Cryocourse 2019

8. - 19. September, Zemplínska Šírava and Košice, Slovakia

Introduction to Superconductivity and Superfluidity

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1. Introduction

These are lecture notes for one 90 minute lecture on superconductivity and another 90 minute lecture on superfluidity. These lectures are given as part of Cryocourse 2019, 8. - 19. September, in Zemplínska Šírava and Košice, Slovakia. Largely the discussion is standard that can be found in the references below, but some parts present also new content. Especially, the presentation of the pairing state as an avoided level crossing (Sec. 2.3) and the discussion of the critical velocity (Sec. 3.7) may not be found elsewhere.

Books on superconductivity

- M. Tinkham, Introduction to Superconductivity (1975, 1996). Very widely used book. More experimental view, and therefore not ideal for this theory course.
- A.L. Fetter and J.D. Walecka, Quantum theory of many-particle systems (1971). Superconductivity studied in chapters 10 and 13. The problem is that most of the microscopic theory is treated using Green's functions, which are not used in this course.
- J.B. Ketterson and S.N. Song, Superconductivity (1999). Just another book.
- A.A. Abrikosov, Fundamentals of the Theory of Metals (1988). Very extensive book. Half of the book discusses normal state metals. Derivation of BCS theory not good.
- P.G. de Gennes, Superconductivity of Metals and Alloys (1961). Old but still useful.
- K. Fossheim and A. Sudbø, Superconductivity: Physics and Applications (2004).
- A.J. Leggett, Quantum Liquids, Bose condensation and Cooper pairing in cond.mat. physics
- Many books on solid state physics include an introduction to superconductivity, for example N. Ashcroft and D. Mermin, Solid state physics (AM).
- E. Thuneberg: Superconductivity. Lecture notes for a course of 26 lecture hours. Available at http://ltl.tkk.fi/~ethuneb/courses/scengl.pdf

Books on superfluidity

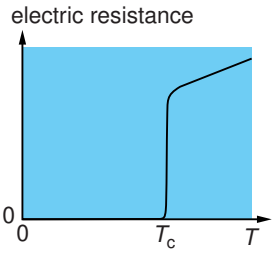
- A.J. Leggett, Quantum Liquids, Bose condensation and Cooper pairing in condensed matter physics (2006).
- T. Guenáult, Basic Superfluids (2003).
- A.L. Fetter and J.D. Walecka, Quantum theory of many-particle systems (1971). Superfluidity studied in chapter 14. The problem is that most of the microscopic theory is treated using Green's functions, which are not used in this course.
- I.M. Khalatnikov, An introduction to the theory of superfluidity.
- References to various properties of superfluid ^{3}He can be found on the web page http://ltl.tkk.fi/research/theory/heliumtheory.html

Short content of the course

- Superconductivity
  - occurrence and basic phenomenology
  - brief history
  - microscopic (BCS) theory
  - Ginzburg-Landau theory
  - Type II superconductivity
  - Josephson effect
- Superfluidity
  - occurrence
  - brief history
  - basic phenomenology
  - microscopic origin
  - hydrodynamics
  - rotating superfluid
  - phase slips, Josephson effect and critical velocity
  - special properties of superfluid ^{3}He

2. Superconductivity

By superconductivity we mean a phenomenon where the electrical resistivity of a material disappears below some temperature.



Occurrence of superconductivity

- several metallic elements : Al, Nb, Sn, (but not in magnetic metals and in noble metals: Cu, Au, Ag)
- many alloys, e.g. Nb-Ti
- some compounds: Nb_3Ge, MgB_2, Y-Ba-Cu-O etc.

The temperature below which superconductivity occurs is called *critical temperature*, T_c. The list gives some critical temperatures.

material	T_c (K)	$\mu_0 H_c(T=0)$ (mT)
Al	1.196	9.9
Hg	4.15	41
In	3.40	29.3
Pb	7.19	80.3
Nb	9.25	
Nb_3Ge	23	
MgB_2	39	
$YBa_2Cu_3O_{6+x}$	98	
$Tl_2Ca_2Ba_2Cu_3O_{10}$	125	

2.1 Basic properties

Infinte conductivity

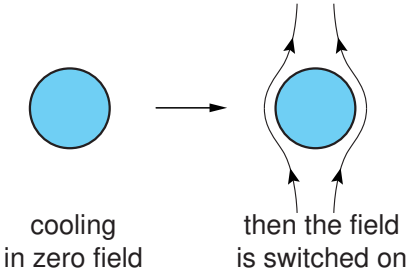
In normal state metals the electric current $\boldsymbol{j}$ is proportional to the electric field $\boldsymbol{E}$:

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{1}$$

If $\sigma \to \infty$ then $\boldsymbol{E} \to 0$. Maxwell's equation

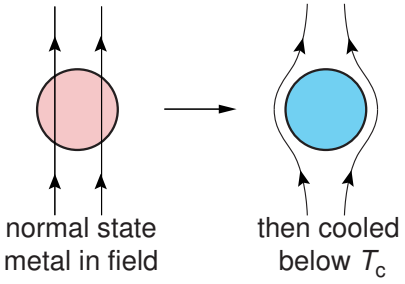
$$\boldsymbol{\nabla} \times \boldsymbol{E} \;\; = \;\; -\frac{\partial \boldsymbol{B}}{\partial t} \tag{2}$$

then gives that the magnetic field $\boldsymbol{B}$ is constant. Let us apply this to the case that a superconductor is cooled below T_c while $\boldsymbol{B} = 0$. When the field is switched on, it will not penetrate into the superconductor.



Meissner effect

A more fundamental phenomenon than infinite conductivity is seen when a normal state metal is first placed in magnetic field, and is then cooled into the superconducting state. It is observed that the magnetic field is expelled from the sample. This is called *Meissner effect*.

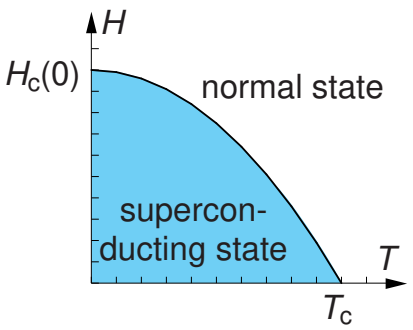


Thus the fundamental property is that the magnetic field is zero inside a superconductor. (Not only constant, as would follow from infinite conductivity.)

Critical field

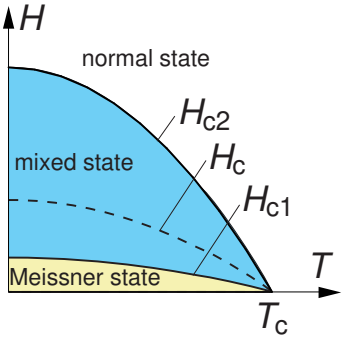
The Meissner effect is observed only if the field is not too large. Let us for simplicity consider a thin bar sample that is parallel to the field. (In this case the magnetizing field $\boldsymbol{H}$ is constant.) It is observed that a transition between superconducting state and normal state takes place in *critical field* H_c, whose dependence on temperature is approximately

$$H_c(T) = H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]. \tag{3}$$



Material showing this behavior are called *type I superconductors*.

Some superconducting materials have a mixed state between Meissner and normal states. These are called *type II superconductors*.

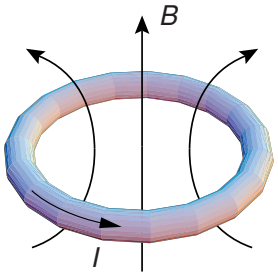


Persistent currents and flux quantization

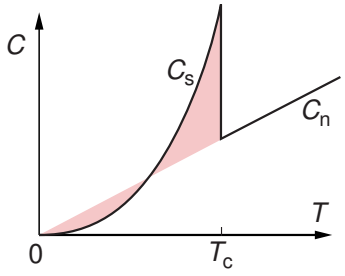
Let us place a normal state ring in perpendicular magnetic field. When it is cooled below T_c, the magnetic field is expelled form the inside of the superconductor, but a magnetic flux through the ring remains. When the external field is removed, this flux remains unchanged. Thus a persistent current I is induced in the superconducting ring that generates the magnetic field $\boldsymbol{B}$. In addition, the magnetic flux $\Phi = \int d\boldsymbol{a} \cdot \boldsymbol{B}$ through the ring is quantized: it is an integer multiple of the flux quantum

$$\Phi_0 = \frac{h}{2|e|} = 2.07 \times 10^{-15} \text{ Wb}. \tag{4}$$

Here h is Planck's constant and e the charge of an electron. [Because $e < 0$, the absolute value is taken in (4).]



Specific heat



The transition between normal and superconducting states is also seen in thermodynamic properties. The specific heat has discontinuity but no latent heat (in zero field). This kind of change of state is called second order phase transition. In the normal state the specific heat is linear at low temperatures. In superconducting state the specific heat is exponential when $T \to 0$:

$$C_s \propto \exp\left(-\frac{\Delta_0}{k_B T}\right). \tag{5}$$

This can be understood so that there is an energy gap Δ_0 between the ground state and the lowest excited states. This gap is somewhat less than $2k_B T_c$ in most superconductors.

Isotope effect

Different isotopes of the same element have difference in T_c, which depends on the ion mass M approximately as

$$T_c \propto M^{-1/2}. \tag{6}$$

This can be used to deduce that the motion of the ions is important for superconductivity.

2.2 History

- 1911 H. Kamerlingh Onnes finds superconductivity in mercury
- 1933 Meissner effect
- 1935 London theory
- 1950 Ginzburg-Landau theory
- 1957 Bardeen-Cooper-Schrieffer theory
- 1957 theory of type II superconductivity
- 1962 Josephson effect
- 1986 Bednorz and Müller find "high temperature superconductors"
- 1999 Superconducting qubit

2.3 Microscopic theory

The superconducting state results from interactions between a large number of particles. The standard way to represent this *many-body state* is to make so-called *anomalous Hartree-Fock approximation* to the many-body Hamiltonian and then diagonalize it with a *Bogoliubov transformation*. Alternatively, a more general justification could be achieved using Green's functions and Fermi-liquid theory (so-called quasiclassical theory). Instead of reproducing these, we present here a simple picture of the superconducting state, which (hopefully) allows physical insight with less formalism.

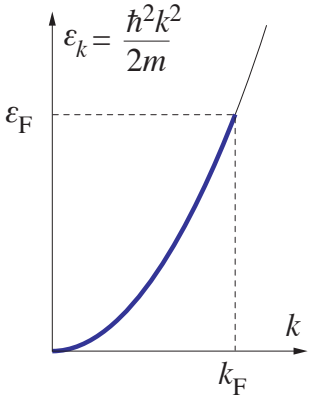
Normal state metal

The properties of metals are more thoroughly studied in courses on condensed matter physics. Here we only consider a very simplified model of a metal, which still is sufficient as a starting point for understanding basic properties of superconductivity.

