Vortex sheets and solitons in superfluid ^{3}He-A

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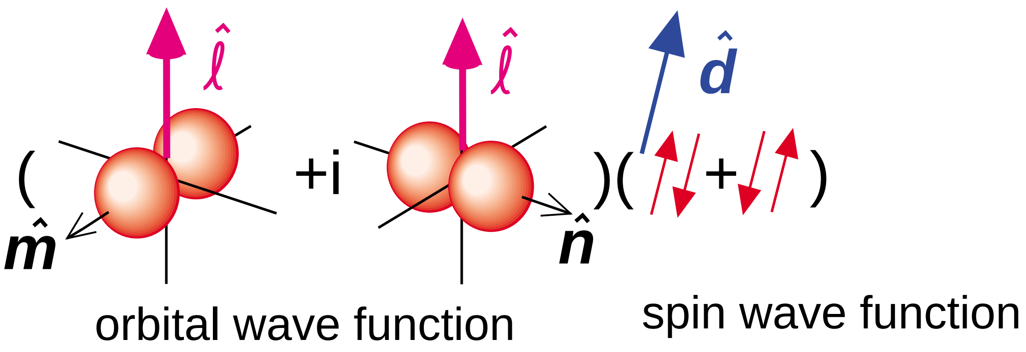
Introduction to superfluid ^{3}He-A

Vortex sheet: analytic results

Solitons: dissipation in NMR absorption

The A phase

The order parameter $A_{\mu j} = \Delta \hat{d}_\mu (\hat{m}_j + i\hat{n}_j)$



A phase factor $e^{i\chi}$ corresponds to rotation of $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ around $\hat{\mathbf{l}}$:

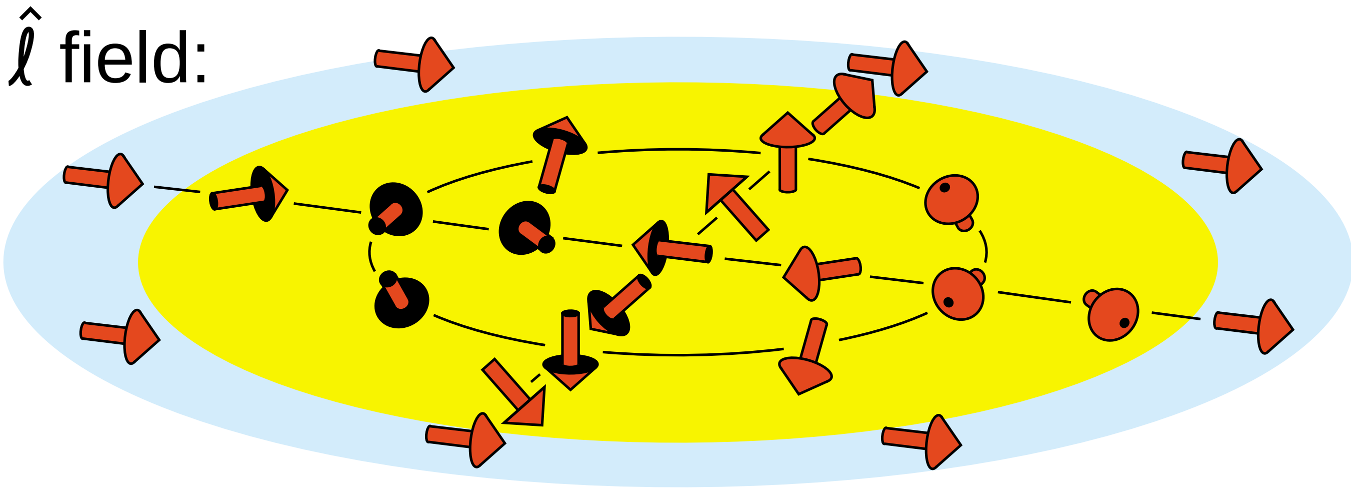
$$
\begin{aligned}
e^{i\chi}(\hat{\mathbf{m}} + i\hat{\mathbf{n}}) &= (\cos\chi + i\sin\chi)(\hat{\mathbf{m}} + i\hat{\mathbf{n}}) \\
&= (\hat{\mathbf{m}}\cos\chi - \hat{\mathbf{n}}\sin\chi) + i(\hat{\mathbf{m}}\sin\chi + \hat{\mathbf{n}}\cos\chi).
\end{aligned}
$$

Superfluid velocity

$$\mathbf{v}_{\mathrm{s}} = \frac{\hbar}{2m}\boldsymbol{\nabla}\chi = \frac{\hbar}{2m}\sum_j \hat{m}_j \boldsymbol{\nabla}\hat{n}_j. \tag{1}$$

Vortices in the A phase

Consider the structure



Here $\hat{\mathbf{l}}$ sweeps once trough all orientations (once a unit sphere).

⇒ $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ circle twice around $\hat{\mathbf{l}}$ when one goes around this object.

