f(r) e^{\mathrm{i}\phi} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Symmetry classification

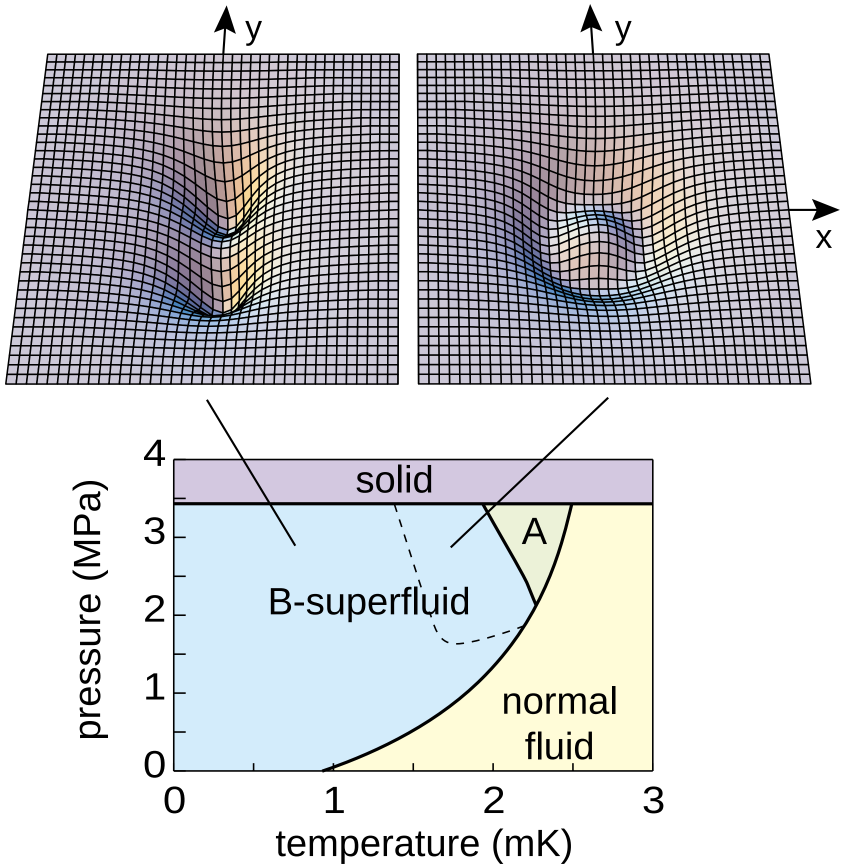
Axially symmetric vortices (Salomaa & Volovik 1983)

Symmetry	$\mathsf{A}(r,\phi) = \begin{pmatrix} A_{rr} & A_{r\phi} & A_{rz} \\ A_{\phi r} & A_{\phi\phi} & A_{\phi z} \\ A_{zr} & A_{z\phi} & A_{zz} \end{pmatrix}$
$\frac{\infty}{m}\frac{2'}{m'}$	$e^{\mathrm{i}\phi}\begin{pmatrix} R_1(r) & I_1(r) & 0 \\ I_2(r) & R_2(r) & 0 \\ 0 & 0 & R_3(r) \end{pmatrix}$
$\frac{\infty}{m}$	$e^{\mathrm{i}\phi}\begin{pmatrix} C_1(r) & C_2(r) & 0 \\ C_3(r) & C_4(r) & 0 \\ 0 & 0 & C_5(r) \end{pmatrix}$
$\infty m'$	$e^{\mathrm{i}\phi}\begin{pmatrix} R_1(r) & I_1(r) & R_2(r) \\ I_2(r) & R_3(r) & I_3(r) \\ R_4(r) & I_4(r) & R_5(r) \end{pmatrix}$
$\infty 2'$	$e^{\mathrm{i}\phi}\begin{pmatrix} R_1(r) & I_1(r) & I_2(r) \\ I_3(r) & R_2(r) & R_3(r) \\ I_4(r) & R_4(r) & R_5(r) \end{pmatrix}$
∞	$e^{\mathrm{i}\phi}\begin{pmatrix} C_1(r) & C_2(r) & C_3(r) \\ C_4(r) & C_5(r) & C_6(r) \\ C_7(r) & C_8(r) & C_9(r) \end{pmatrix}$

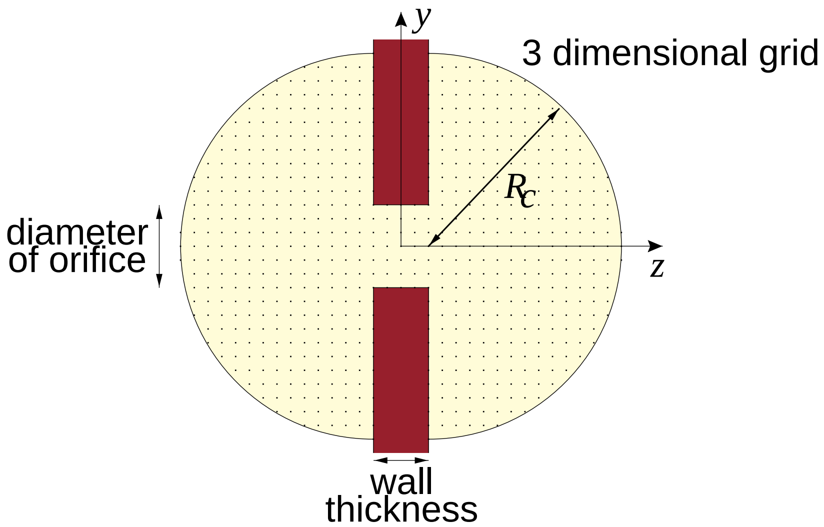
Vortex cores in ^{3}He-B

Numerical minimization of Ginzburg-Landau functional (1) with boundary condition (8) in 2 D