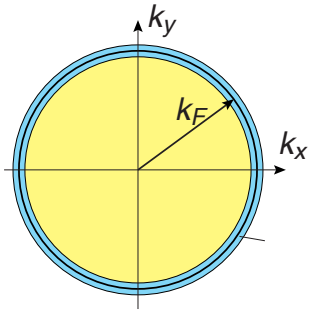
We assume that there are *conduction electrons*, which can move freely like ideal gas through the metal. We assume that the rest of the electrons called *valence electrons* are bound to the atomic nuclei so that they do not contribute to the electric conductivity. The kinetic energy $\varepsilon_{\boldsymbol{k}}$ of a conduction electron depends on the wave vector $\boldsymbol{k}$. In the simplest case this dependence has the same form as for a free particle,

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m}. \tag{7}$$

Here m is the effective mass in the conduction band. The momentum of an electron in this simple model is $\boldsymbol{p} = \hbar\boldsymbol{k}$. At $T = 0$ the levels with k smaller than the Fermi wave vector k_F are filled. In energy the levels are filled up to the Fermi energy $\varepsilon_F = \hbar^2 k_F^2/2m$.



The Fermi energy ε_F in a typical metal is on the order of a few eV. In superconductivity the typical temperature is $T \sim T_c$ a few kelvin. The corresponding thermal energy is $k_B T_c \approx 10^{-4}$ eV. Thus there are two very different energy scales, $k_B T_c/\varepsilon_F \sim 10^{-4}$. We will discuss below that superconductivity concerns the electrons only in this tiny shell around the Fermi surface.



Since we are only interested in the neighborhood of the Fermi surface, we can approximate the electron energy (7) by

$$\varepsilon_k = \varepsilon_F + \hbar v_F (k - k_F), \tag{8}$$

where $v_F = \hbar k_F/m$ is the Fermi velocity.

Ideas about superconductivity

Materials can be studied on two very different scales:

- atomic scale. Quantum mechanics is essential. There is no friction.
- macroscopic scale. Material is described by laws of classical physics (e.g. theory of elasticity). Friction is essential.

The phenomena on atomic scale are not usually visible on the macroscopic scale. The reason for this is that macroscopic bodies consist of enormously large number of particles ($\sim 10^{23}$). The different particles are in general in different quantum levels, and only the average of them is visible. Friction is needed because it is not possible to take into account the motion of individual particles.

Idea: Superconductivity is an exception to the rule above: it is a quantum phenomenon that is still preserved on a macroscopic scale.

How is this possible? As a simple example, let us consider ideal Bose gas. At zero temperature its distribution function reduces to the form

$$f(\varepsilon) = \begin{cases} N & \text{lowest level } (\varepsilon_i = \varepsilon_0) \\ 0 & \text{other levels } (\varepsilon_i > \varepsilon_0). \end{cases} \tag{9}$$

Also at finite temperatures below so-called Bose condensation temperature, the occupation of the lowest level $N_0/N > 0$, whereas for all other levels $f_i/N \to 0$ when $N, V \to \infty$. It is expressed by saying that the wave function of the lowest level becomes *macroscopic wave function* because a macroscopic number of particles is in the same level.

It can be said that *the superfluid phases of alkali atom gases and* ^{4}*He liquid are based on Bose condensation.* These superfluids will be discussed more in the second part of these lectures.

Electrons are fermions. One can put only one fermion into a single level. Thus the discussion above as such cannot explain the superconductivity of metals.

Let us study the wave functions of spin-$\frac{1}{2}$ fermions. The wave function $\phi(\boldsymbol{r}\sigma)$ of a single particle, which we call a *level*, depends on the location $\boldsymbol{r}$ and on the spin index $\sigma = \pm\frac{1}{2}$. In general form the Pauli exclusion principle says that the wave function has to be antisymmetric in any exchange of two electron coordinates. For a two-electron wave function $\Psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)$ this requirement is

$$\Psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = -\Psi(\boldsymbol{r}_2\sigma_2, \boldsymbol{r}_1\sigma_1). \tag{10}$$

Starting from an arbitrary function $\psi_0(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2)$, one can always by antisymmetrizing construct a function that

satisfies this condition:

$$\Psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = \psi_0(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) - \psi_0(\boldsymbol{r}_2\sigma_2, \boldsymbol{r}_1\sigma_1). \quad (11)$$

Thus the pair state formed from two levels ϕ_1 and ϕ_2 has the wave function

$$\Psi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_1) = \phi_1(\boldsymbol{r}_1\sigma_1)\phi_2(\boldsymbol{r}_2\sigma_2) - \phi_1(\boldsymbol{r}_2\sigma_2)\phi_2(\boldsymbol{r}_1\sigma_1), \quad (12)$$

so called Slater determinant. We see immediately that this vanishes, $\Psi \equiv 0$, if the levels are the same: $\phi_1 = \phi_2$. Thus two fermions cannot be placed into the same level.

In case of many particles, the antisymmetry is required in any pairwise exchange of the coordinates of two particles

$$\begin{aligned}&\Psi(\ldots, \boldsymbol{r}_i\sigma_i, \boldsymbol{r}_{i+1}\sigma_{i+1}, \ldots, \boldsymbol{r}_k\sigma_k, \ldots)\\ &= -\Psi(\ldots, \boldsymbol{r}_k\sigma_k, \boldsymbol{r}_{i+1}\sigma_{i+1}, \ldots, \boldsymbol{r}_i\sigma_i, \ldots) \quad (13)\end{aligned}$$

Idea: fermions form *pairs*.

$$\begin{aligned}\psi_0(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2, \boldsymbol{r}_3\sigma_3, \ldots) = \phi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) \times\\ \phi(\boldsymbol{r}_3\sigma_3, \boldsymbol{r}_4\sigma_4)\phi(\boldsymbol{r}_5\sigma_5, \boldsymbol{r}_6\sigma_6)\ldots, \quad (14)\end{aligned}$$

All *pair states are the same*!. This function does *not* vanish in antisymmetrization as long as the pair function is antisymmetric,

$$\phi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = -\phi(\boldsymbol{r}_2\sigma_2, \boldsymbol{r}_1\sigma_1). \quad (15)$$

In exchanging pairs one gets the factor $(-1)^2 = 1$, similarly to bosons.

Pair state

Consider a pair of particles. Assuming the pair is at rest, the wave function can depend only on the relative location $\boldsymbol{r}_1 - \boldsymbol{r}_2$, not on the center of mass coordinate $\boldsymbol{r} = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)$. We also assume the spin state is a singlet, $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$, which is believed to be the case in most superconductors. Thus

$$\phi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = \psi(\boldsymbol{r}_1 - \boldsymbol{r}_2)\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow). \quad (16)$$

If we now look the orbital part ψ in k space, we see that it is a superposition of states, where, if one particle is in state $\boldsymbol{k}$, the other is in the opposite state $-\boldsymbol{k}$:

$$\psi(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \sum_{\boldsymbol{k}} c_{\boldsymbol{k}} e^{\boldsymbol{k}\cdot\boldsymbol{r}_1} e^{-\boldsymbol{k}\cdot\boldsymbol{r}_2} \quad (17)$$

This is understandable because the pair is assumed to be at rest, and therefore the total momentum should vanish.

A special case of the pair state (17) is that $c_{\boldsymbol{k}}$ vanishes for all wave vectors except one, say $\boldsymbol{k} = \boldsymbol{k}_0$. Such a state, which describes two fermions with wave vectors $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$, is an energy eigenstate if there is no interaction between the particles. In the presence of interaction, the two particles will collide and scatter to some other wave vectors $\boldsymbol{k}_1$ and $-\boldsymbol{k}_1$. These then scatter to wave vectors $\boldsymbol{k}_2$ and $-\boldsymbol{k}_2$, and so on. Thus the energy eigenstate in the presence of interactions is a coherent superposition of opposite wave-vector states as in (17) with many non-vanishing $c_{\boldsymbol{k}}$'s.

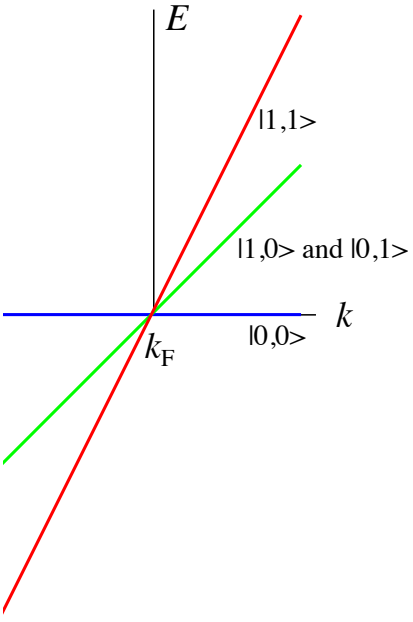
Let us look the pair state from the point of a fixed single $\boldsymbol{k} = \boldsymbol{k}_0$. The states included in (17) are $|1,1\rangle$ and $|0,0\rangle$. Here the two numbers mean the occupation of levels with wave vectors $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$, respectively. The presence of $|0,0\rangle$ state means that the particles are scattered into some other wave-vector states leaving the levels $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$ empty. In order to have complete basis states for wave vectors $\boldsymbol{k}_0$ and $-\boldsymbol{k}_0$, we also need states $|1,0\rangle$ and $|0,1\rangle$. These mean states where only the former or the latter level is occupied, repectively.

Let us study the energies of the states $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$ and $|1,1\rangle$. These contain different number of particles. The standard method to deal with such a case is to connect the system with a particle reservoir. An ideal particle reservoir can store particles at a constant energy. The energy μ is called chemical potential. In the free electron model (7) μ is equal to ε_F at $T = 0$.

We can now calculate the energies four states in the absence of interactions. It is convenient to choose μ as zero of energy, and to express other energies relative to that. We get

$$\begin{aligned}|0,0\rangle &: E = 0\\ |1,0\rangle &: E = \xi_k\\ |1,0\rangle &: E = \xi_k\\ |1,1\rangle &: E = 2\xi_k \quad (18)\end{aligned}$$

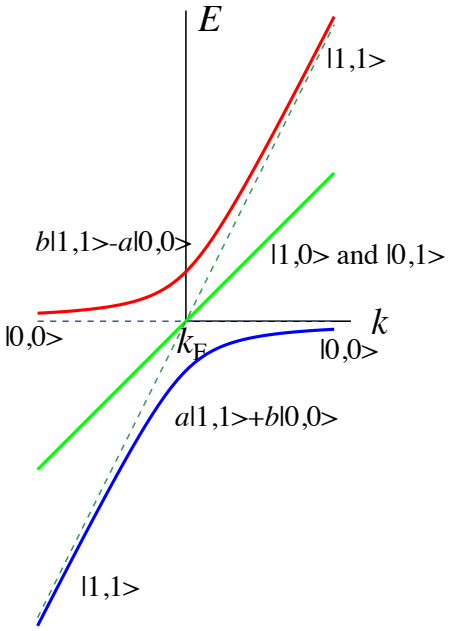
where $\xi_k = \hbar v_F(k - k_F)$. The energies are illustrated in the figure as function of k.