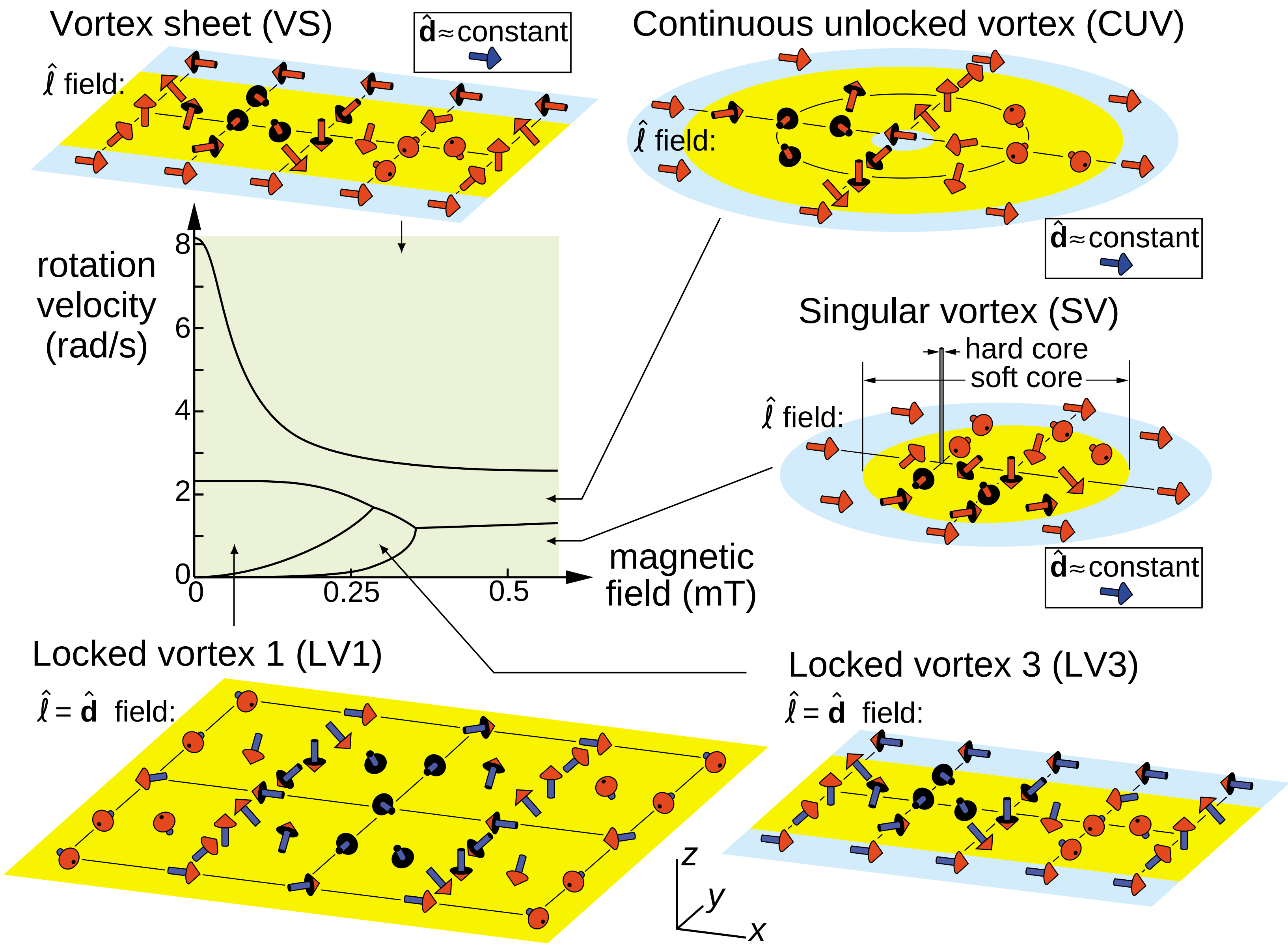
⇒ This is a two-quantum vortex. It is called *continuous*, because Δ (the amplitude of the order parameter) vanishes nowhere.

Hydrostatic theory of ^{3}He-A

Assume the order parameter ($\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{l}}$, $\hat{\mathbf{d}}$) changes slowly in space. Then we can make gradient expansion of the free energy

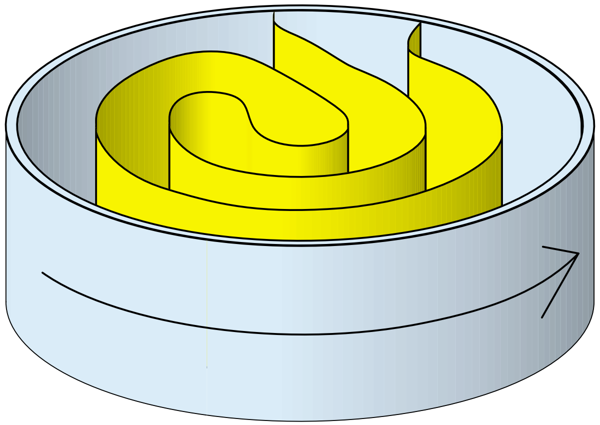
$$\begin{aligned} F \;=\; & \int d^3r \Big[-\tfrac{1}{2}\lambda_{\mathsf{D}}(\hat{\mathbf{d}}\cdot\hat{\mathbf{l}})^2 + \tfrac{1}{2}\lambda_{\mathsf{H}}(\hat{\mathbf{d}}\cdot\mathbf{H})^2 \\ & +\tfrac{1}{2}\rho_{\perp}\mathbf{v}^2 + \tfrac{1}{2}(\rho_{\|}-\rho_{\perp})(\hat{\mathbf{l}}\cdot\mathbf{v})^2 + C\mathbf{v}\cdot\nabla\times\hat{\mathbf{l}} - C_0(\hat{\mathbf{l}}\cdot\mathbf{v})(\hat{\mathbf{l}}\cdot\nabla\times\hat{\mathbf{l}}) \\ & +\tfrac{1}{2}K_{\mathsf{s}}(\nabla\cdot\hat{\mathbf{l}})^2 + \tfrac{1}{2}K_{\mathsf{t}}|\hat{\mathbf{l}}\cdot\nabla\times\hat{\mathbf{l}}|^2 + \tfrac{1}{2}K_{\mathsf{b}}|\hat{\mathbf{l}}\times(\nabla\times\hat{\mathbf{l}})|^2 \\ & +\tfrac{1}{2}K_5|(\hat{\mathbf{l}}\cdot\nabla)\hat{\mathbf{d}}|^2 + \tfrac{1}{2}K_6[(\hat{\mathbf{l}}\times\nabla)_i\hat{\mathbf{d}}_j)]^2 \;\Big]. \end{aligned} \tag{2}$$

