# Ship waves

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The purpose of this note is to show that many propeties of ship waves, such as shown in Fig. 1, can be explained simply by the fact that the group velocity of the waves is only a half of the phase velocity. The physics has been understood since the 1800's. It can be found in the literature, but most likely not quite in the same form as presented here.

## I. SOUND WAVES

For a background, it is good to take a look at sound waves generated by a supersonic object. At some initial time the object is at point A in Fig. 2. It generates spherical waves that at time $\Delta t$ later form a sphere of radius $v\Delta t$, where $v$ is the sound velocity. In the same time the object moving at velocity $u$ has arrived at point B. The waves the object has generated amplify each other at line BC, if $u > v$. This forms a cone with opening angle determined by

$$\sin\theta = \frac{v}{u} \tag{1}$$

Thus the cone angle is determined by the speed of the object. This argument does not tell anything about the wave length of the waves.

## II. SHIP WAVES

The surface waves generated by a ship differ essentially from sound waves in two respects. 1) The velocity of the waves is not constant but depends on the wave length. 2) The group velocity $v_g$ and the phase velocity $v_p$ are not the same. For waves in deep water (depth $\gg$ wavelength) and sufficiently long wavelength ($\gg$ 10 cm so that surface tension is not important) have the dispersion relation $\omega = \sqrt{gk}$ and thus[1]

$$v_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_p. \tag{2}$$

Thus the group velocity is only half of the phase velocity.

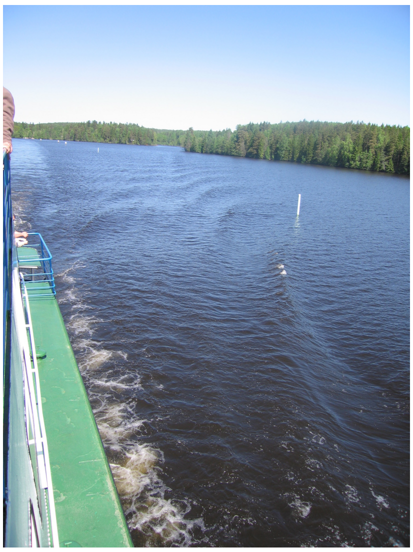

FIG. 1: Waves formed by a ship on a lake at the Saimaa canal.

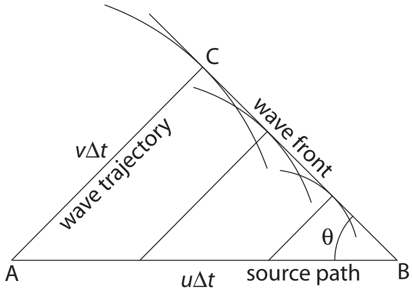


FIG. 2: Wavefront formed by a supersonic object in a fluid with constant sound velocity $v$ .

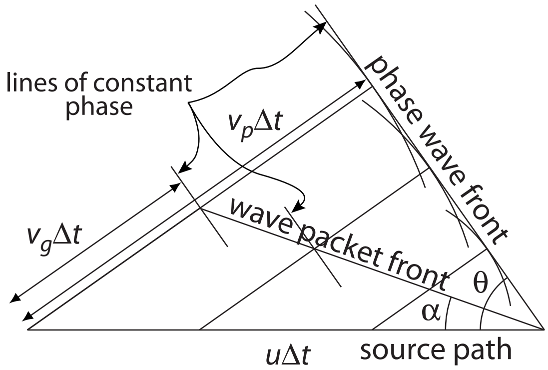


FIG. 3: Sketch of surface wave fronts generated by a ship.

Let us first consider waves of a fixed velocity (and wavelength) such that $v_p < u$. Similarly as for sound waves, the waves generated by the object form a front of constant phase at angle $\theta$, see Fig. 3. Note that $\theta$ is a function of the phase velocity

$$\sin\theta = \frac{v_p}{u} \tag{3}$$

However, no waves are visible at the phase wave front (except just at the tip of the ship) since wave packets travel with the slower group velocity. As a result, there is another front formed at angle $\alpha$ determined by the group velocity. From the figure one can deduce the condition

$$\tan\alpha = \frac{\sin\theta\cos\theta}{v_p/v_g - \sin^2\theta} \tag{4}$$

At $v_p/v_g = 2$ this fixes $\alpha$ at a given $v_p$.

Now we must consider that $v_p$ is not a constant. Thus there will be infinitely many of the front angles $\theta$ and $\alpha$ corresponding to different $v_p$. However, looking at more closely one finds that $\alpha$ has a maximum value. By the usual density of states argument near an extremum, we deduce that different values of $v_p$ near this point produce nearly the same wave packet front. Therefore this front will be much stronger than others. Quantitatively, it corresponds to $\tan\alpha = 1/\sqrt{8}$, $\cos\theta = 1/\sqrt{3}$, $\alpha = 19°$ and $\theta = 55°$. (These values are used in drawing Fig. 3.)

Thus one should see waves forming a front at lines forming an angle $\alpha$, but the constant phase direction of the these waves is given by angle $\theta$. The waves are stationary in the frame of the ship. Note that both angles are independent of the speed of the ship. The phase velocity $v_p = u/\sqrt{3}$ and thus the wave length $\lambda = 2\pi/k = 2\pi u^2/3g$ are determined by the speed of the ship. The distance between two successive wave amplitude maxima measured along the direction of a constant-phase front is $d = \lambda/\tan(\theta - \alpha) = \sqrt{2}\lambda$.

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[1] This is in most books of hydrodynamics. Those knowing finnish might read it from my lecture notes at http://cc.oulu.fi/~tf/tiedostot/pub/hydro/luennot/ 1