It is obvious that the system ground state consist of

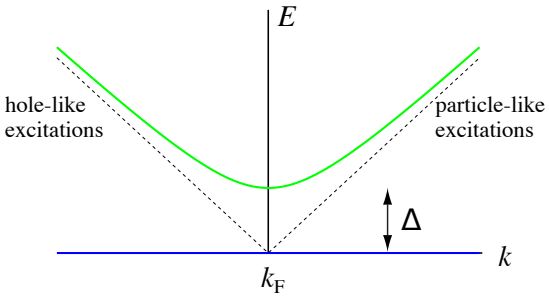
states $|1,1\rangle$ at $k < k_F$ and of states $|0,0\rangle$ at $k > k_F$. This is the Fermi sphere of noninteracting particles.

Let us then consider what happens when the attractive interaction is switched on. As discussed above the formation of pair state leads to coupling between different momentum states, and this leads to coupling of states $|1,1\rangle$ and $|0,0\rangle$. It is a standard problem in quantum mechanics that such a coupling leads strongest mixing of the states when the unperturbed energies are close to each other. In the present case this happens just at the Fermi surface ($k = k_F$), where the states $|1,1\rangle$ and $|0,0\rangle$ are degenerate in the absence of interaction. This leads to the *avoided crossing*, which is a common phenomenon in quantum mechanics (or in any wave theory).



The ground state (blue in the figure) is now a superposition of states $|1,1\rangle$ and $|0,0\rangle$. However, deep inside the Fermi sphere ($k \ll k_F$) it is to a good approximation $|1,1\rangle$, and far outside ($k \gg k_F$) it is to a good approximation $|0,0\rangle$, as in the normal state. Thus pairing only changes the electron state near the Fermi surface, $k \approx k_F$. But this is just the region that is essential for transport properties at low temperatures.

In addition to the superconducting ground state, the picture above gives also the elementary excited states. Namely, the lowest energy exited state is to break the pair a some $\boldsymbol{k}$ and replace it by state $|1,0\rangle$ or $|0,1\rangle$. The excitation energy is just the energy difference between the green and blue lines in the figure above. The excited pair states (red in the figure) correspond to double excitation at the same $\boldsymbol{k}$. The excitation energy of the elementary excitations is replotted in the figure below.

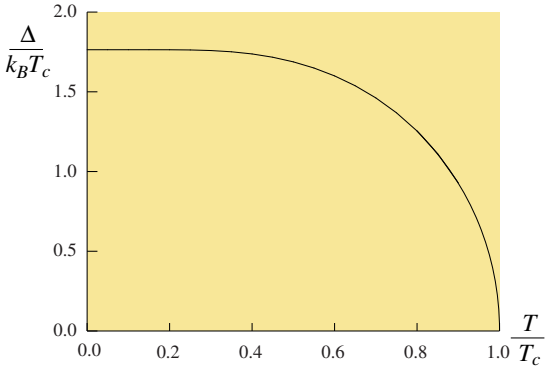


We see that there is minimum of the excitation energy. It is called the energy gap and denoted by Δ. This is in contrast to excitations in the normal state which have no gap (shown by dashed line).

The formation of the paired state requires attractive interaction. In most superconductors, the attractive interaction comes from coupling of electrons to lattice vibrations (phonons). In order to a material to be superconducting, this attractive interaction must be stronger that the direct Coulomb repulsion between the electrons. Whether this is the case or not, depends on the detailed structure of the material, which is difficult to predict theoretically.

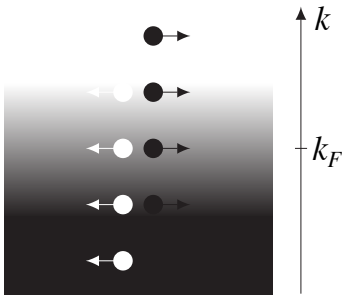
Also repulsive interaction causes scattering between the wave vector states. This, however, does not lead to formation of a paired state. Instead, it turns the ideal gas into a *Fermi liquid*, where, for example, the effective mass is modified.

The formation of excitations limits the number of the wave vectors $\boldsymbol{k}$ that contribute to the pair state (17). With increasing number of excitations the pair state is suppressed, leading to reduced Δ. With increasing temperature, the energy gap vanishes at T_c, and the system transits to the normal state. The temperature dependence is shown in the figure, calculated in the limit of weak pairing interaction (weak coupling approximation).



The excitation spectrum with the energy gap and its temperature dependence gives explanation of the specific heat, as mentioned in the introduction. The energy gap also explains the infinite conductivity. (We return to this topic in section 3.7.)

Let us look at the excitation in more detail. For wave vectors far above the Fermi surface, $k \gg k_F$, the elementary excitation corresponds to one more particle to the ground state ($|1,0\rangle$ or $|0,1\rangle$ compared to $|0,0\rangle$). For wave vectors deep inside the Fermi surface, $k \ll k_F$, it is the opposite: the excited state has one particle less than the ground state ($|1,0\rangle$ or $|0,1\rangle$ compared to $|1.1\rangle$). For this reason, the excitations at $k > k_F$ are called *particle like*, and the excitations at $k < k_F$ are called *hole like*. The figure illustrates the behavior of excitation $|1,0\rangle$ as a function of k.



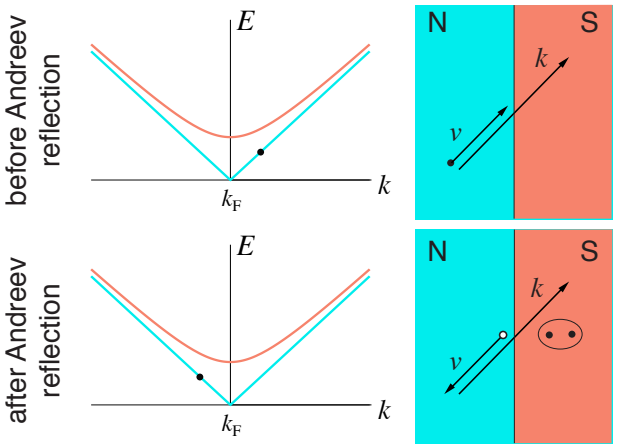
The propagation velocity of the excitations can be computed by the group-velocity formula

$$v = \frac{1}{\hbar}\frac{dE}{dk}, \tag{19}$$

(remembering that the angular velocity $\omega = E/\hbar$ in quantum mechanics). We see that the velocity is in the direction of the momentum for particle-type excitations and in the opposite direction for a hole-type excitation This holds for both $|1,0\rangle$ and $|0,1\rangle$ excitations. The only difference is that the momentum of $|1,0\rangle$ excitation is $\hbar\boldsymbol{k}$ and the momentum of $|0,1\rangle$ excitation is the opposite, $-\hbar\boldsymbol{k}$. Note that the momentum depends only weakly on the magnitude k, because a particle moving in one direction and a hole moving in the opposite direction have nearly the same momentum at $k \approx k_F$.

Andreev Reflection

An interesting process taking place in inhomogeneous superconductors is *Andreev reflection*. Consider the interface between a normal state (N) and a superconducting state (S) of a metal. Consider a particle type excitation on the normal side approaching the interface. Assume that the excitation energy is smaller than the energy gap on the superconducting side, $\xi_k < \Delta$. In this case the excitation cannot continue to the superconducting side since there are no states at the same energy. What can happen is that the excitation forms a Cooper pair in the superconductor. But a Cooper pair needs two electrons. This means that a hole-type excitation is created at the interface and its velocity $\boldsymbol{v}$ is opposite to that of the initial particle excitation. The extra Cooper pair is depicted by a circled pair of dots in the figure.



Energy and momentum are conserved in the process. The momentum change of the excitation is small, $\Delta p \ll p_F$, in contrast to usual reflection (from a potential barrier or impurities), where the momentum change is on the order of p_F. Note that the reflection direction differs from specular (=mirror like) reflection, where only the normal component of the velocity changes sign. To emphasize this, Andreev reflection is called retroreflection. Andreev reflection appears predominantly when the contact between the metals is good. The opposite case of a weakly transmitting contact is discussed later in Sec. 2.6.

2.4 Inhomogenous states

In the previous section we assumed the pairs to be at rest. In many situations we also need to consider more general cases, where the pair wave function depends on the location of the pair. That is, instead of (16), we have

$$\phi(\boldsymbol{r}_1\sigma_1, \boldsymbol{r}_2\sigma_2) = \Psi\left(\frac{\boldsymbol{r}_1+\boldsymbol{r}_2}{2}\right)\psi(\boldsymbol{r}_1-\boldsymbol{r}_2)\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow). \tag{20}$$

The new quantity here is $\Psi(\boldsymbol{r})$, which describes the center of mass of the pair. Because its macroscopic character, it is often called *the order parameter*. The simplest approach in the case of nontrivial $\Psi(\boldsymbol{r})$ is to use phenomenological theories, such as the London theory, or the Ginzburg-Landau (GL) theory.

The GL theory can be derived as a special case of the more general *Fermi-liquid theory of superconductivity*. It is valid at temperatures close to the transition temperature, $T \approx T_c$. In this region Ψ is assumed to be small. Thus the free energy density $f = F/V$ can be written as Taylor series in Ψ and Ψ^* (the complex conjugate),

$$f = f_0 + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \ldots. \tag{21}$$

The terms appearing here are restricted by the fact that F_s has to be real valued for arbitrary complex valued Ψ. Therefore, the term $c\Psi$ cannot appear. Instead, Ψ has to appear in product with Ψ^*: $\Psi^*\Psi = |\Psi|^2$. Also the term $c\,\mathrm{Re}\,\Psi$ is not accepted. The reason is that we require F_s

to remain unchanged in the transformation $\Psi \to e^{i\phi}\Psi$, where ϕ is a real-valued constant.