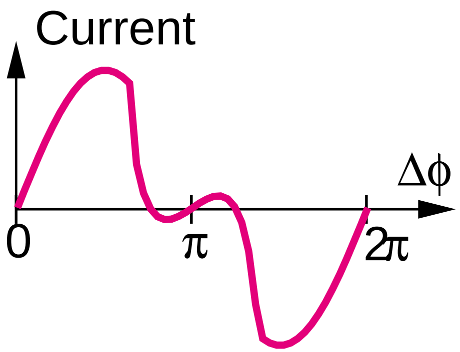
$$\partial_j\partial_j A_{\mu i} + (\gamma-1)\partial_i\partial_j A_{\mu j} = [-\mathsf{A} + \beta_1 \mathsf{A}^* Tr(\mathsf{A}\mathsf{A}^T) + \beta_2 \mathsf{A} Tr(\mathsf{A}\mathsf{A}^{T*}) + \beta_3 \mathsf{A}\mathsf{A}^T\mathsf{A}^* + \beta_4 \mathsf{A}\mathsf{A}^{T*}\mathsf{A} + \beta_5 \mathsf{A}^*\mathsf{A}^T\mathsf{A}]_{\mu i}. \quad (9)$$



Josephson effect

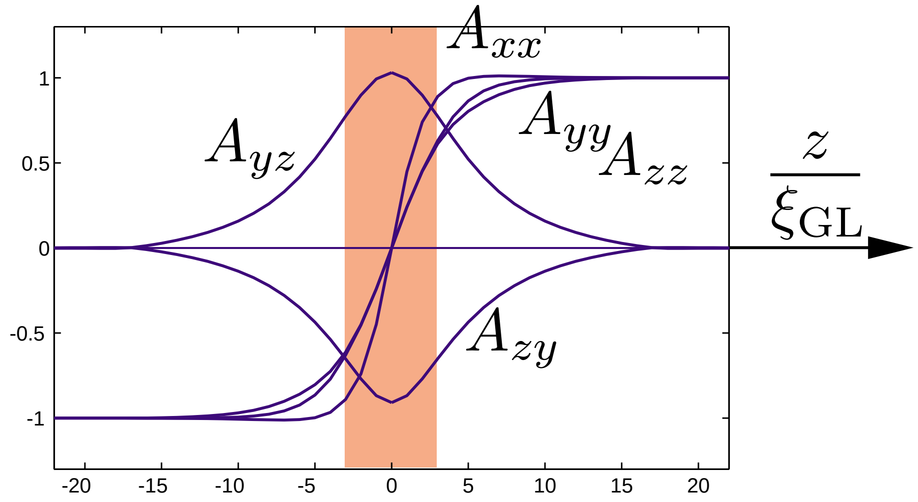


Numerical calculation with Ginzburg-Landau theory (Janne Viljas)



⇒ π-state

The order parameter in the orifice for a phase difference π



This is very similar as the order parameter in the double-core vortex on an axis that passes between the two cores.

The two cores can be interpreted as two *half-quantum* vortices.

Other line defects in ^{3}He-B

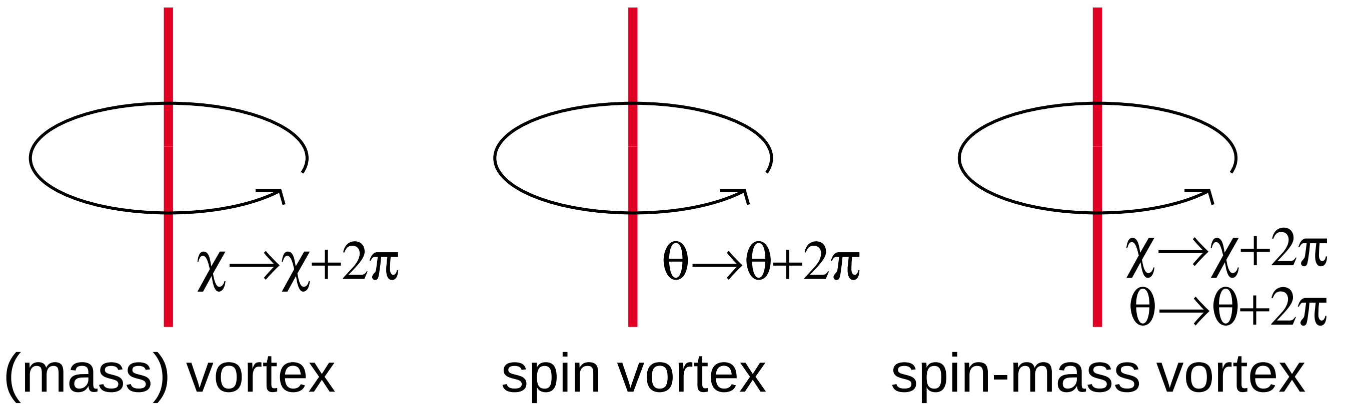
In the order parameter

$$A = \Delta \exp(i\chi) R(\hat{\mathbf{n}}, \theta) \tag{5}$$

the rotation matrix R can be parametrized with an axis of rotation $\hat{\mathbf{n}}$ and an angle θ.

Symmetry $U(1) \times SO_3$. Homotopy groups $\pi(U(1)) = Z$, $\pi(SO_3) = Z_2$.

Variation of R results in spin currents.



Ginzburg-Landau calculation
$\Rightarrow$ The spin-mass vortex is stable against dissociation: $F_{\mathrm{sm}} < F_{\mathrm{s}} + F_{\mathrm{m}}$.

Spin current is not conserved on a large scale.

