Tutorial lecture, QFS&MBT
Niagara Falls, NY, USA, 15.8.2015

Introduction to Landau's Fermi Liquid Theory

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1. Introduction

The principal problem of physics is to determine how bodies behave when they interact. Most basic courses of classical and quantum mechanics treat the problem of one or two particles or bodies. (An external potential can be considered as one very heavy body.) The problem gets more difficult when the number of bodies involved is larger. In particular, in condensed matter we are dealing with a macroscopic number $N \sim 10^{23}$ of particles, and typically hundreds of them directly interact with each other. This problem is commonly known as the *many-body problem*.

There is no general solution to the many-body problem. Instead there is a great number of approximations that successfully explain various limiting cases. Here we discuss one of them, *the Fermi-liquid theory*. This type of approximation for a fermion many-body problem was invented by Landau (1957). It was originally proposed for liquid ^{3}He at very low temperatures. Soon it was realized that a similar approach could be used to other fermion systems, most notably to the conduction electrons of metals. The Fermi-liquid theory allows to understand very many properties of metals. A generalization of the Fermi-liquid theory also allows to understand the superconducting state, which occurs in many metals at low temperatures. Even when Landau's theory is not valid, it forms the standard against which to compare more sophisticated theories. Thus Fermi-liquid theory is a paradigm of many-body theories, and it is presented in detail in many books and articles discussing the many-body problem.

- L.D. Landau and E.M. Lifshitz, *Statistical Physics, Part 2* (Pergamon, Oxford, 1980).
- P. Nozieres, *Theory of interacting Fermi systems* (Bejamin, New York 1964) [N].
- D. Pines and P. Nozieres, *The theory of quantum liquids, Vol. 1 Normal Fermi liquids* (Bejamin, New York 1966).
- G. Baym and C. Pethick, *Landau Fermi-liquid theory* (Wiley, Ney York 1991) [BP].
- A.A. Abrikosov and I.M. Khalatnikov, Rep. Progr. Phys. **22**, 329 (1959), an early review.
- A.J. Leggett, Rev. Mod. Phys. **47**, 331 (1975), and *Quantum liquids* (Oxford, 2006).

In this lecture I present an introduction to the Landau theory. I present the theory as if it could have logically developed, but this does not necessarily reflect historical facts. I will start with the calculation of specific heat in an ideal gas, and compare the result with the measurement in liquid ^{3}He. This leads to the concept of quasiparticles with an effective mass differing from the atomic mass. It is then shown that in order to make a consistent theory, one has to allow an interaction between the quasiparticles. After having formulated the theory, we shortly mention the main applications. Some more recent results are discussed in more length. Various generalizations of the theory are briefly mentioned.

2. Preliminary topics

Many-body problem

The many-body problem for identical particles can be formulated as follows. Consider particles of mass m labeled by index $i = 1, 2, \ldots, N$. Their locations and momenta are written as $\boldsymbol{r}_k$ and $\boldsymbol{p}_k$. The Hamiltonian is

$$H = \sum_{k=1}^{N} \frac{p_k^2}{2m} + V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots). \tag{1}$$

Here V describes interactions between any particles, and it could in many cases be written as a sum of pairwise interactions $V = V_{12} + V_{13} + \ldots + V_{23} + \ldots$. The classical many-body problem is to solve the Newton's equations.

In quantum mechanics the locations and momenta become operators. In the Schrödinger picture $\boldsymbol{p}_k \to -i\hbar\boldsymbol{\nabla}_k$, where

$$\boldsymbol{\nabla}_k = \hat{\boldsymbol{x}}\frac{\partial}{\partial x_k} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y_k} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z_k}. \tag{2}$$