Vortex phase diagram in ^{3}He-A



Vortex sheet

Vortex sheets are possible in ^{3}He-A



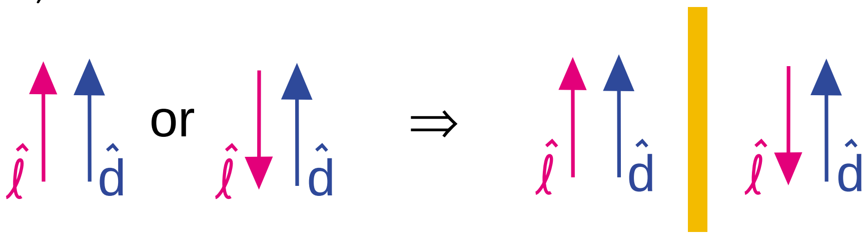
Sheets were first suggested to exist in ^{4}He, but they were found to be unstable.

Why stable in ^{3}He-A?

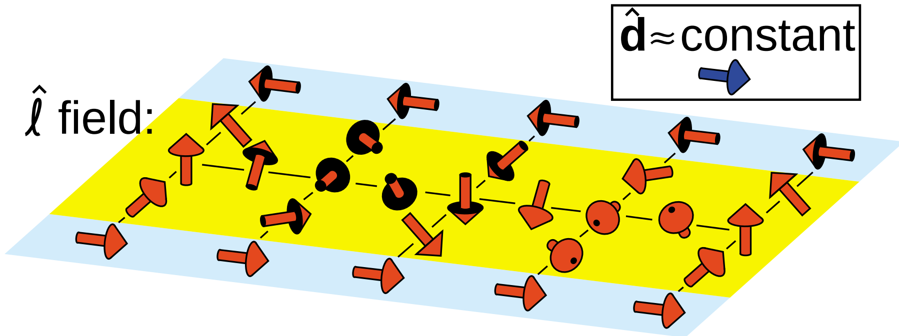
Dipole-dipole interaction (2)

$$f_D = -\frac{1}{2}\lambda_D(\hat{\mathbf{d}}\cdot\hat{\mathbf{l}})^2$$

$\Rightarrow$



Vortex sheet = soliton wall to which the vortices are bound.



Simple model of the vortex sheet

Minimize

$$F = \int d^3r \tfrac{1}{2}\rho_s v^2 + \sigma A. \qquad (3)$$

with constraints

$$\nabla \cdot \mathbf{v} = 0, \ \nabla \times \mathbf{v} = 2\mathbf{\Omega}. \qquad (4)$$

Here $\mathbf{v} \equiv \mathbf{v}_s - \mathbf{v}_n$. A is the area of the sheet and σ its surface tension. $\mathbf{v}$ has tangential discontinuity at the sheet.

Warning: this neglects the bending energy of the texture.

Exact corollary

$$\sigma K + \tfrac{1}{2}\rho_s(v_1^2 - v_2^2) = 0. \qquad (5)$$

where $K/2$ is the mean curvature of the sheet ($K = R_a^{-1} + R_b^{-1}$, where R_a and R_b are the principal radii of curvature). v_1 and v_2 are the (tangential) velocities on the two sides.

Applications

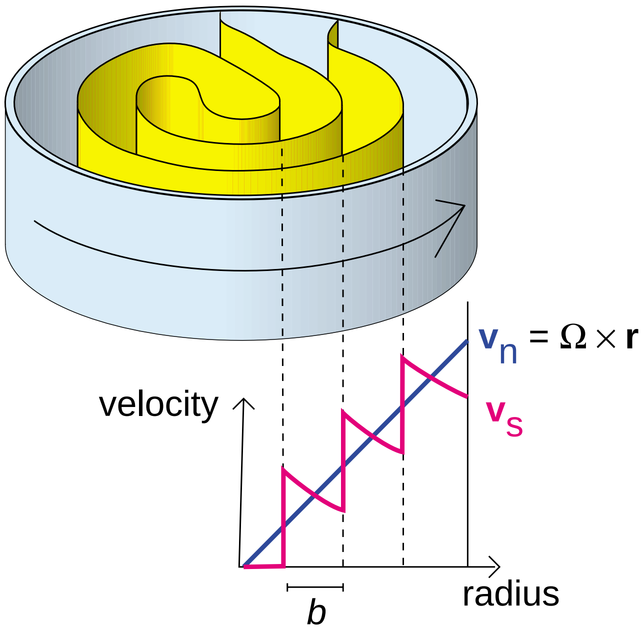
1) **Planar sheets** with distance b:

$$\mathbf{v} = 2\Omega x \hat{\mathbf{y}} \qquad (6)$$

$$\frac{F}{V} = \frac{1}{b}\int_{-b/2}^{b/2} dx \frac{1}{2}\rho_s v^2 + \frac{\sigma}{b} = \frac{1}{6}\rho_s \Omega^2 b^2 + \frac{\sigma}{b} \qquad (7)$$

Minimization with respect to b gives the equilibrium distance between sheets

$$b = \left(\frac{3\sigma}{\rho_s \Omega^2}\right)^{1/3}. \qquad (8)$$



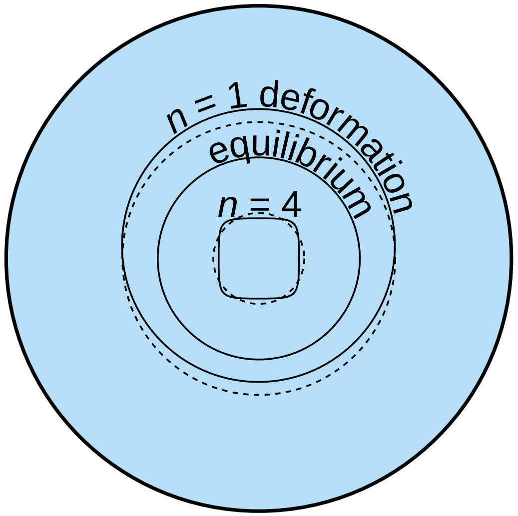
2) **One cylindrical sheet** (cylindrical container, radius R)