The expansion (21) is incomplete because nothing in it prevents a spatial dependence $\Psi(\boldsymbol{r})$. Such a spatial dependence can be limited by adding a term $|\boldsymbol{\nabla}\Psi|^2$ that increases the energy of inhomogeneous states. However, also this is unsatisfactory in the case of a nonzero magnetic field. The magnetic field $\boldsymbol{B}$ can be described with a vector potential $\boldsymbol{A}$:

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}. \tag{22}$$

The real momentum $m\boldsymbol{v} = \boldsymbol{p} - q\boldsymbol{A}$. Here $\boldsymbol{p}$ is a canonical momentum, which in quantum mechanics is replaced by the operator $\frac{\hbar}{i}\boldsymbol{\nabla}$. Analogously to this, GL chose the additional energy term to be

$$\gamma \left| \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\boldsymbol{A} \right) \Psi \right|^2, \tag{23}$$

where q turns out to be the charge of an electron pair, $q = 2e$. Let us take into account also the energy density of the magnetic field in the sample

$$\frac{1}{2\mu_0} B^2. \tag{24}$$

In this way we obtain the total energy in Ginzburg-Landau theory as

$$\begin{aligned} F &= F_0 + \int d^3r\, f, \\ f &= \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \gamma \left| \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\boldsymbol{A} \right) \Psi \right|^2 \\ &\quad + \frac{1}{2\mu_0} B^2. \end{aligned} \tag{25}$$

Often one wants to study a system in a given external magnetic field. Then, instead of F, on should minimize G, in this case

$$G = F - \int d^3r\, \boldsymbol{H} \cdot \boldsymbol{B}. \tag{26}$$

Let us still write the G in GL theory in its full form

$$\begin{aligned} G &= F_0 + \int d^3r\, g, \\ g &= \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4 + \gamma \left| \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\boldsymbol{A} \right) \Psi \right|^2 \\ &\quad + \frac{1}{2\mu_0} B^2 - \boldsymbol{B} \cdot \boldsymbol{H}. \end{aligned} \tag{27}$$

GL differential equations

In equilibrium the free energy must be minimized. In a given external field one must thus minimize G (27) both with respect to Ψ and to $\boldsymbol{A}$. {In minimizing with respect to Ψ the independent variables [e.g. $(\mathrm{Re}\,\Psi, \mathrm{Im}\,\Psi)$ or $(|\Psi|, \arg\Psi)$] can be chosen arbitrarily. The shortest calculation follows by treating Ψ and Ψ^* as independent variables.} Let us leave the minimization as an exercise. As a result we obtain the GL differential equations

$$\gamma \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\boldsymbol{A} \right)^2 \Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0, \tag{28}$$

$$\begin{aligned} &\frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{q\hbar\gamma}{i} \left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^* \right) \\ &-2q^2\gamma|\Psi|^2\boldsymbol{A}. \end{aligned} \tag{29}$$

The surface terms arising from integration by parts must also vanish. From this we get the boundary conditions at the surface of a superconductor

$$\hat{\boldsymbol{n}} \cdot \left(\frac{\hbar}{i}\boldsymbol{\nabla} - q\boldsymbol{A} \right) \Psi = 0, \tag{30}$$

$$\hat{\boldsymbol{n}} \times (\boldsymbol{B} - \mu_0\boldsymbol{H}) = 0. \tag{31}$$

[It is noted in passing that the transformation (26) is essential only for the surface terms.]

It is noted that based on the Maxwell equation

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \epsilon_0\mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\boldsymbol{j} \tag{32}$$

we identify the quantity appearing in (29) as an electric current density

$$\boldsymbol{j} = \frac{q\hbar\gamma}{i} \left(\Psi^*\boldsymbol{\nabla}\Psi - \Psi\boldsymbol{\nabla}\Psi^* \right) - 2q^2\gamma|\Psi|^2\boldsymbol{A}. \tag{33}$$

Exercise: show that the equations guarantee current conservation

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \quad \hat{\boldsymbol{n}} \cdot \boldsymbol{j} = 0. \tag{34}$$

Special cases

The GL equations (28) and (29) constitute a coupled set of differential equations, whose solution gives $\Psi(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$. In the general case this is very complicated. Let us start by considering simple special cases.

1) Homogeneous superconductor, $\boldsymbol{H} = \boldsymbol{A} = 0$. Equation (28) gives as possible solutions

$$\Psi = 0, \tag{35}$$

$$|\Psi|^2 = -\frac{\alpha}{\beta}. \tag{36}$$

The former solution describes normal state. The latter, superconducting state, is possible only if $\alpha/\beta < 0$. In order for F (25) to be sensible (minimum energy must be achieved with a finite Ψ) we must always have $\beta > 0$. The condition for the latter state is therefore $\alpha < 0$. The energies corresponding to the states (35) and (36) are found by inserting into the functional (25):

$$F = F_0, \tag{37}$$

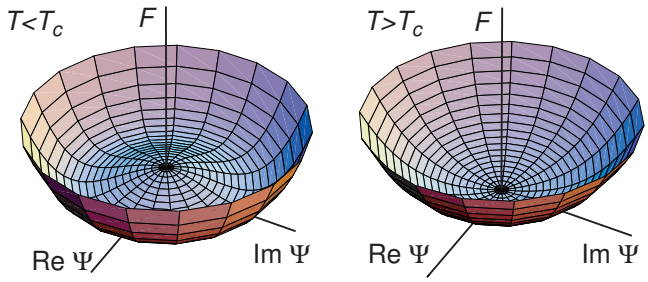
$$F = F_0 - V\frac{\alpha^2}{2\beta}. \tag{38}$$

We thus see that if $\alpha > 0$, only the normal state is possible, while in the case $\alpha < 0$ the superconducting state has the lowest energy. The transition temperature T_c thus corresponds to the point where $\alpha = 0$. In GL theory it is assumed that the temperature dependence of the coefficient α is linear

$$\alpha(T) = \alpha' \left(\frac{T}{T_c} - 1 \right), \tag{39}$$

and the other coefficients (β, γ, q) are temperature-independent.

The dependence of the free energy on the order parameter can be illustrated with the following pictures.



It is observed that in the normal state $\Psi = 0$ is completely determined, but in the superconducting state only the absolute value $|\Psi|$ of the order parameter is fixed while the phase $\arg \Psi$ is arbitrary.

2) Changing $|\Psi|$, $\boldsymbol{A} = 0$. From the boundary condition (30) it follows that $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla}\Psi = 0$. Thus a position-independent $|\Psi|$ (36) is a valid solution everywhere in the superconductor, also close to boundaries. Despite this we consider a case where Ψ deviates from its equilibrium value. From equation (28) we find

$$\hbar^2 \gamma \nabla^2 \Psi - \alpha \Psi - \beta |\Psi|^2 \Psi = 0. \tag{40}$$

Assuming Ψ to be real and writing $\Psi = \sqrt{|\alpha|/\beta} f$ we put this in the form

$$\xi_{GL}^2 \nabla^2 f + f - f^3 = 0, \tag{41}$$

where we have defined the *GL coherence length*

$$\xi_{GL} = \sqrt{\frac{\hbar^2 \gamma}{|\alpha|}}. \tag{42}$$

We see that ξ_{GL} determines that length scale on which Ψ can vary essentially. As an example we give the solution of equation (41) in the case of a one-dimensional dependence:

$$f(x) = \tanh \frac{x}{\sqrt{2}\xi_{GL}}. \tag{43}$$

3) Let us investigate the case

$$\Psi(\boldsymbol{r}) = e^{i\phi(\boldsymbol{r})} |\Psi| \tag{44}$$

where $|\Psi|^2 \approx |\alpha|/\beta$ is constant. By inserting into the expression of current (33) we find

$$\boldsymbol{j} = 2q\gamma |\Psi|^2 \left(\hbar \boldsymbol{\nabla}\phi - q\boldsymbol{A} \right). \tag{45}$$

By taking the rotor of this we have the *London equation*

$$\boldsymbol{\nabla} \times \boldsymbol{j} = -2q^2 \gamma |\Psi|^2 \boldsymbol{B}. \tag{46}$$

By using the Maxwell equations (32) and $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ this yields

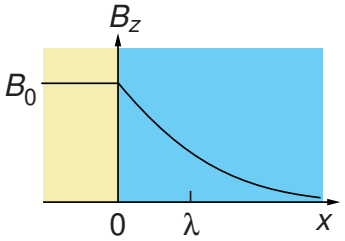
$$\begin{aligned} \boldsymbol{B} &= -\frac{\boldsymbol{\nabla} \times \boldsymbol{j}}{2q^2\gamma|\Psi|^2} = -\frac{\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B})}{2\mu_0 q^2 \gamma |\Psi|^2} \\ &= \frac{\nabla^2 \boldsymbol{B}}{2\mu_0 q^2 \gamma |\Psi|^2}. \end{aligned} \tag{47}$$

Thus we have

$$\boldsymbol{B} = \lambda^2 \nabla^2 \boldsymbol{B}, \tag{48}$$

where

$$\lambda = \sqrt{\frac{\beta}{2\mu_0 q^2 \gamma |\alpha|}}. \tag{49}$$

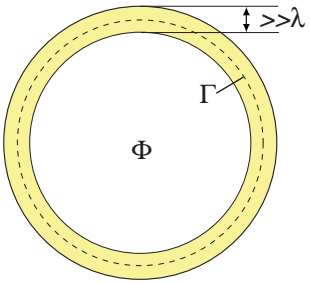


Now we study a superconducting half-space $x > 0$. Let us assume that outside the superconductor ($x < 0$) there is a field $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$ parallel to the surface of the superconductor. The solution of equation (48) inside the superconductor is

$$B_z(x) = B_0 e^{-x/\lambda}. \tag{50}$$

Thus we explain the Meissner effect: the magnetic field does not penetrate into the superconductor, apart from a layer with thickness on the order of λ. In this layer a current is flowing (calculate it), which cancels the external field inside the superconductor. The result thus also implies the existence of a dissipationless current. Below some experimental values for the penetration depth are given.

	$\lambda(T \ll T_c)$ (nm)
Al	49
Sn	51
Pb	39



4) We look at the superconducting state in a ring (loop, torus), whose cross-section is considerably larger than the penetration depth. Then inside the ring $\boldsymbol{j} = 0$. From (45) we find that for a path Γ going around inside the ring we have

$$0 = \oint d\boldsymbol{l} \cdot (\hbar \boldsymbol{\nabla} \phi - q\boldsymbol{A}) = \hbar 2\pi N - q \int d\boldsymbol{a} \cdot \boldsymbol{\nabla} \times \boldsymbol{A}. \quad (51)$$

Here N is an integer, which follows from that the fact that a unique single-valued Ψ (44) only allows for ϕ to change by a multiple of 2π when going around Γ. Thus for the magnetic flux threading the loop we find

$$\Phi = \int d\boldsymbol{a} \cdot \boldsymbol{B} = N \frac{2\pi\hbar}{q}. \quad (52)$$

It has been experimentally observed that the flux is quantized according to this formula. From the magnitude from the observed flux quantum, $\Phi_0 = \frac{h}{2|e|}$ (4), we deduce that q is twice the charge e of an electron (the sign of the charge cannot be deduced from this).

5) Above we have defined two lengths: the GL coherence length $\xi_{\rm GL}$ (42) and the penetration length λ (49). Both have the temperature dependence

$$\lambda(T),\ \xi_{\rm GL}(T) \propto \frac{1}{\sqrt{|\alpha|}} \propto \frac{1}{\sqrt{1 - T/T_c}}, \quad (53)$$

so that they diverge when $T \to T_c$. The ratio of the lengths is called the *GL parameter*

$$\kappa = \frac{\lambda(T)}{\xi_{\rm GL}(T)} = \sqrt{\frac{\beta}{2\mu_0 q^2 \hbar^2 \gamma^2}}. \quad (54)$$

It is a temperature-independent constant. By writing the GL equations in a dimensionless form we observe that this is the only dimensionless parameter in the theory.