Additional terms in Ginzburg-Landau functional

$$
\begin{aligned}
F_{\mathrm{basic}} \;=\; & \int d^3r \Big[\alpha A^*_{\mu i} A_{\mu i} \\
& +\beta_1 |A_{\mu i} A_{\mu i}|^2 + \beta_2 (A_{\mu i} A^*_{\mu i})^2 + \beta_3 A^*_{\mu i} A^*_{\nu i} A_{\nu j} A_{\mu j} \\
& +\beta_4 A^*_{\mu i} A_{\nu i} A^*_{\nu j} A_{\mu j} + \beta_5 A^*_{\mu i} A_{\nu i} A_{\nu j} A^*_{\mu j} \\
& +(\gamma - 1)\partial_i A^*_{\mu i} \partial_j A_{\mu j} + \partial_i A^*_{\mu j} \partial_i A_{\mu j}\Big].
\end{aligned}
$$

Dipole-dipole interaction of ^{3}He nuclear moments ($g_D \sim 10^{-6}\alpha'$)

$$
F_{\mathrm{D}} = g_{\mathrm{D}} \int d^3r (A^*_{ii} A_{jj} + A^*_{ij} A_{ji} - \tfrac{2}{3} A^*_{\mu i} A_{\mu i}). \tag{10}
$$

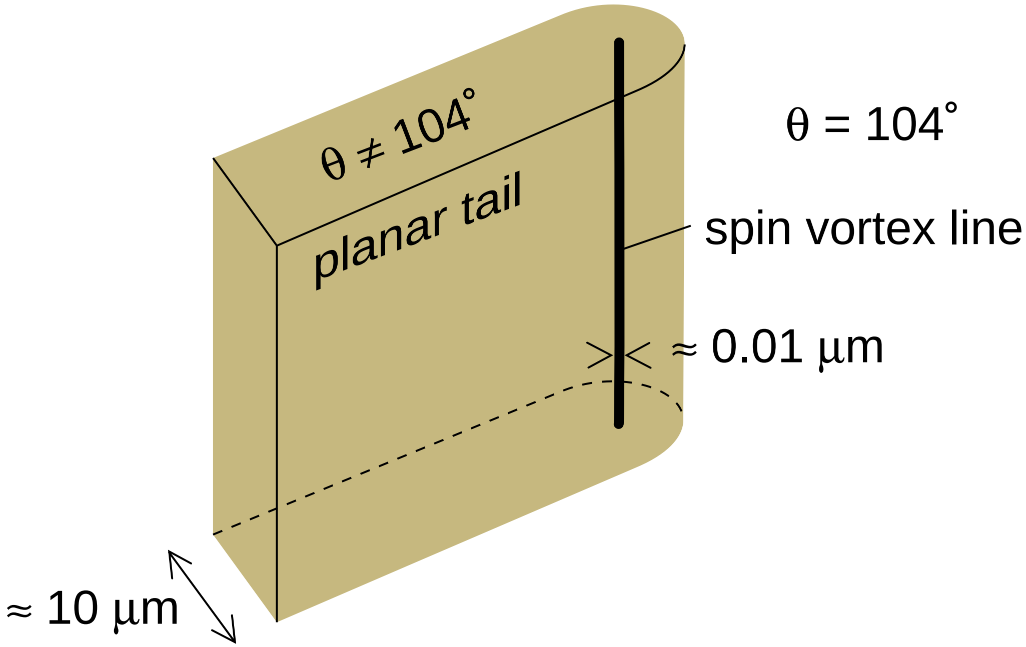
External magnetic field

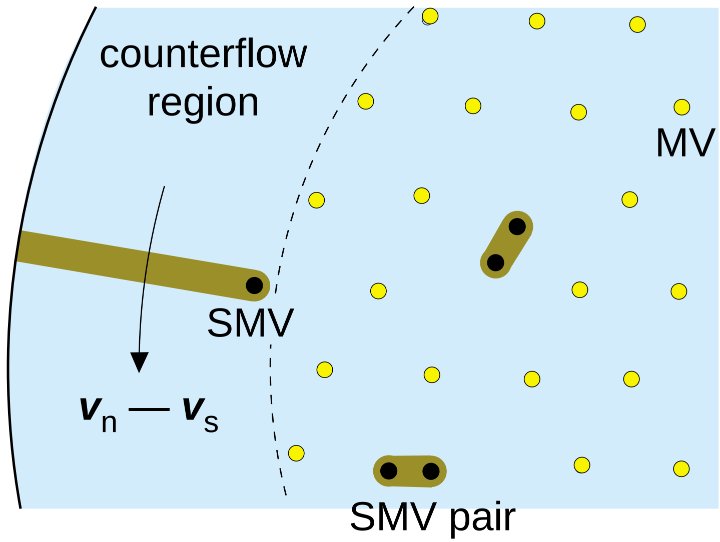
$$
F_{\mathrm{H}} = \int d^3r \left(-\mathrm{i} g_{\mathrm{H1}} \epsilon_{\kappa\mu\nu} H_\kappa A^*_{\mu i} A_{\nu i} + g_{\mathrm{H}} H_\mu A^*_{\mu i} A_{\nu i} H_\nu + g'_{\mathrm{H}} H^2 A^*_{\mu i} A_{\mu i}\right). \tag{11}
$$

Rotation of the liquid

$$
F_{\mathrm{G}} \;=\; -\mathrm{i}(2\eta - 1) K \epsilon_{kij} \int d^3r (\boldsymbol{\nabla} \times \mathbf{v}_{\mathrm{n}})_k A^*_{\mu i} A_{\mu j}. \tag{12}
$$

Large-scale structure of a spin-mass vortex

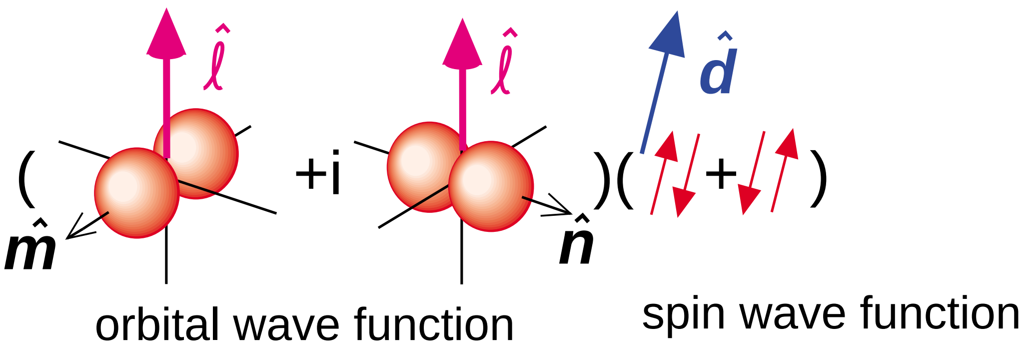




A few spin-mass vortices are created in fast A→B transitions.