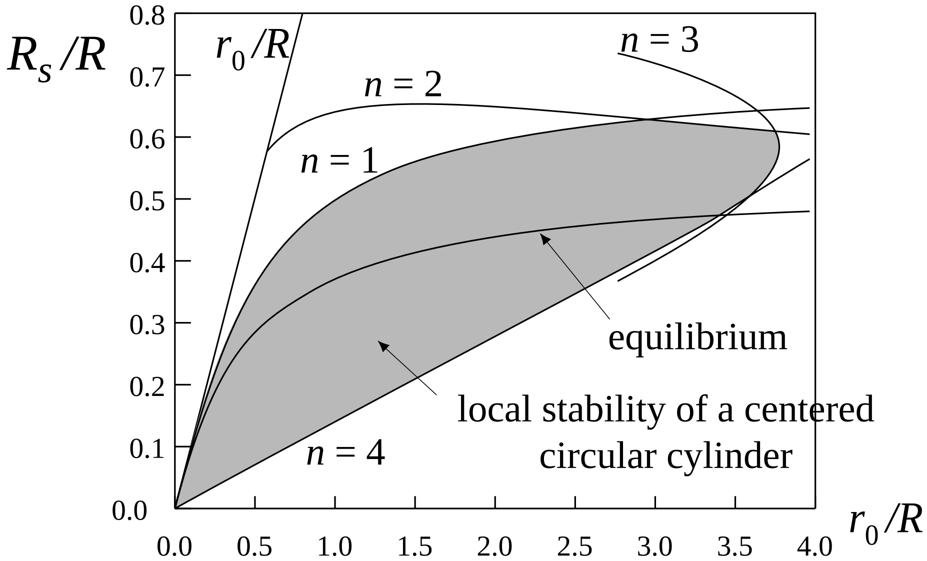
Radius of equilibrium sheet R_s for given circulation $\kappa = \oint d\mathbf{r} \cdot \mathbf{v}_s = 2\pi\Omega R_v^2$ obeys

$$\frac{\sigma}{R_s} + \rho_s \Omega^2 R_v^2 (1 - \frac{R_v^2}{2R_s^2}) = 0, \tag{9}$$

Stability against small deformations $r_s(\phi) = R_s + \mathcal{A}\cos n\phi$ is determined by

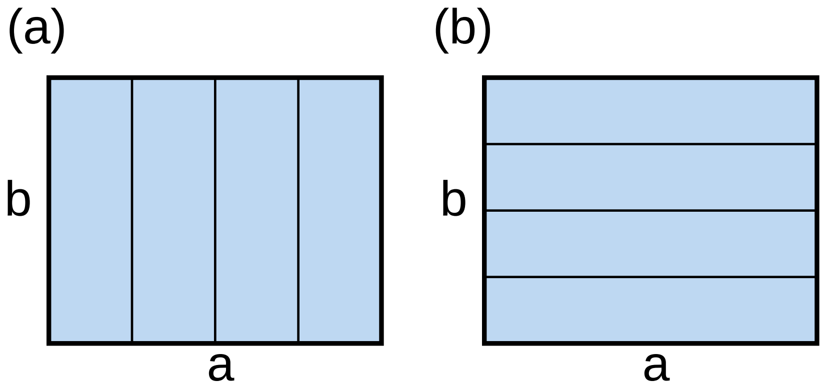
$$\frac{(n^2-1)\sigma}{R_s^2} + \rho_s \Omega^2 R_s \left[1 - n - (1 - \frac{R_v^2}{R_s^2})[1 + \frac{R_v^2}{R_s^2} + n(1 - \frac{R_v^2}{R_s^2})\frac{R^{2n} + R_s^{2n}}{R^{2n} - R_s^{2n}}]\right] = 0 \tag{10}$$





$r_0 = \rho_s \kappa^2 / (8\pi^2 \sigma)$

3) **Rectangular container**



Kinetic energy for vortexfree rectangle (area $d_1 d_2$) (Fetter 1974)

$$f_{\mathsf{k}} = \frac{F_{\mathsf{k}}}{d_1 d_2 d_3} = \rho_s \Omega^2 d_1 d_2 g(\frac{d_1}{d_2}), \quad (11)$$

$$g(x) = g(\frac{1}{x}) = \frac{x}{6} - \frac{x^2}{\pi^5} \sum_{j=1}^{\infty} \frac{1}{(j-\frac{1}{2})^5} \tanh \frac{\pi(j-\frac{1}{2})}{x}. \quad (12)$$