An additional feature is that generally particles have spin. This is described by an additional index σ that takes values $-s, -s+1, \ldots, s-1, s$ for a particle of spin s. Thus the Hamiltonian operator is

$$H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m}\nabla_k^2 + V(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots). \tag{3}$$

and the state of the system is described by a wave function $\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots; t)$. Particles having integral spin are called *bosons*. Their wave function has to be symmetric in the exchange of any pairs of arguments. For example

$$\begin{aligned}&\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_4, \sigma_4, \ldots)\\ &= +\Psi(\boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_4, \sigma_4, \ldots),\end{aligned} \tag{4}$$

where coordinates 1 and 3 have been exchanged. Particles having half-integral spin are called *fermions*. Their wave function has to be anti-symmetric in the exchange of any pairs of arguments. For example

$$\begin{aligned}&\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_4, \sigma_4, \ldots)\\ &= -\Psi(\boldsymbol{r}_3, \sigma_3, \boldsymbol{r}_2, \sigma_2, \boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_4, \sigma_4, \ldots).\end{aligned} \tag{5}$$

The quantum many body problem is to solve time-dependent Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}(\boldsymbol{r}_1, \sigma_1, \ldots, \boldsymbol{r}_N, \sigma_N, t) = H\Psi(\boldsymbol{r}_1, \sigma_1, \ldots, \boldsymbol{r}_N, \sigma_N, t) \tag{6}$$

or to solve energy eigenvalues and eigenstates.

Ideal Fermi gas

As stated in the introduction, there is no general solution of the many-body problem. What one can do is to study some limiting cases. One particularly simple case is *ideal gas*, where we assume no interactions, $V \equiv 0$. Below we concentrate on ideal spin-half ($s = 1/2$) Fermi gas.

In the absence of interactions we can assume a factorizable form

$$\Psi_0(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots; t) = \phi_a(\boldsymbol{r}_1, \sigma_1)\phi_b(\boldsymbol{r}_2, \sigma_2)\ldots. \tag{7}$$

consisting of a product of single-particle wave functions $\phi_\alpha(\boldsymbol{r}, \sigma)$. This does not yet satisfy the antisymmetry requirement (5) but permuting the arguments in Ψ_0 and summing them all together multiplied by $(-1)^{n_P}$, where n_P is the number of pairwise permutations in a permutation P, one can generate a proper wave function

$$\begin{aligned}&\Psi(\boldsymbol{r}_1, \sigma_1, \boldsymbol{r}_2, \sigma_2, \ldots) = \frac{1}{\sqrt{N}}\sum_P (-1)^{n_P}\\ &\times\phi_a(\boldsymbol{r}_{P(1)}, \sigma_{P(1)})\phi_b(\boldsymbol{r}_{P(2)}, \sigma_{P(2)}))\ldots.\end{aligned} \tag{8}$$

This is known as Slater determinant since it can also be presented as a determinant

$$\Psi = \frac{1}{\sqrt{N}}\begin{vmatrix} \phi_a(\boldsymbol{r}_1, \sigma_1) & \phi_b(\boldsymbol{r}_1, \sigma_1) & \cdots \\ \phi_a(\boldsymbol{r}_2, \sigma_2) & \phi_b(\boldsymbol{r}_2, \sigma_2) & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}. \tag{9}$$

We can now see if any two of the single-particle wave functions ϕ_a, ϕ_b, ... are identical, the resulting wave function vanishes identically. (Verify this in the case of two fermions.) This is the *Pauli exclusion principle*, which states that a single state can be occupied by one fermion only.

The natural choice for wave functions of a single free particle are plane wave states. In order to incorporate the spin, we have "spin-up states"

$$\phi_{\boldsymbol{p}\uparrow}(\boldsymbol{r}, \sigma) = \begin{cases} \frac{1}{\sqrt{V}}e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} & \text{if } \sigma = \frac{1}{2} \\ 0 & \text{if } \sigma = -\frac{1}{2} \end{cases} \tag{10}$$

and "spin-down states"

$$\phi_{\boldsymbol{p}\downarrow}(\boldsymbol{r}, \sigma) = \begin{cases} 0 & \text{if } \sigma = \frac{1}{2} \\ \frac{1}{\sqrt{V}}e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar} & \text{if } \sigma = -\frac{1}{2} \end{cases}. \tag{11}$$