The A phase

The order parameter $A_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$



A phase factor $e^{i\chi}$ corresponds to rotation of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ around $\hat{\mathbf{l}}$:

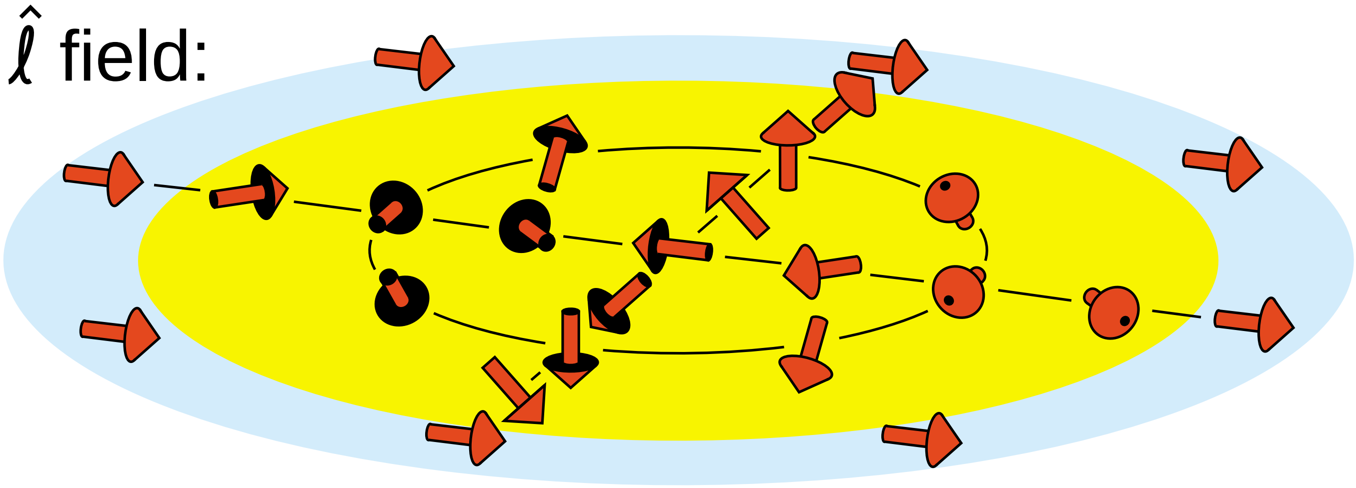
$$\begin{aligned} e^{i\chi}(\hat{\mathbf{m}} + i\hat{\mathbf{n}}) &= (\cos\chi + i\sin\chi)(\hat{\mathbf{m}} + i\hat{\mathbf{n}}) \\ &= (\hat{\mathbf{m}}\cos\chi - \hat{\mathbf{n}}\sin\chi) + i(\hat{\mathbf{m}}\sin\chi + \hat{\mathbf{n}}\cos\chi). \end{aligned}$$

Superfluid velocity

$$\mathbf{v}_\mathrm{s} = \frac{\hbar}{2m}\boldsymbol{\nabla}\chi = \frac{\hbar}{2m}\sum_j \hat{m}_j \boldsymbol{\nabla}\hat{n}_j. \qquad (13)$$

Vortices in the A phase

Consider the structure



Here $\hat{\mathbf{l}}$ sweeps once trough all orientations (once a unit sphere).

Claim: $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ circle twice around $\hat{\mathbf{l}}$ when one goes around this object.

Exercise: verify this by drawing $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ on the figure.

Conclusion: this is a two-quantum vortex. It is called *continuous*, because Δ (the amplitude of the order parameter) vanishes nowhere.

Hydrostatic theory of ^{3}He-A

Assume the order parameter ($\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{l}}$, $\hat{\mathbf{d}}$) changes slowly in space. Then we can make gradient expansion of the free energy

$$
\begin{aligned}
F \;=\; & \int d^3r \Big[-\tfrac{1}{2}\lambda_{\mathrm{D}}(\hat{\mathbf{d}}\cdot\hat{\mathbf{l}})^2 + \tfrac{1}{2}\lambda_{\mathrm{H}}(\hat{\mathbf{d}}\cdot\mathbf{H})^2 \\
& + \tfrac{1}{2}\rho_{\perp}\mathbf{v}^2 + \tfrac{1}{2}(\rho_{\parallel}-\rho_{\perp})(\hat{\mathbf{l}}\cdot\mathbf{v})^2 + C\mathbf{v}\cdot\nabla\times\hat{\mathbf{l}} - C_0(\hat{\mathbf{l}}\cdot\mathbf{v})(\hat{\mathbf{l}}\cdot\nabla\times\hat{\mathbf{l}}) \\
& + \tfrac{1}{2}K_{\mathrm{s}}(\nabla\cdot\hat{\mathbf{l}})^2 + \tfrac{1}{2}K_{\mathrm{t}}|\hat{\mathbf{l}}\cdot\nabla\times\hat{\mathbf{l}}|^2 + \tfrac{1}{2}K_{\mathrm{b}}|\hat{\mathbf{l}}\times(\nabla\times\hat{\mathbf{l}})|^2 \\
& + \tfrac{1}{2}K_5|(\hat{\mathbf{l}}\cdot\nabla)\hat{\mathbf{d}}|^2 + \tfrac{1}{2}K_6[(\hat{\mathbf{l}}\times\nabla)_i\hat{\mathbf{d}}_j)]^2 \Big].
\end{aligned}
$$

Needs as input $\lambda_{\mathrm{D}} = 4g_D\Delta^2$, λ_{h}, $\rho_{\perp}$, $\rho_{\parallel}$, C, C_0, K_{t}, K_{s}, K_{b}, K_5, and K_6.