Applying to n sheets in configurations (a) and (b) gives

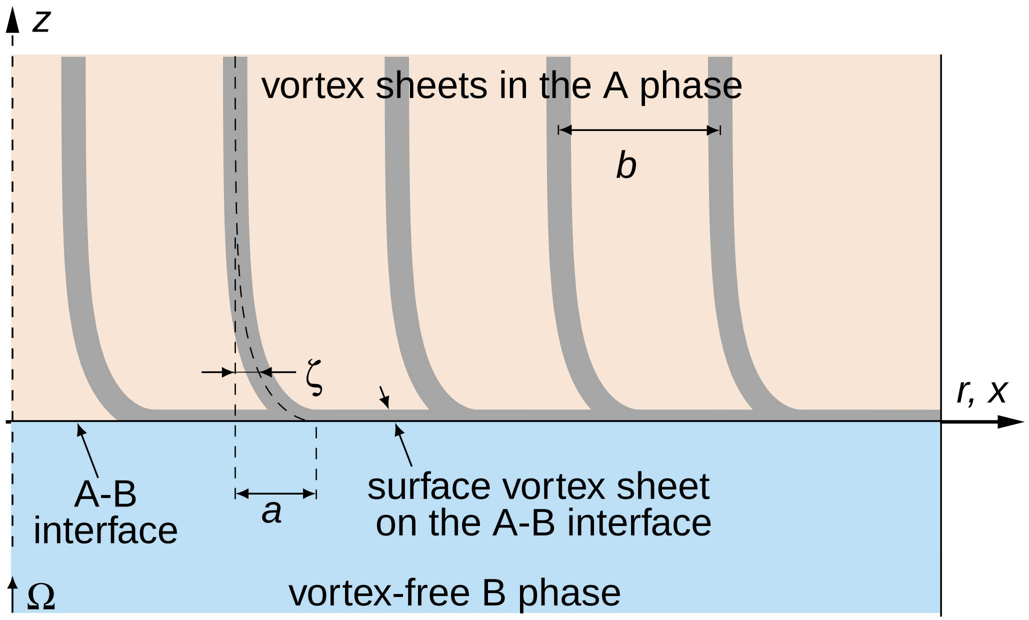
$$f_{\mathsf{a},n} = \frac{f_{\mathsf{s}}}{b}\left[\frac{nb}{a} + \frac{\alpha}{n+1}\frac{a}{b} g(\frac{a}{(n+1)b})\right], \quad (13)$$

$$f_{\mathsf{b},n} = \frac{f_{\mathsf{s}}}{b}\left[n + \frac{\alpha}{n+1}\frac{a}{b} g(\frac{b}{(n+1)a})\right], \quad (14)$$

where $\alpha = \rho_{\mathsf{s}} \Omega^2 b^3 / f_{\mathsf{s}}$ and $f_{\mathsf{s}} \approx \sigma$.

Sequence for increasing Ω ($b = 0.9a$): 0, 1a, 1b, 2a, 2b, . . . , 5a, 5b, 6b, 7b, . . .

4) **Bending of sheets at A-B interface** (Hänninen et al 2003)



$$\mathbf{v} = 2\Omega x \hat{\mathbf{y}} \tag{15}$$

$$\begin{aligned}\frac{F}{L_x L_y} &= \int_0^\infty dz \left(\frac{1}{b} \int_{-b/2+\zeta}^{b/2+\zeta} dx \frac{1}{2}\rho_s v^2 + \frac{\sigma}{b}\sqrt{1 + \left(\frac{d\zeta}{dz}\right)^2} \right) \\ &= \int_0^\infty dz \left[\frac{1}{6}\rho_s \Omega^2 (b^2 + 12\zeta^2) + \frac{\sigma}{b}\sqrt{1 + \left(\frac{d\zeta}{dz}\right)^2} \right]\end{aligned} \tag{16}$$

$$\Rightarrow \frac{z}{a} = 1 - \sqrt{2-(\zeta/a)^2} - \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2}-\sqrt{2-(\zeta/a)^2}}{(\sqrt{2}-1)\zeta/a} \tag{17}$$

where $a = b/\sqrt{6}$ (b is the equilibrium distance).

Surprisingly the Bekarevich-Khalatnikov model gives exactly the same form for vortex lines.

Summary

Analytical calculations for sheets are simpler than for vortex lines.

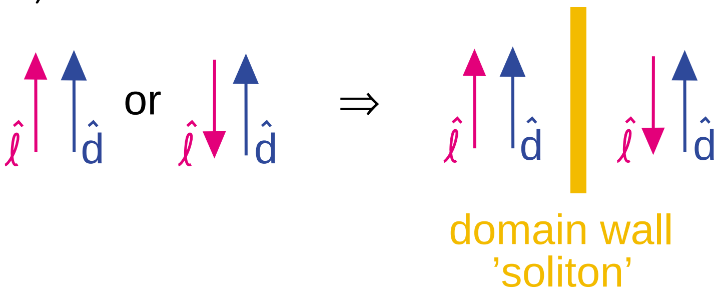
Textural bending energy?

Solitons

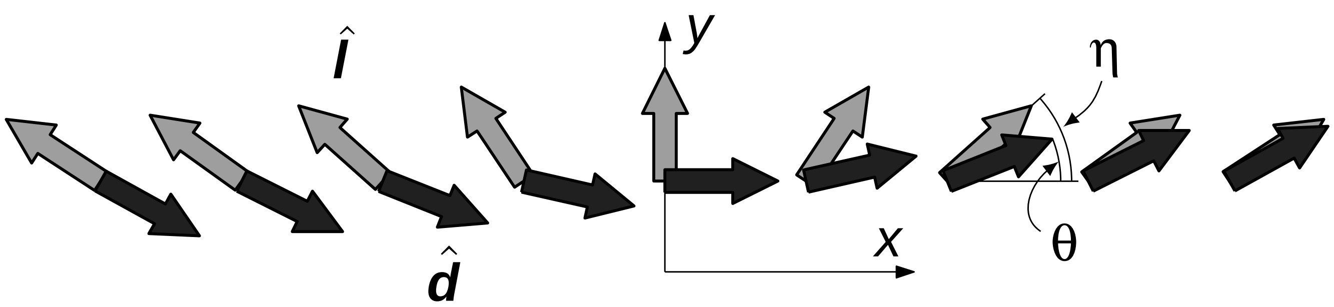
Dipole-dipole interaction (2)

$$f_D = -\tfrac{1}{2}\lambda_{\mathrm{D}}(\hat{\mathbf{d}}\cdot\hat{\mathbf{l}})^2$$

$\Rightarrow$



Structure of splay soliton ($\mathbf{B} = B\hat{\mathbf{z}}$)