6) The equilibrium between normal and superconducting states in an external field was studied already in the beginning of the course, but it is instructive to see the same by starting from the GL functional (27). In the superconducting state we obtain from the terms $\alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4$ a negative contribution that was calculated above (38). This is independent of the field H, because $B \equiv 0$. In the normal state only the terms $\frac{1}{2\mu_0}B^2 - \boldsymbol{B} \cdot \boldsymbol{H}$ are nonzero. By minimizing G with respect to $\boldsymbol{B}$ we find

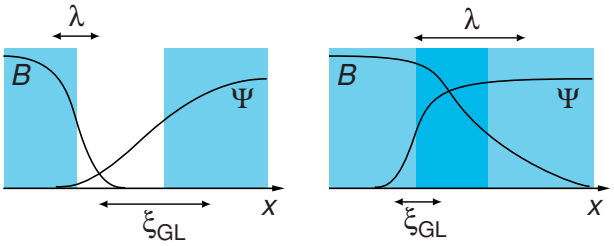
$$G = F_0 - \frac{1}{2} V \mu_0 H^2, \quad \boldsymbol{B} = \mu_0 \boldsymbol{H}. \quad (55)$$

In order for the energies to be equal when $H = H_c$ we have

$$\frac{1}{2}\mu_0 H_c^2 = \frac{\alpha^2}{2\beta}. \quad (56)$$

From this and the linearity (39) of $\alpha(T)$ it is concluded that $H_c(T)$ is linear close to T_c. This is consistent with the experimental observation (3).

7) Next we investigate the interface between normal and superconducting states. This requires the field H to be of the critical magnitude H_c, because the interface can only be stable if the two phases are in equilibrium. The structure of the interface can be solved exactly from the GL equations, be here we are satisfied by a qualitative analysis. In the previous point we identified the essential terms of normal and superconducting states in the functional (27). Let us see how these are involved in the interface.



In the figure a light shading roughly describes the regions where the energy is lowered from F_0 by the amount (56).

The situation in an S-N interface depends essentially on the ratio of the penetration depth and the GL coherence length. If $\lambda \ll \xi_{\rm GL}$ a region of thickness $\approx \xi_{\rm GL}$ is formed, where neither of the negative contributions is reached. This means an interface energy $\sigma \approx \frac{1}{2}\xi_{\rm GL}\mu_0 H_c^2$. In the opposite case $\lambda \gg \xi_{\rm GL}$, both negative contributions are present within a thickness $\approx \lambda$ and we find a negative interface energy $\sigma \approx -\frac{1}{2}\lambda\mu_0 H_c^2$. This latter case leads to completely new types of properties. The description of the intermediate state given in the beginning if the course is clearly not valid in this case.

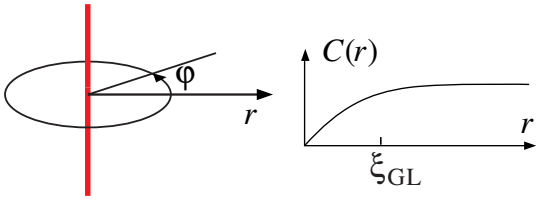
A superconductor where the interface energy in negative is called a type II superconductor, as opposed to the type I superconductor that has a positive interface energy. By solving the GL equations we find that the limit between the two cases goes at the value $\kappa = 1/\sqrt{2}$ of the GL parameter. Thus for a type I superconductor $\kappa < 1/\sqrt{2}$ and for a type II superconductor $\kappa > 1/\sqrt{2}$.

2.5 Type II superconductivity

In type II superconductor the interface energy is negative. It follows that the magnetic field penetrates into the sample in as small units as possible in order to maximize the amount of the interface. Because of flux quantization (52) we deduce that the smallest unit is one flux quantum Φ_0. We sketch the corresponding solution of the GL

equations. In cylindrical coordinates

$$\Psi(r,\varphi,z) = C(r)e^{i\varphi}. \tag{57}$$



Here the phase ϕ of the order parameter is the same as the azimuthal angle φ of the cylindrical coordinates. Because the order parameter is independent of z, we consider it in the x-y plane. Ψ has to be continuous everywhere. It has a zero at $r = 0$, where it is analytic in spite of the singularity of the coordinate system, $\Psi(x,y,z) = a(x+iy) + O(r^2)$.

The dependence $e^{i\varphi}$ (57) on the phase causes a current (45) that circulates around the z axis. We suppose that also the vector potential $\boldsymbol{A}$ is in the direction of the azimuthal angle, $\boldsymbol{A} = A(r)\hat{\boldsymbol{\varphi}}$. The current

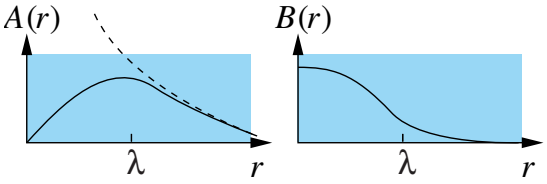
$$\boldsymbol{j} = 2q\gamma C^2(r)\left[\frac{\hbar}{r} - qA(r)\right]\hat{\boldsymbol{\varphi}}. \tag{58}$$

At large r the order parameter approaches its equilibrium value (36). There the current (58) must vanish (exponentially). Thus

$$\boldsymbol{A}(r) = \frac{\hbar}{q}\frac{\hat{\boldsymbol{\varphi}}}{r} \quad (r \gg \lambda). \tag{59}$$

Requiring that $\boldsymbol{A}(r)$ is regular at origin, we can guess its shape. Finally we calculate

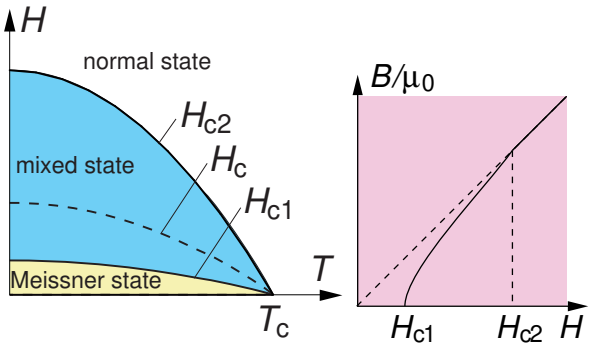
$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \frac{\hat{\boldsymbol{z}}}{r}\frac{d(rA)}{dr}. \tag{60}$$



The accurate forms of the functions are obtained by solving the GL equations, which generally is possible only numerically.

The solution of the type (57) is called a *quantized vortex* or *vortex* or *flux line*. We see from equation (59) that the magnetic flux associated to a vortex is precisely one flux quantum Φ_0 (4).

For a type II superconductor one gets the following phase diagram.

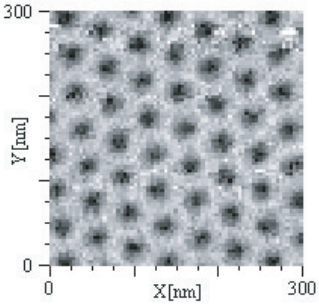


Between the critical fields H_{c1} and H_{c2} the magnetic field partly penetrates to the sample. The density of vortices is $n = B/\Phi_0$. The solution for one vortex describes the situation at fields near H_{c1}, where the vortices are far apart from each other. In increasing field also the density of vortices increases until at H_{c2} they are so dense that there is no space for superconductivity in between. Because the vortex core size is approximately ξ_{GL}, we estimate from this that $B_{c2} \sim \Phi_0/\xi_{GL}^2$. An accurate calculation with GL theory gives

$$B_{c2} = \mu_0 H_{c2} = \frac{\hbar}{2|e|\xi_{GL}^2} = \frac{\Phi_0}{2\pi\xi_{GL}^2}. \tag{61}$$

Near H_{c2} the order parameter goes continuously to zero. In this case the third order term $\Psi|\Psi|^2$ in the GL equation (28) can be dropped, and the remaining equation is the same as the Schrödinger equation for a charged particle in constant magnetic field $\boldsymbol{B}$. We leave the mathematics of this problem to the condensed matter course.

In the equilibrium state the vortices fill the sample as uniformly as possible. This leads to a lattice that is hexagonal.



Spectroscopic image of the vortex lattice in NbS_2 at 4.2 Kelvin and 1 Tesla. Dark corresponds to the normal vortex cores, and bright to the superconducting regions. The vortex lattice imaging by scanning tunneling spectroscopy relies on spatial variations of the density of levels in the mixed state. Indeed, the local density of levels is different at the center of vortex cores compared to the surrounding superconducting regions. Plotting these differences as a function of position yields a spectroscopic real space image of the Abrikosov vortex lattice. (figure from http://dpmc.unige.ch/gr_fischer/)

Check as an exercise that the dimensions of the figure and the given field are consistent.

Force on a flux line

A Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{62}$$

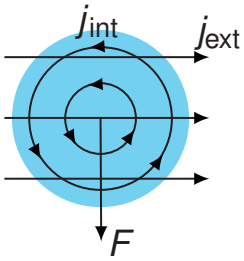
acts on a charged particle. This can be generalized to continuously distributed matter as

$$\boldsymbol{F} = \int d^3r(\rho\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B}), \tag{63}$$

where ρ is the charge density and $\boldsymbol{j}$ the electric current density. We apply this to a single flux line in an applied flow field $\boldsymbol{j}_{\text{ext}}(\boldsymbol{r})$. Because of charge neutrality $\rho = 0$. Supposing that $\boldsymbol{j}_{\text{ext}}(\boldsymbol{r})$ is approximately constant in the cross section of the vortex (area $\approx \lambda^2$), we can calculate the integral in transverse plane and get the force acting on the flux line

$$\boldsymbol{F} = \Phi_0 \int \boldsymbol{j}_{\text{ext}} \times d\boldsymbol{l}, \tag{64}$$

where $d\boldsymbol{l}$ is the line element of the flux line. This force drives the vortex in direction that is perpendicular to the applied current.

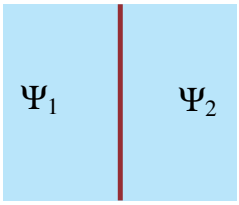


Notice that the direction of the force is such that it tends to decrease the region where the total flow velocity is largest (compare to the Magnus force).

One application of superconductivity is to build strong magnets. Because H_c is relatively small (see the table on page 2), superconductors of type II are used. A large part of applications use Nb-Ti alloy, where 45 weight per cent is titanium. Because the alloy is disordered, the mean free path is very short and therefore H_{c2} is high, $B_{c2} \approx 10$ T at $T = 4$ K. Notice that if the alloy would be ordered, the mean free path could in principle be as long as in a pure element (see condensed matter course for justification).

2.6 Josephson effect

We consider two superconductors that are weakly coupled to each other. We assume that $\boldsymbol{B} = \boldsymbol{A} = 0$.