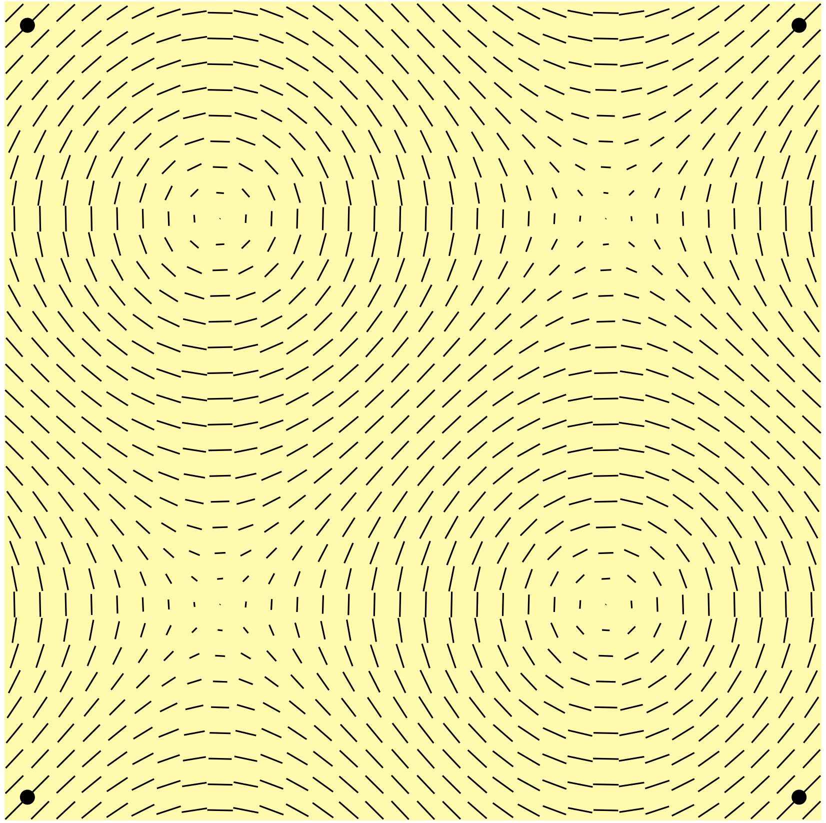
Low field ($H < 1$ mT): $\hat{\mathbf{d}} \parallel \hat{\mathbf{l}} \Rightarrow$ vortex cores fills all space

High field ($H > 1$ mT): $\hat{\mathbf{d}} \perp \mathbf{H} \Rightarrow$ vortex cores limited by g_D to size $\xi_D \approx 10\ \mu$m.

Continuous vortex lattices

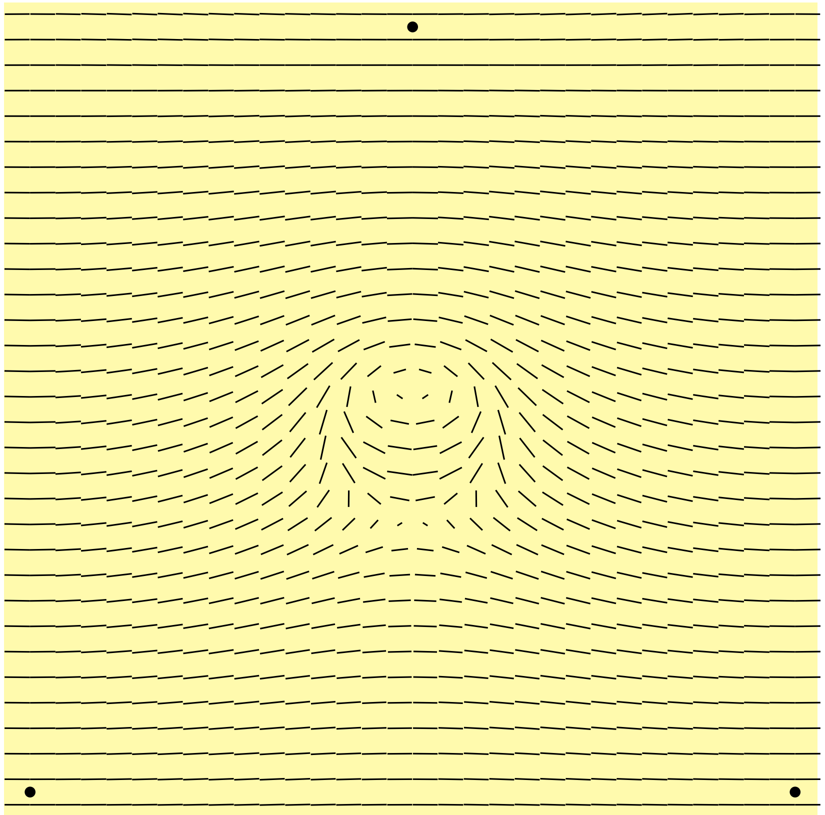
Projection of $\hat{\mathbf{l}}$ field on x-y plane

low field vortex
Locked vortex 1 (LV1)



square lattice
$P\frac{4}{n}b'm'$

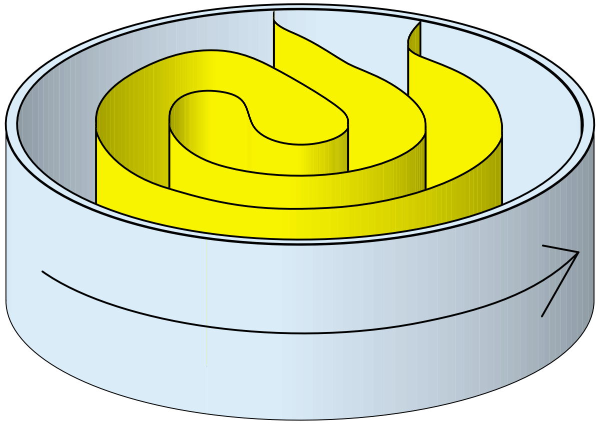
high field vortex
Continuous unlocked vortex (CUV)