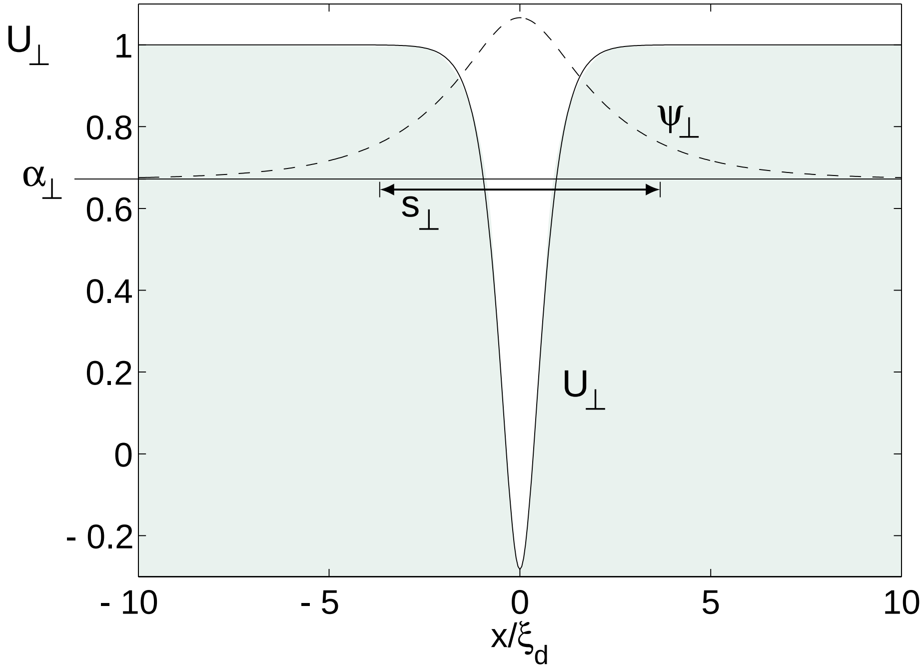


NMR resonance frequencies

$$(\mathcal{D} + U_\parallel) d_\theta = \alpha_\parallel d_\theta \tag{18}$$
$$(\mathcal{D} + U_\perp) d_z = \alpha_\perp d_z. \tag{19}$$

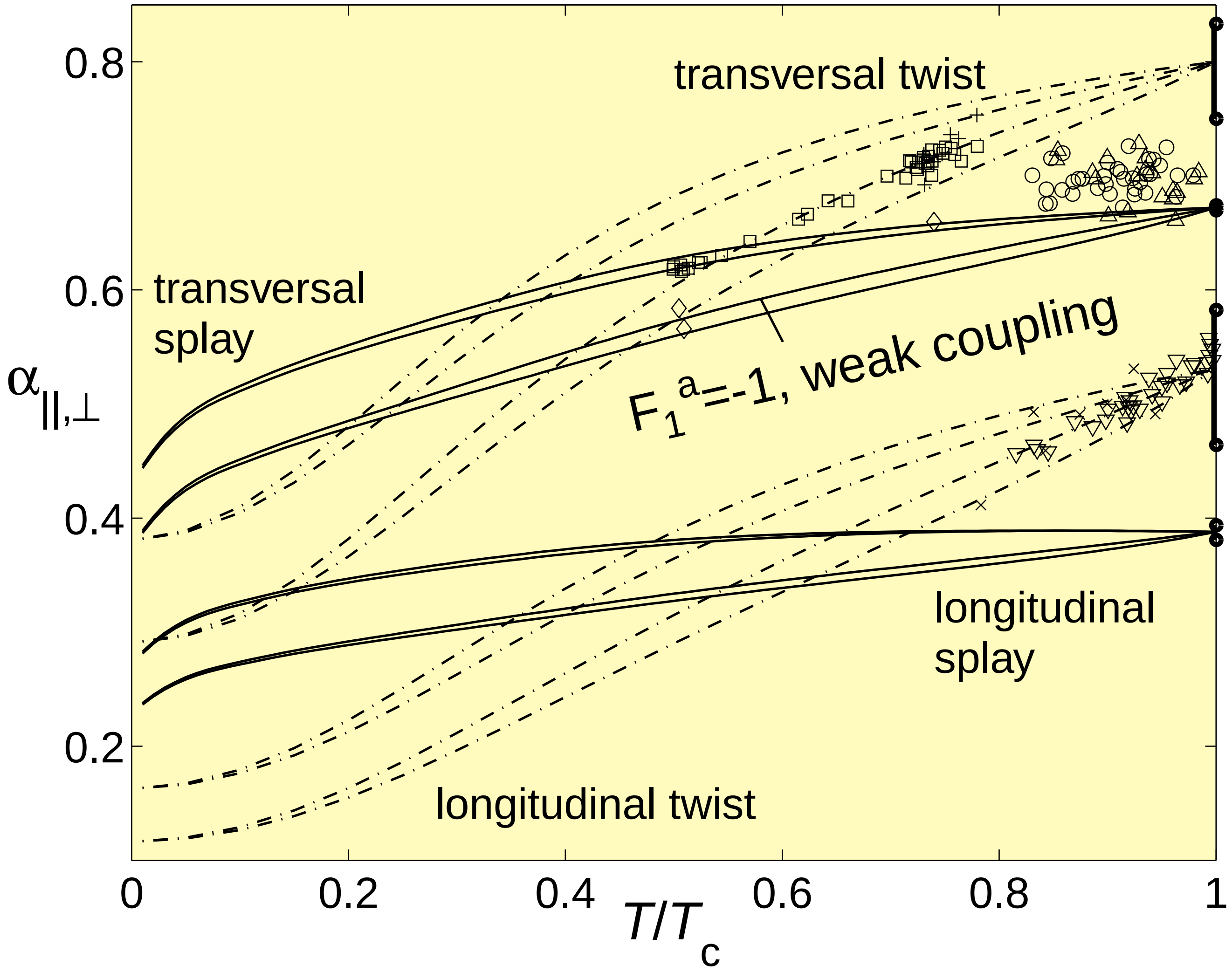
$$\mathcal{D} f = -\frac{K_6}{\lambda_d} \nabla^2 f - \frac{K_5 - K_6}{\lambda_d} \boldsymbol{\nabla} \cdot \left[\hat{\mathbf{l}} (\hat{\mathbf{l}} \cdot \boldsymbol{\nabla}) f \right]. \tag{20}$$