Analogously to the phenomenological derivation of the GL theory, we form an expression for the energy associated with such a junction. Let Ψ_1 and Ψ_2 denote the order parameters on the left and right hand sides. We require 1) reality, 2) independence of a constant phase factor $\exp(i\phi)$. 3) independence on exchange of the two sides, and 4) take only the leading order terms. This way we get the *Josephson energy*

$$F_J = -a(\Psi_1^*\Psi_2 + \Psi_1\Psi_2^*) = -2a\,\mathrm{Re}(\Psi_1^*\Psi_2). \tag{65}$$

We substitute

$$\Psi_1 = Ce^{i\phi_1}, \quad \Psi_2 = Ce^{i\phi_2}, \tag{66}$$

and get

$$F_J = -E_J \cos\Delta\phi. \tag{67}$$

We have defined the *phase difference*

$$\Delta\phi = \phi_2 - \phi_1 \tag{68}$$

and $E_J = 2aC^2$.

From the Josephson energy (67) we get the electric current through the junction

$$J = J_c \sin\Delta\phi, \tag{69}$$

where $J_c = (q/\hbar)E_J = (2e/\hbar)E_J$.

Justification of (69): We substitute (44) in to the GL energy (25). We make variation of it with respect ϕ on both sides (1-dimensional model is sufficient). The variation gives surface terms, and these counted together with the Josephson energy (67) should vanish, which gives relation (69). ∎

Another important relation is the following, which gives the time derivative of the phase,

$$\frac{d\phi}{dt} = -\frac{2\mu}{\hbar}. \tag{70}$$

Justification of (70): the order parameter Ψ of the GL theory was interpreted as the wave function describing a Cooper pair. In equilibrium the Cooper pairs are in equilibrium with electrons so that the energy of a Cooper pair is 2μ, twice the electron chemical potential. The time dependence of an energy eigenstate in quantum mechanics comes from the factor $\exp(-iEt/\hbar)$, which for the order parameter $\Psi = e^{i\phi}|\Psi|$ is $\exp(-i2\mu t/\hbar)$, and thus one gets (70). ∎

We apply (70) to a Josephson junction. For the phase difference (68) we get

$$\frac{d\Delta\phi}{dt} = \frac{2eV}{\hbar}, \tag{71}$$

since the difference in the chemical potentials is related to the voltage V by $\Delta\mu = \mu_2 - \mu_1 = -Ve$.

The equations (69) and (71) are known as Josephson equations. The first gives that in equilibrium ($V = 0$) a constant current flows trough that depends sinusoidally on the phase difference $\Delta\phi \equiv \phi_2 - \phi_1$. This is known as *dc Josephson effect*.

If the voltage V is constant, one gets from equation (71) that the phase grows linearly in time,

$$\Delta\phi = \frac{2eV}{\hbar}t. \tag{72}$$

Substituting this into equation (69) one gets alternating current at angular frequency

$$\omega = \frac{2e}{\hbar}V. \tag{73}$$

This is known as *ac Josephson effect*.

At voltage 0.1 mV the equation (73) gives the frequency $\nu = \omega/2\pi = 48$ GHz.

Using Josephson junctions it is possible to make sensitive measuring devices. For example, equation (73) makes possible a voltage standard, as the frequency can accurately be measured.

The current source driving the junction makes in time dt the work $VJdt$. According to energy conservation we must have

$$\frac{dF_J}{dt} = VJ. \tag{74}$$

We see that using this relation we can derive the third relation based on any pair of the relations (67), (69) and (71).

3. Superfluidity

By superfluidity we mean a phenomenon where a fluid can flow without friction.

3.1 Occurrence of superflidity

- Liquid ^{4}He (^{4}He atom is a boson)
- Liquid ^{3}He (the rare isotope of helium, ^{3}He atom is a fermion)
- dilute gases of alkali metal atoms. These are mainly bosons like ^{87}Rb, ^{7}Li, ^{23}Na, ^{1}H, but also some fermions like ^{6}Li, ^{40}K under special conditions. These gases are metastable (not thermodynamically stable) but they persists sufficiently long that experiments can be made. The gases are generated in magnetic or optical traps and are cooled by evaporation and laser cooling.
- possibly: neutron fluid inside a neutron star (fermion), ...
- superconductivity of metals can be understood as superfluidity of the conduction electrons (fermion)

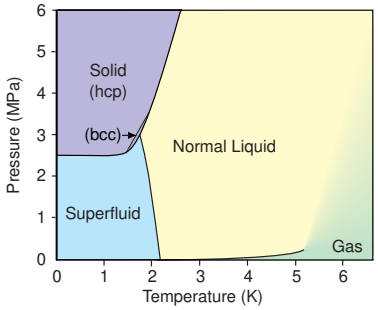
There are two essential differences between superconductors and other superfluids.

- Electrons have electric charge and therefore their motion is essentially coupled with electromagnetic field. Other superfluids are charge neutral, and therefore their coupling to electromagnetic field is weaker.
- The electrons move in the background of the crystal lattice. Other superfluids do not have such a preferred frame of reference.

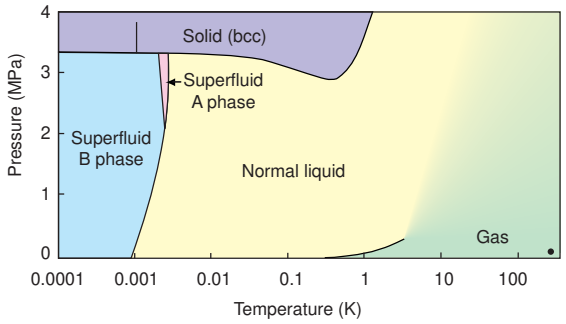
As superconductivity was considered in the previous chapter, we limit here to lattice-free charge-neutral superfluids.

The temperature below which superfluidty occurs is called *critical temperature*, T_c.

- Liquid ^{4}He: $T_c \approx 2$ K



- Liquid ^{3}He: $T_c \approx 2$ mK



  ^{3}He has three different superfluid phases, A, A_1, and B. The A_1 phase only appears in magnetic field, and therefore is not visible here.

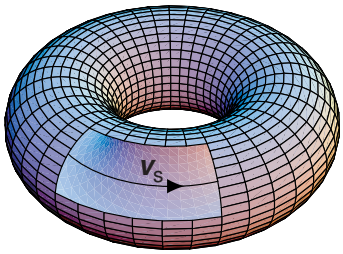
- dilute gases of alkali metals $T_c \sim 1$ μK depending on density.

3.2 History

- 1908 liquefaction of helium
- 1938 discovery of superfluid state of ^{4}He
- 1972 discovery of superfluid states of ^{3}He
- 1995 Bose-Einstein condensation in alkali atom gases

3.3 Basic properties

Once put into motion, the flow in a closed tube does not decay.



Similarly, superfluid can flow through narrow pores or as a thin film.

Viscous behavior is observed, e.g., for oscillating disks immersed in superfluid.

Critical velocity: the superfluid state persists only up to some velocity, beyond which dissipation occurs. The critical velocity depends essentially on the experimental conditions.

Two sound modes: the usual (first) sound and a *second sound*

Anomalous heat conduction and thermomechanical effects.

3.4 Microscopic origin

In short, superfluidity can be explained as a quantum mechanical effect that shows up on a macroscopic scale.

Further analysis depends essentially whether the particles are *bosons* or *fermions*. The defining property of bosons is that their many-body wave function is symmetric in the exchange of coordinates of two particles. This means that there is no restriction on many bosons to be in the same level. Consider an ideal boson gas. At $T = 0$ all the particles condense into the lowest energy level, see Eq. (9). This is known as Bose-Einstein condensation. It can be shown that the occupation of the lowest level stays macroscopic up to the temperature

$$T_{\mathrm{BE}} = \frac{h^2}{2\pi m k_B} \left(\frac{N}{2.612V} \right)^{2/3} . \tag{75}$$

Here N is the number of particles, V the volume, and m the particle mass. We claim that *superfluidity in boson systems is based on macroscopic occupation of a single level.*

While the ideal gas model explains Bose-Einstein condensation, it is quite insufficient in other respects. The interactions between particles are essential for the system to show superfluidity (see later). In interacting system the macroscopically occupied state $\psi(\boldsymbol{r})$ need not be the lowest energy state, and thus the macroscopic wave function can be nontrivial. In dilute bose gases (^{87}Rb etc.) the interactions are weak, and a quantitative description can be achieved by a relatively simple Gross-Pitaevskii equation. In ^{4}He the interactions are much stronger, and a quantitative theory is not easily achieved. But even in that case, formula (75) gives 3.1 K, which is not too far from the measure value.

Let us now turn to fermions. This case was already discussed above in connection of superconductivity. Pauli exclusion principle prohibits more that one fermion to be in a level. But this does not exclude all fermions to be in a single pair state. Thus we can propose, that *superfluidity in fermion systems is caused by macroscopic occupation of a pair state.*

Let us consider liquid ^{3}He. This system has Fermi temperature (the Fermi energy expressed in temperature units) $T_F = \varepsilon_F / k_B \approx 1$ K. The superfluid transition temperature is much lower, $T_c \sim 10^{-3} T_F$. In this respect the behavior of ^{3}He atoms is very similar to conduction electrons in superconducting metals.

Above we have discussed the boson and fermion cases separately. But in fact, one can think to go continuously from one case to the other. First note that T_{BE} and T_F are of the same order of magnitude for gases of the same density and particle mass. In superconductors and in liquid ^{3}He T_c is much smaller. This is because the pairs (often called Cooper pairs) are very weakly bound. Increasing the attractive interaction would increase T_c towards $T_{\mathrm{BE}} \sim T_F$ and there would be a continuous

cross-over to Bose condensation, when the binding of the pair becomes stronger. In fact, all bose particles are composite particles made of fermions. Part of this cross-over has been been tested experimentally in dilute alkali gases.

3.5 Hydrodynamics

Many properties of superfluids can be understood in terms of the *two-fluid model*. The basic assumption is that the liquid consists of two parts. These are called the *superfluid* and *normal components*. The current density $\boldsymbol{j}$ can be represented as a sum

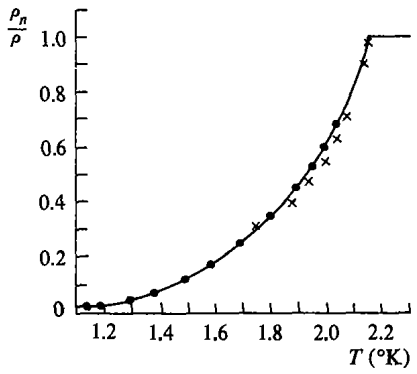
$$\boldsymbol{j} = \rho_s \boldsymbol{v}_s + \rho_n \boldsymbol{v}_n. \tag{76}$$

Here ρ_s and $\boldsymbol{v}_s$ are the density and velocity of the superfluid component and ρ_n and $\boldsymbol{v}_n$ are the corresponding quantities for the normal part. The liquid density is the sum of the two densities, $\rho = \rho_s + \rho_n$. The superfluid component can flow without viscosity and it carries no heat or entropy. Moreover it is curl free,

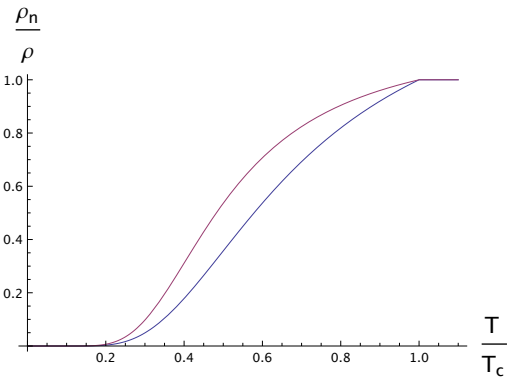
$$\boldsymbol{\nabla} \times \boldsymbol{v}_s = 0. \tag{77}$$

This holds for ^{4}He and ^{3}He-B but ^{3}He-A is a more tricky case. The normal component behaves more like a usual viscous fluid.