centered rectangular lattice
$C2'$

Vortex sheet

Also vortex sheets are possible in ^{3}He-A



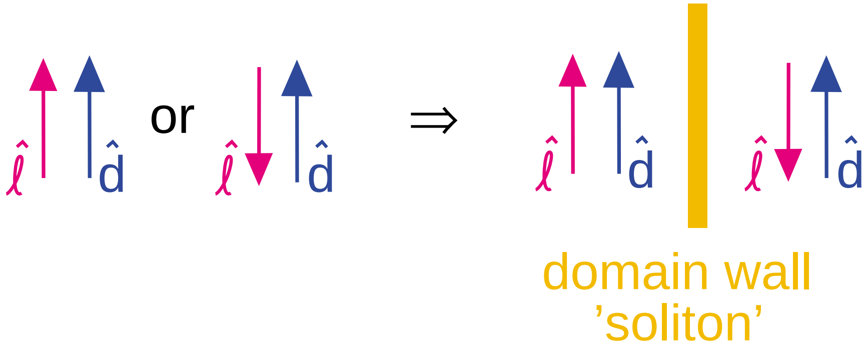
Sheets were first suggested to exist in ^{4}He, but they were found to be unstable.

Why stable in ^{3}He-A?

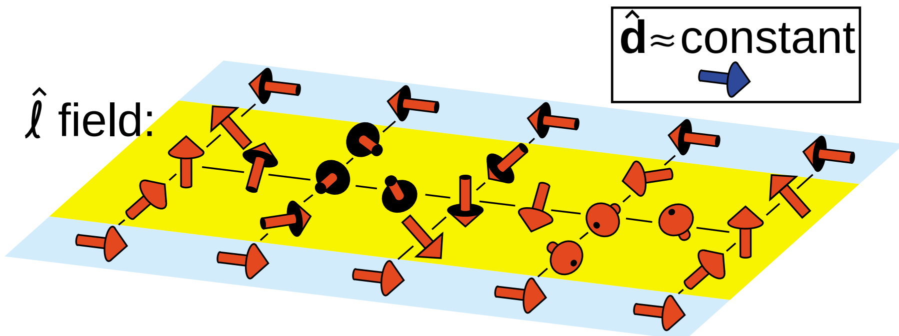
Dipole-dipole interaction (10)

$$f_D = -\frac{1}{2}\lambda_D(\hat{\mathbf{d}}\cdot\hat{\mathbf{l}})^2$$

$\Rightarrow$



Vortex sheet = soliton wall to which the vortices are bound.



Shape of the vortex sheet

The equilibrium configuration of the sheet is determined by the minimum of

$$F = \int d^3r \tfrac{1}{2}\rho_s(\mathbf{v}_n - \mathbf{v}_s)^2 + \sigma A.$$

Here A is the area of the sheet and σ its surface tension.

The equilibrium distance b between two sheets is

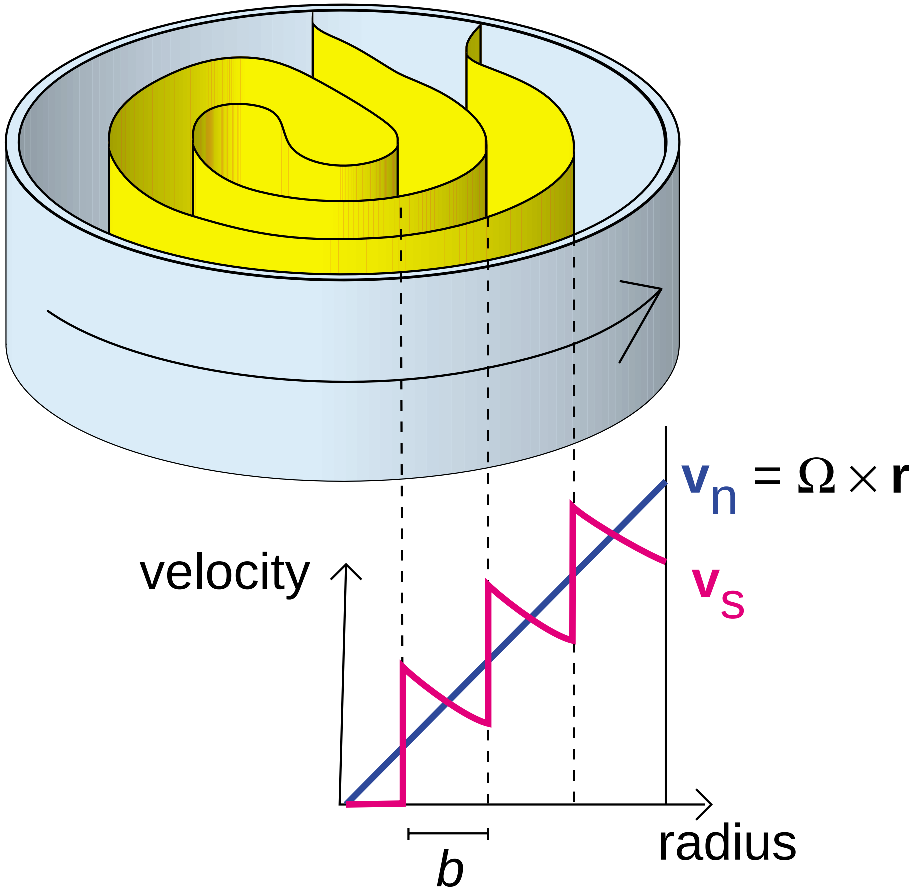
$$b = \left(\frac{3\sigma}{\rho_s \Omega^2}\right)^{1/3}. \qquad (14)$$

This gives 0.36 mm at $\Omega = 1$ rad/s.