$$U_\parallel = 1 - l_z^2 - 2(\hat{\mathbf{l}} \times \hat{\mathbf{d}}_0)_z^2 \tag{21}$$
$$U_\perp = 1 - 2\hat{l}_z^2 - (\hat{\mathbf{l}} \times \hat{\mathbf{d}}_0)_z^2 - \frac{K_6}{\lambda_d} (\boldsymbol{\nabla}\theta)^2 - \frac{K_5 - K_6}{\lambda_d} (\hat{\mathbf{l}} \cdot \boldsymbol{\nabla}\theta)^2. \tag{22}$$



Results (no dissipation)

Effect of $F_1^a = 0$, -1 and strong coupling



Dissipation

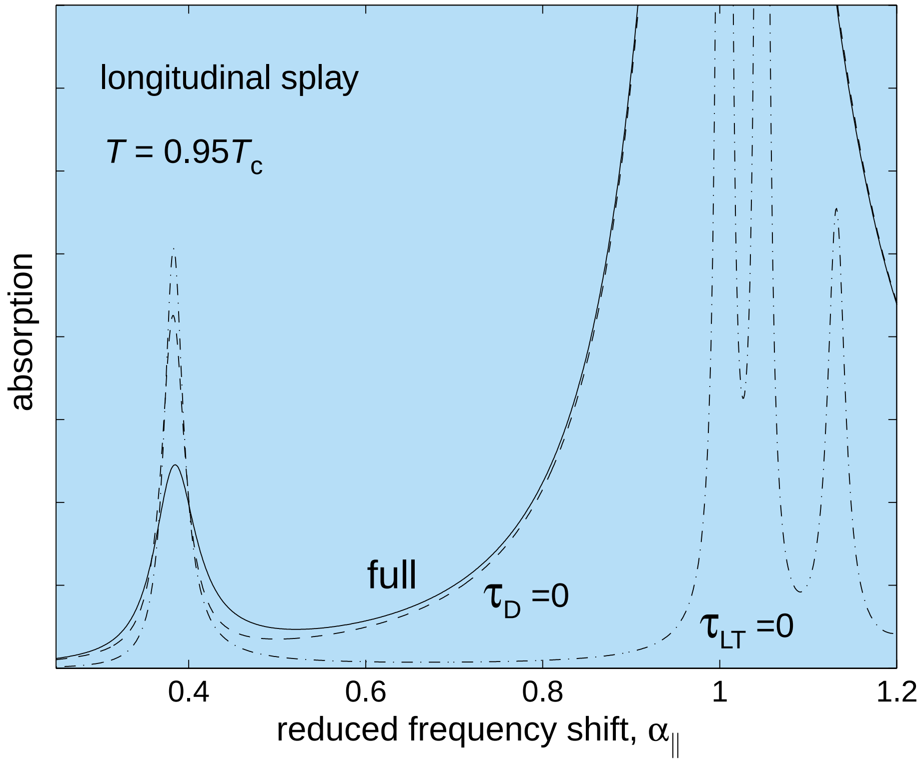
Normal-superfluid conversion (Leggett-Takagi) and spin diffusion (simple model)

$$\dot{\mathbf{S}}_q = \gamma \mathbf{S}_q \times \left(\mathbf{B} - \mu_0 \gamma \frac{F_0^a}{\chi_0} \mathbf{S}_p \right) + \frac{1}{\tau} \left[(1-\lambda) \mathbf{S}_p - \lambda \mathbf{S}_q \right] + \kappa \nabla^2 \mathbf{S}_q \tag{23}$$

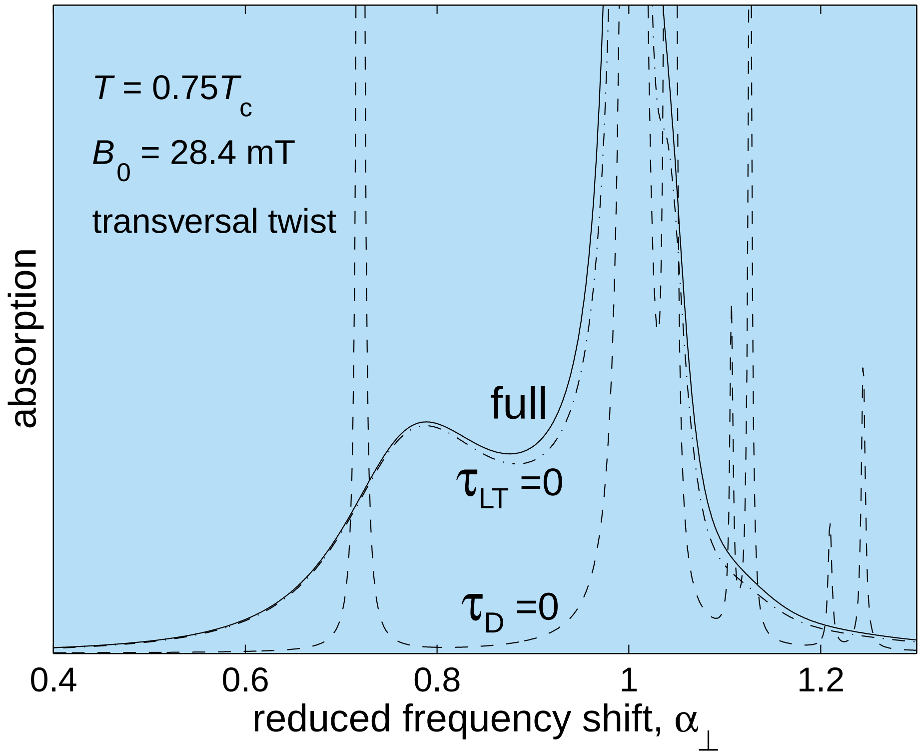
$$\dot{\mathbf{S}}_p = \gamma \mathbf{S}_p \times \left(\mathbf{B} - \mu_0 \gamma \frac{F_0^a}{\chi_0} \mathbf{S}_q \right) - \frac{1}{\tau} \left[(1-\lambda) \mathbf{S}_p - \lambda \mathbf{S}_q \right] - \hat{\mathbf{d}} \times \frac{\delta f}{\delta \hat{\mathbf{d}}} \tag{24}$$