The two-fluid model can be justified from the microscopic considerations above. The superfluid component corresponds to particles in the macroscopic wave function, and the normal component to particles in excited levles. The densities of the two components depend on temperature. With increasing temperature $\rho_s(T)$ drops continuously from $\rho_s(0) = \rho$ and vanishes at the superfluid transition temperature T_c.



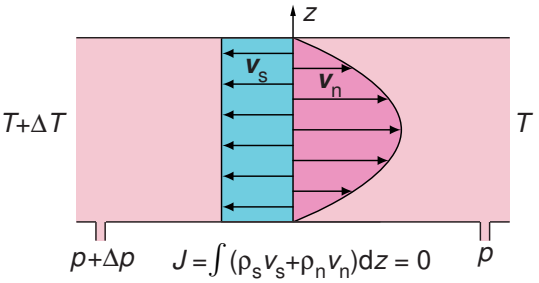
ρ_n/ρ of superfluid ^{4}He measured by Peshkov (1946)



ρ_n/ρ of superfluid ^{3}He-B at melting pressure (upper) and vapor pressure (lower) according to weak coupling theory

The two-fluid model can explain many properties of superfluids. In particular, the existence of frictional forces depends on the type of experiment. On one hand, the flow in a ring-shaped container persists because it is only the superfluid component that flows. On the other hand, nonvanishing viscosity is measured with a rotational viscometer, where the superfluid is placed between two coaxial cylinders that rotate at different angular velocities. Here the normal component is driven into motion and causes dissipation.

Superfluids show peculiar mixing of thermal and mechanical properties. Consider superfluid in a channel which is heated at one end.



The superfluid component is attracted to the hot region because the chemical potential is lower there. As a consequence a pressure difference appears. This drives the normal component in the direction of decreasing temperature and convects the heat away from the source. The viscosity of the normal component causes v_n to vanish at walls. The superfluid velocity $v_s(z)$ has to be constant in order to be curl free (77). Assuming the geometry does not allow net mass transfer, the mass transported by the normal and superfluid components in opposite directions are equal in magnitude.

In addition to usual sound wave, superfluids have another propagating mode. This *second sound* is an oscillation where normal and superfluid components move in opposite directions. This leads to oscillation of temperature whereas the density remains nearly constant. Second sound can be generated by heating the superfluid periodically, and standing waves of temperature have

been demonstrated experimentally.

In addition to the mass current $\boldsymbol{j}$, there can be persistent spin currents. This is possible in superfluids whose order parameter is more complicated than scalar (^{3}He). Spin current is described by a tensor $j^{\rm spin}_{\mu j}$. The index $\mu = x, y, z$ indicates the direction of spin angular momentum that is flowing, and $j = x, y, z$ indicates the direction of the flow. Even in equilibrium the order parameter of ^{3}He has nontrivial spatial variation called *texture*. This is associated with persistent spin currents and, in case of ^{3}He-A, also with persistent mass currents.

Quantization of circulation

Consider a superfluid with order parameter Ψ. The superfluid velocity $\boldsymbol{v}_s$ can be expressed as a function of the order parameter as

$$\boldsymbol{v}_s = \frac{\hbar}{M}\boldsymbol{\nabla}\phi. \tag{78}$$

Here $\phi(\boldsymbol{r})$ is the phase of the order parameter, $\Psi(\boldsymbol{r}) = Ae^{i\phi(\boldsymbol{r})}$, and the amplitude A is assumed constant. M is the boson mass, i.e. the mass of a particle in a boson superfluid and the mass of a pair in a fermion superfluid. Eq. (78) can be justified starting from the expression of current in quantum mechanics.

An alternative form of Eq. (78) is obtained by taking line integral along a closed path,

$$\oint \boldsymbol{v}_s \cdot d\boldsymbol{l} = N\frac{h}{M}. \tag{79}$$

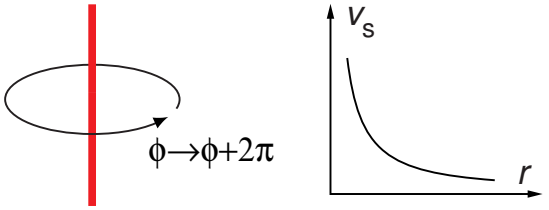
Here we have used the property that ϕ is defined modulo 2π, and N is an integer. Eq. (79) is known as *quantization of circulation*. The curl-free condition (77) is a direct consequence of Eq. (78) or (79).

Consider again superfluid in a ring-shaped container. We can apply Eq. (79) to a path in the ring. We see that, in addition to being persistent, the superfluid velocity can only have discrete values. (Compare with flux quantization in superconductors).

3.6 Rotating superfluid

Let us consider superfluid in a container that is rotated with angular velocity $\boldsymbol{\Omega}$. The normal component will follow this motion because of its viscosity. In equilibrium it rotates uniformly with the container, $\boldsymbol{v}_n = \boldsymbol{\Omega} \times \boldsymbol{r}$. This is not possible for the superfluid component because it has to be curl free (77). [Eq. (77) should be compared to $\boldsymbol{\nabla} \times \boldsymbol{v}_n = 2\boldsymbol{\Omega}$.]

The rotating state of a superfluid is most commonly realized by *vortex lines*. On a path around the vortex line, the phase ϕ changes by 2π (or an integral multiple of it). This is illustrated in the figure.



Equivalently, the circulation of superfluid velocity (79) around the vortex line is h/M. Assuming cylindrical symmetry, the phase ϕ is the same as the azimuthal angle in the cylindrical coordinate system (r, ϕ, z). The velocity field can be calculated from Eq. (78):

$$\boldsymbol{v}_s = \frac{\hbar}{Mr}\hat{\boldsymbol{\phi}}, \tag{80}$$

where $\hat{\boldsymbol{\phi}}$ is a unit vector in the azimuthal direction.

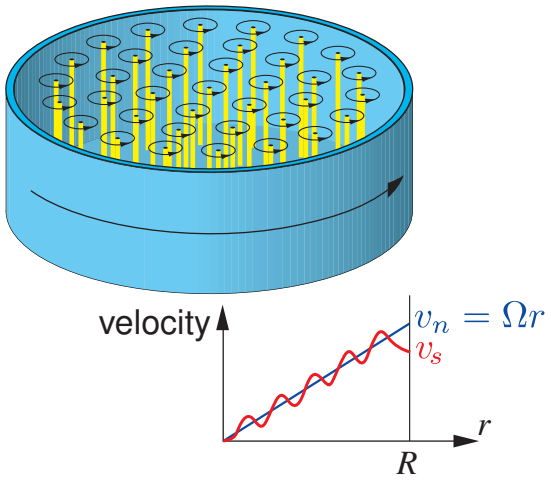
The structure of the rotating state is determined by minimum of free energy. The rotation of the container is taken into account by minimizing $F = F_0 - \boldsymbol{L} \cdot \boldsymbol{\Omega}$, where F_0 is the free energy functional in the stationary case and $\boldsymbol{L}$ the angular momentum. In the two-fluid model this reduces to

$$F = \int d^3r \frac{1}{2}\rho_s(\boldsymbol{v}_s - \boldsymbol{v}_n)^2 + \text{constant}. \tag{81}$$

Thus the optimal solution corresponds to $\boldsymbol{v}_s$ as equal as possible to $\boldsymbol{v}_n = \boldsymbol{\Omega} \times \boldsymbol{r}$, but subject to condition (78). This is achieved by a regular array of vortex lines. The number of vortex lines n per unit area is determined by the condition that the circulations of normal and superfluid velocities are the same over an area containing many vortex lines. This yields

$$n = \frac{2M\Omega}{h}. \tag{82}$$

There are approximately 1000 vortex lines in a circular container of radius 1 cm that is rotating 1 round per minute.



Vortex lines in an uncharged superfluid are analogous to flux lines, which occur in type II superconductors. Flux

lines of superconductors appear in magnetic field, which is analogous to rotation of an uncharged superfluid.

[One can ask what happens when a superconductor is rotated. The answer is that effect of rotation is cancelled by generating a uniform magnetic field (known as London moment), so that no flux lines need to be generated.]

The velocity field (80) of a vortex diverges at the vortex line. Thus there must be a *vortex core*, where the two-fluid description is insufficient. A finite energy in the vortex core is achieved if the amplitude of the order parameter vanishes at the vortex line. This is the case for a scalar order parameter. For a matrix order parameter it is not necessary that all components of the matrix vanish at the line. Such vortex lines are realized in superfluid ^{3}He.

The quantization of the superfluid velocity (79) is not always true for uncharged superfluids. This happens when there is an additional contribution to the superfluid velocity (78) coming from the matrix form of the order parameter. Such a case is realized in superfluid ^{3}He-A, and careful reanalysis of the rotating state is needed. It turns out that, in addition to one-dimensional vortex lines, the vorticity may be arranged as two-dimensional vortex sheets and three-dimensional textures. All these have been confirmed experimentally. In any case, a homogeneous rotation of the superfluid is excluded.

3.7 Phase slip, Josehpson effect and critical velocity

Let us study superflow in a channel under thermal equilibrium ($\boldsymbol{v}_n = 0$). The maximum supercurrent is determined by a process called *phase slip*. Consider that a short piece of vortex line is nucleated at a surface on one side of the channel. This vortex expands, goes through the whole cross section of the channel, and finally disappears on the other side. As a result of this process, the phase difference $\Delta\phi$ between the ends of the channel has changed by 2π. Part of the superfluid kinetic energy is dissipated in the motion of the vortex. This means that the flow ceases to be dissipationless above a critical velocity for phase slips. Phase slips take first place in constrictions of the flow channel, where the superfluid velocity has its maximum value.

A special type of phase slip takes place in very short constrictions, where Eq. (78) ceases to be valid. An ideally short constriction shows the *Josephson effect*, where the supercurrent J_s depends on the phase difference $\Delta\phi$ as

$$J = J_c \sin(\Delta\phi), \tag{83}$$

and J_c is a constant. Moreover, the time derivative of $\Delta\phi$ is proportional to the difference of the chemical potential $\Delta\mu$ on the two sides of the constriction,

$$\frac{d\Delta\phi}{dt} = -\frac{2\Delta\mu}{\hbar}. \tag{84}$$

Combining the two equations, one sees that a constant $\Delta\mu$ generates an oscillating current at the frequency $2\Delta\mu/h$.