The area of the sheet

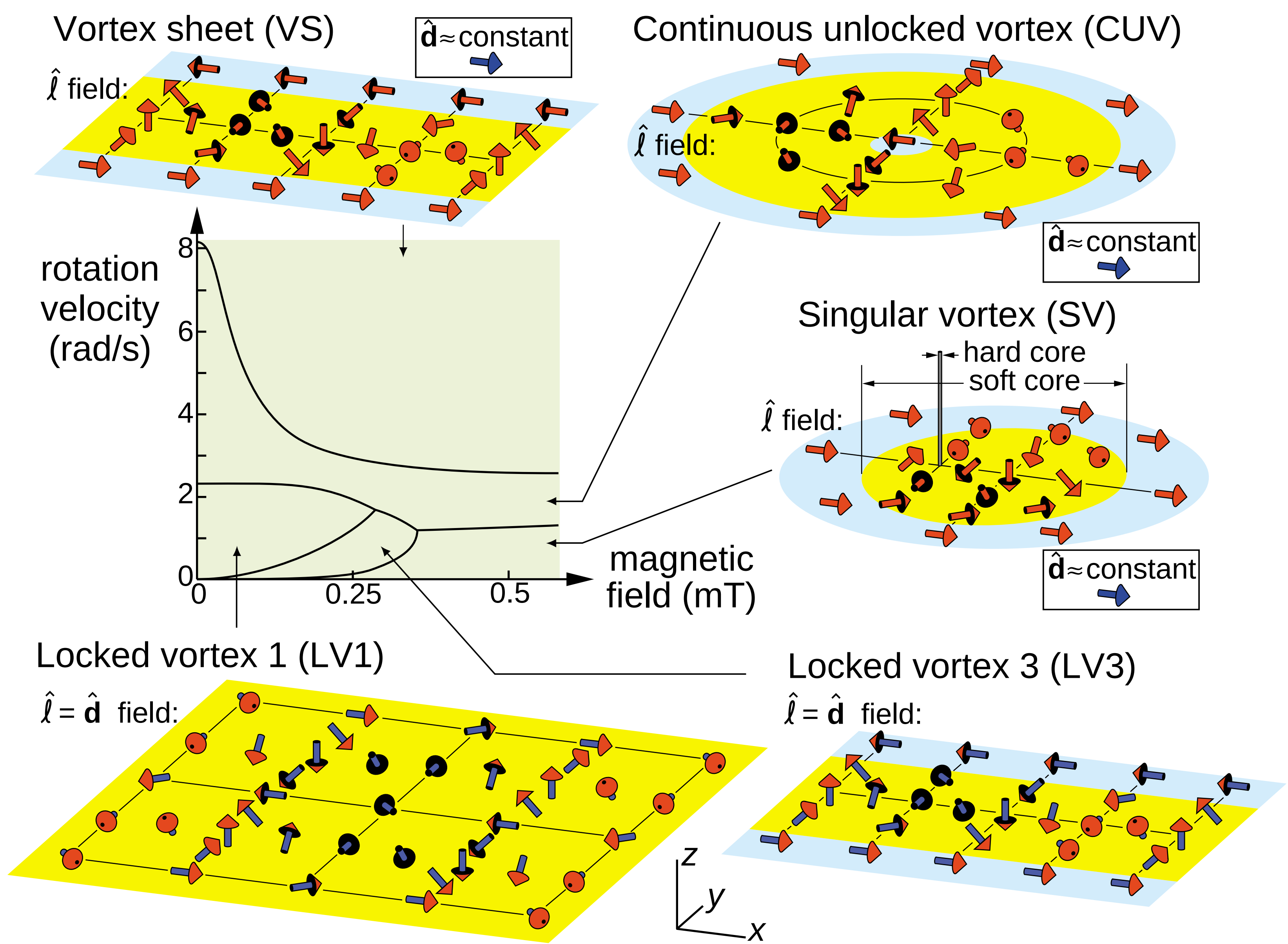
$$A \propto \frac{1}{b} \propto \Omega^{2/3}, \qquad (15)$$

as compared to $N_{\text{vortex line}} \propto \Omega$ (7).