$$\dot{\hat{\mathbf{d}}} = \gamma \hat{\mathbf{d}} \times \left[\mathbf{B} - \mu_0 \gamma \frac{F_0^a}{\chi_0} \mathbf{S}_q - \mu_0 \gamma \left(\frac{F_0^a}{\chi_0} + \frac{1}{\lambda \chi_0} \right) \mathbf{S}_p \right]. \tag{25}$$

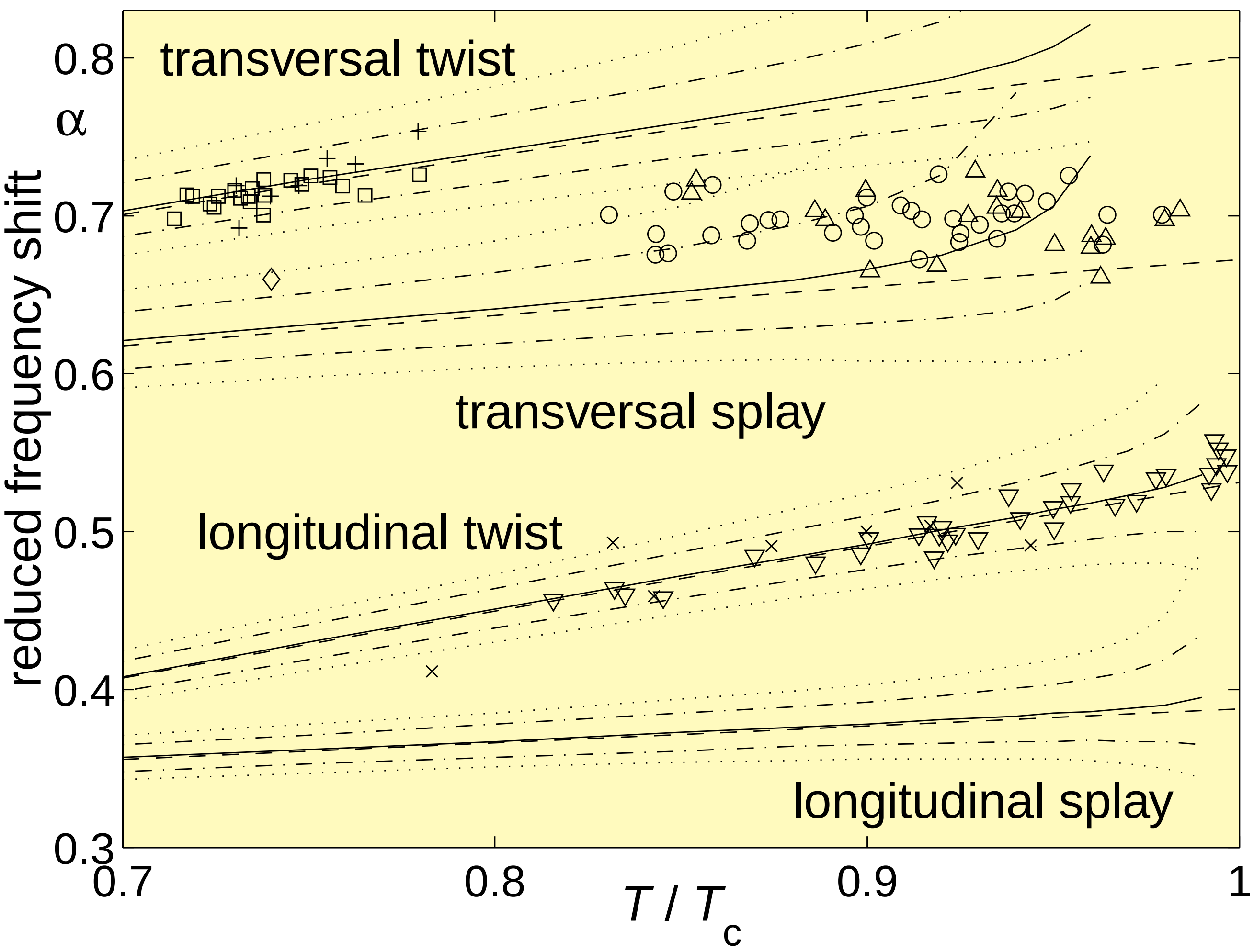
Results



longitudinal splay
T = 0.95T_c
absorption
full
τ_D =0
τ_{LT} =0
0.4
0.6
0.8
1
1.2
reduced frequency shift, $\alpha_{\parallel}$



T = 0.75T_c
B_0 = 28.4 mT
transversal twist
absorption
full
τ_{LT} =0
τ_D =0
0.4
0.6
0.8
1
1.2
reduced frequency shift, $\alpha_{\perp}$



R. Hänninen and E. T. cond-mat/0103052

Conclusion

Vortex sheet: analytic results

- maybe can be tested at high rotation speed

Solitons: including dissipation

- a problem in transverse splay resonance
- measurement of longitudinal resonance of splay soliton?

These lecture notes will be available at
http://boojum.hut.fi/research/theory/