The Josephson effect has been extensively studied in superconductors. In helium superfluids it is more difficult to fabricate constrictions that are small enough, but the Josephson effect has been seen there as well.

Let us now consider a moving object in a stationary superfluid. In this case there will be a drag force on the object caused by the normal fluid component. However, if the temperature is low, say below $0.2T_c$, the normal fraction is vanishingly small. In this case the motion should be nearly dissipationless at low velocities. To how large velocities this can be the case?

Before going to superfluids, we can ask what is the maximum dissipationless velocity in an ideal fluid. Ideal fluid obeys Euler's equation and the equation of continuity. The answer is sound velocity. At a velocity v exceeding the sound velocity c, there will be a conical wave front at angle α satisfying $\sin\alpha = c/v$. The wave carry energy away from the object and lead to drag.

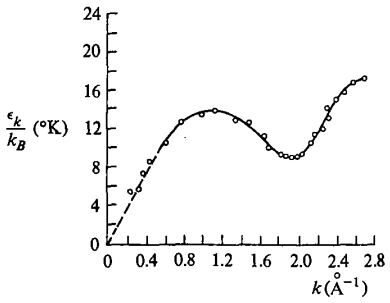
Ideal fluid is a bit too trivial case, since it has linear dispersion of waves. That is, the angular frequency $\omega = ck$ with a constant c. In the case of nonlinear dispersion the condition should be stated as follows: there are waves having the component of the phase velocity in the direction of the object, which is equal the object velocity. Such waves are stationary in the frame of the object. You probably have noted that this is the case for the main waves generated by a ship. Thus the critical velocity equals the lowest phase velocity of waves:

$$v_c = \min \frac{\omega}{k} = \min \frac{E}{p}. \tag{85}$$

The second equality applies because in quantum mechanics $E = \hbar\omega$ and $p = \hbar k$. In connection of superfluidity (85) is known as Landau criterion. This sounds to me as one more example of Stigler's law.

(Stigler's law states that no scientific discovery is named after its original discoverer. This likely is an overstatement, but at least wikipedia lists several examples.)

Let us apply Landau criterion to a few cases.

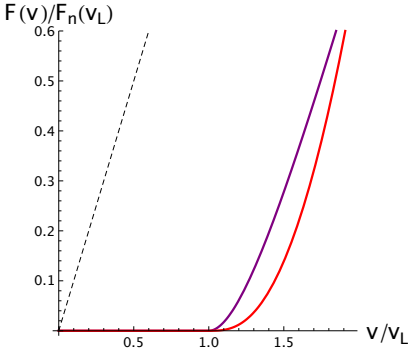


The figure shows the dispersion relation of elementary excitations in ^{4}He based on neutron scattering by Henshaw and Woods (1961). The excitations are divided to phonons (linear part at small k) and rotons (around the minimum). The critical velocity is determined by the roton minimum, which gives $v_c \approx 60$ m/s.

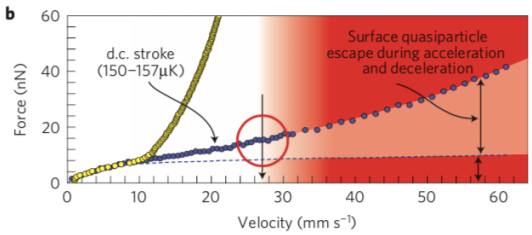
For ideal bose gas the dispersion is as for free particle (7). This gives $v_c = 0$. Thus ideal bose gas is not superfluid in this sense.

^{3}He-B has similar dispersion as shown for superconductors above. This gives $v_c \approx \Delta/p_F \approx 27$ mm/s.

Critical velocities close to Landau velocity has been measured for negative ions in both ^{4}He and ^{3}He-B. In other cases the dissipation sets in earlier. In many cases this believed to be caused by vortices.



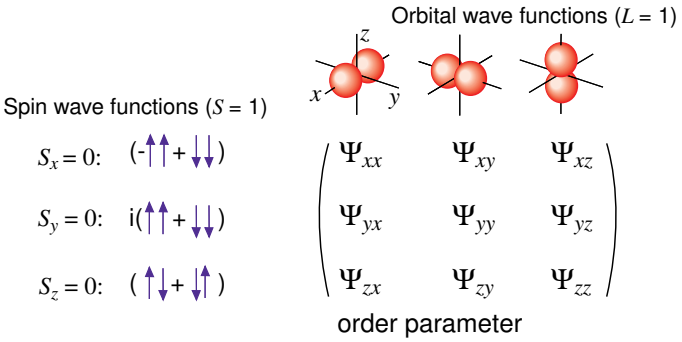
The figure show theoretical curves for drag force on a small (radius $r \ll \xi$) but not too light object in ^{3}He-B at $T = 0$. The solid lines correspond to two different models of quasiparticle scattering on the object surface (Bowley 1977, Kuorelahti et al 2018). The drag in the normal state is shown by dashed line.



There is one experiment that is contrary to the discussion above. Bradley et al (2016) do not see much dissipation with a uniformly moving wire up to twice the Landau velocity. The reason for this is unknown.

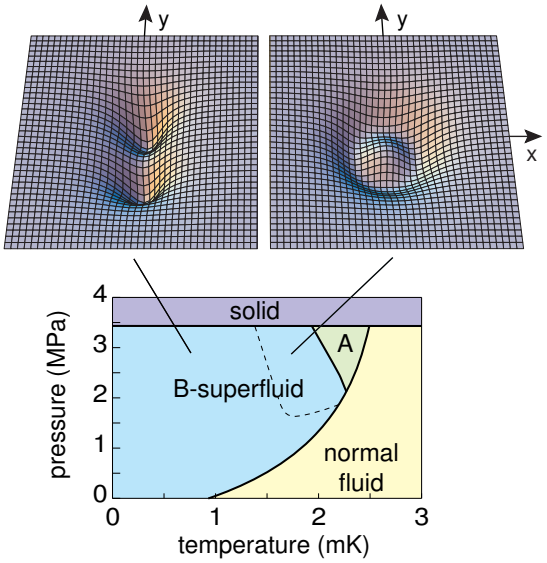
3.8 Special properties of superfluid ^{3}He

Here we briefly discuss superfluid phenomena that do not take place in liquid ^{4}He. The examples are for ^{3}He, but related phenomena can appear in some dilute gases and superconductors.

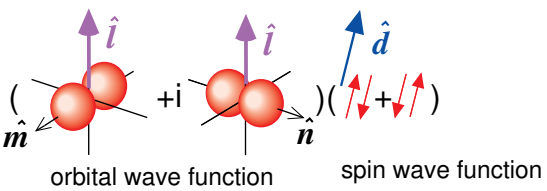


In superfluid ^{3}He the spin part of the pair wave function is spin triplet. Standard choices for the three triplet states are shown in the figure. In addition, the angular part of the wave function corresponds to angular momentum states $L = 1$. The standard choice is to use the three p-wave functions. Thus the order parameter is a matrix with $3 \times 3 = 9$ complex functions that give the pair amplitude projected on all the spin and orbital basis states.

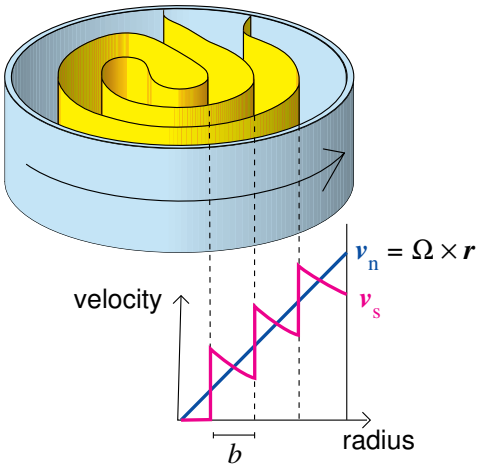
The B phase correspond to unit matrix, or an arbitrary rotation matrix, that can be parametrized by rotation angle and axis $\hat{\boldsymbol{n}}$.



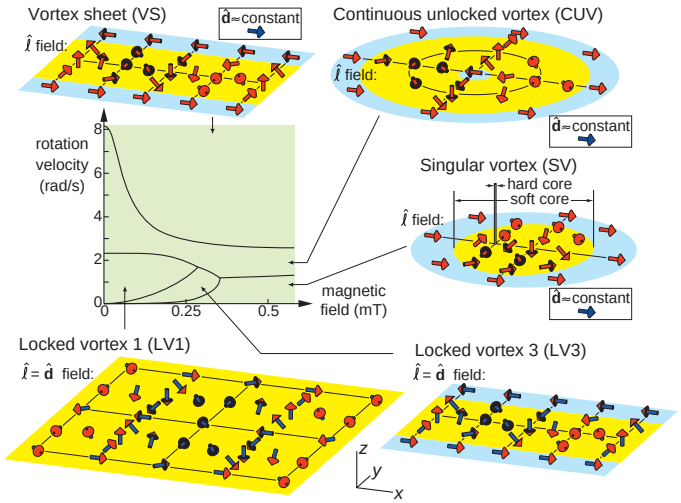
Because of the matrix order parameter, the vortex core can have different structures. Experimentally a transition in the core structure was found (lower part of the figure). GL calculations of the core structure allowed to identify the two structures. In the low temperature structure the core is split into two half cores (illustrated by $\sum_{ij} |\Psi_{ij}|^2$ in the upper part of the figure).



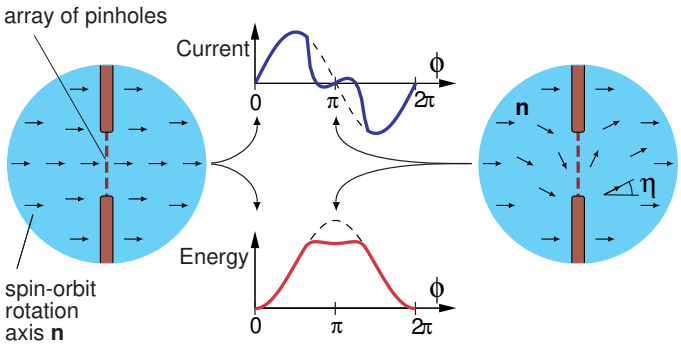
The figure illustrates the pairing state in ^{3}He-A. It has angular momentum in the direction of vector $\hat{\boldsymbol{l}}$ and the orientation of the spin part is given by vector $\hat{\boldsymbol{d}}$



The A phase has several vortex structures. One of them has sheet structure instead of the usual line structure.



Theoretical phase diagram of different vortex structures in the A phase. It is partly confirmed experimentally.



The figure illustrates Josehpson coupling between two volumes of ^{3}He-B. Depending on the phase difference ϕ, the B-phase rotation matrix axis $\hat{\boldsymbol{n}}$ may flip, which leads to unusual π-state in the Josephson current-phase relation.