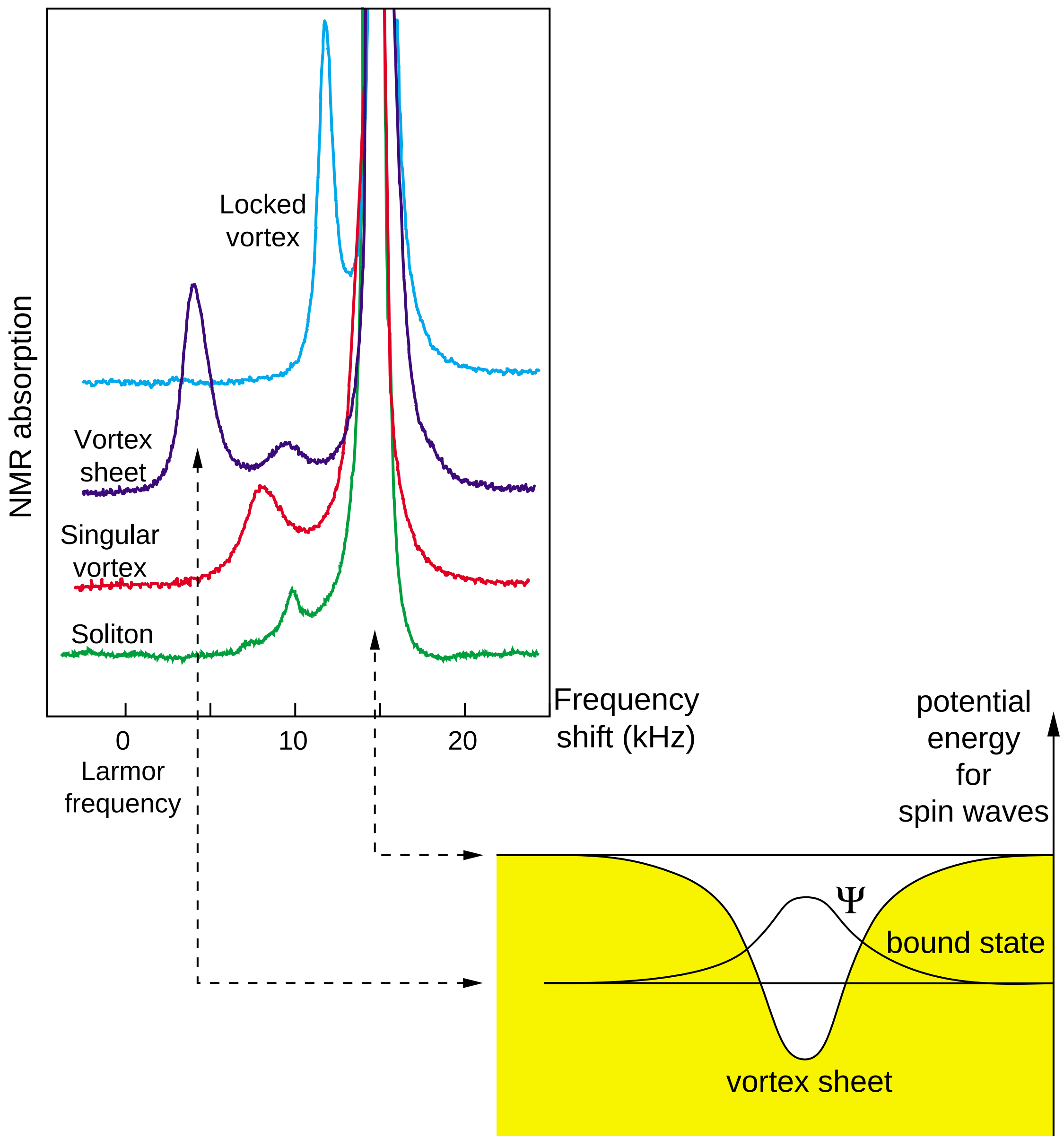


Connection lines with the side wall allow the vortex sheet to grow and shrink when angular velocity Ω changes.

Vortex phase diagram in ^{3}He-A

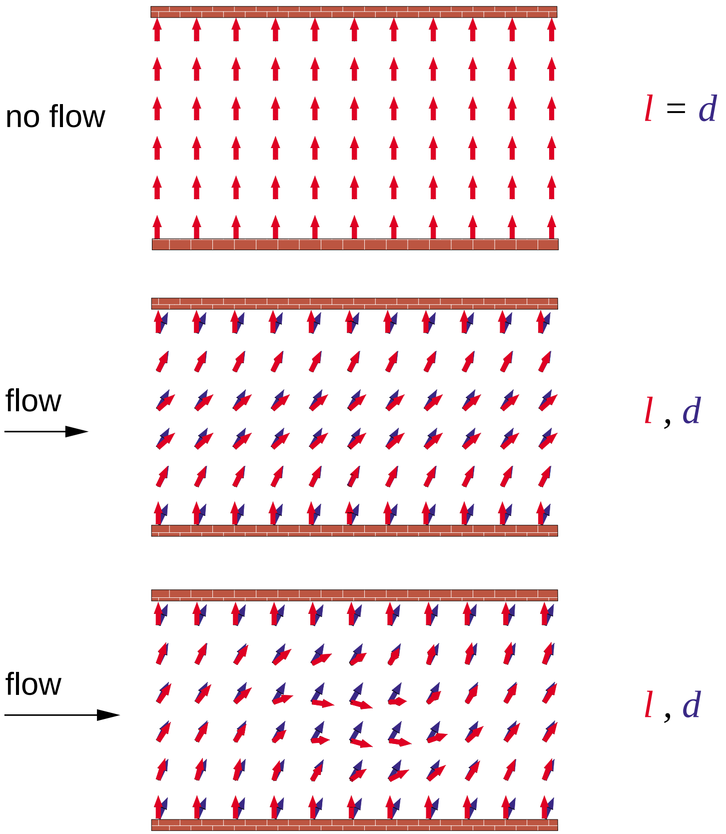


NMR spectra of vortices



Nucleation of vortices

The length scale of vorticity in ^{3}He-A is 10 μm.
Surface roughness can be controlled on this level.
$\Rightarrow$ possibility to study intrinsic vortex formation



Summary of periodic vortex structures

material	vortex name	Bravais lattice	space group	N	ν_d	theory	experiment
^{4}He-II	vortex line	hexagonal	$P6/mm'm'$	1	-	1949-55	1950-79
s.c. metals	flux line	several	several	1	-	1957	1936-66
^{3}He-B	A-phase-core v.	hexagonal	$P6m'm'$	1	-	1983	1982
	double-core v.	cent. rectang.	$Cm'm'2$	1	-	1986	1982
^{3}He-A	locked vortex 1 (LV1)	square	$P4/nb'm'$	4	2	1978	1990
	cont. unlocked v. (CUV)	cent. rectang.	$C2'$	2	0	1983	1982
	singular vortex (SV)	cent. rectang.	Cm'	1	-	1983	1987-95
	locked vortex 2 (LV2)	cent. rectang.	$C2'$	2	1	1985	-
	vortex sheet (VS)	prim. rectang.	$Pb'a'n$	4	0	1994	1985-94

N: number of circulation quanta per unit cell

ν_d: topological quantum number for $\hat{\mathbf{d}}$.

Conclusion

Vorticity in superfluid ^{3}He forms a very rich system because of the several length scales that are relevant in this system.