Here the wave vector $\boldsymbol{k}$ or the momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ appears as a parameter. In order to count the states, it is most simple to require that the wave functions are periodic in a cube of volume $V = L^3$, which allows the momenta $\boldsymbol{p}$ (j_x, j_y and j_z are integers)

$$p_x = \frac{2\pi\hbar j_x}{L}, \quad p_y = \frac{2\pi\hbar j_y}{L}, \quad p_z = \frac{2\pi\hbar j_z}{L}. \tag{12}$$

We suppose that the volume V is very large. Then we can take the limit $V \to \infty$ in quantities that do not essentially depend on V.

The energy of a single-particle states is $\epsilon_p = p^2/2m$. The total energy is this summed over all occupied states

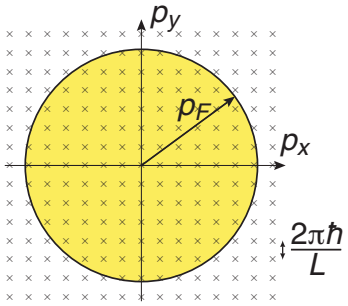
$$E(n_{\boldsymbol{p}\sigma}) = \sum_\sigma \sum_{\boldsymbol{p}} \frac{p^2}{2m} n_{\boldsymbol{p}\sigma}, \tag{13}$$

where $n_{\boldsymbol{p}\sigma} = 1$ for an occupied state and is zero otherwise.

The ground state of a system with N particles has N lowest energy single-particle states occupied and others empty. Here the maximal kinetic energy of an occupied state is called Fermi energy ϵ_F. We also define the Fermi wave vector k_F and Fermi momentum $p_F = \hbar k_F$ so that

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2 k_F^2}{2m}. \tag{14}$$

In momentum space this defines the Fermi surface ($p = p_F$). All states inside the Fermi surface ($p < p_F$) are occupied, and the ones outside are empty.



The number of particles can be calculated as

$$N = 2\sum_{p<p_F} 1 = 2\frac{\frac{4}{3}\pi p_F^3}{(2\pi\hbar/L)^3}, \tag{15}$$

where the factor 2 comes from spin. From this we get a relation between the Fermi wave vector and the particle density,

$$\frac{N}{V} = \frac{p_F^3}{3\pi^2\hbar^3}. \tag{16}$$

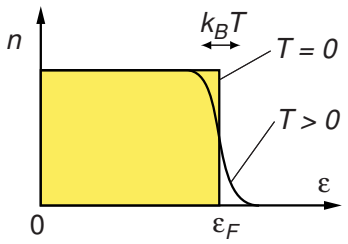
The excited states of the system consist of states where one or more fermions is excited to higher energy states. The average occupations of the states at a given temperature T is given by the Fermi-Dirac distribution

$$n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1}. \tag{17}$$

Here $\beta = 1/k_B T$ and $k_\mathrm{B} = 1.38\times10^{-23}$ J/K is Boltzmann's constant, which is needed to express the temperature T in Kelvins, and μ is the chemical potential. At $T = 0$ the system is in its ground state,

$$n(\epsilon) = \begin{cases} 1 & \text{for } \epsilon < \mu \\ 0 & \text{for } \epsilon > \mu. \end{cases} \tag{18}$$

with $\mu = \epsilon_F$. When $T > 0$, the occupation $n(\epsilon)$ gets rounded so that the change from $n \approx 1$ to $n \approx 0$ takes place in the energy interval $\approx k_\mathrm{B}T$.



Next we calculate the specific heat of the ideal Fermi gas at low temperatures. The average energy is given by

$$E = \sum_\sigma \sum_{\boldsymbol{p}} \epsilon_p n(\epsilon_p) = \frac{2}{(2\pi\hbar/L)^3}\int \epsilon_p n(\epsilon_p) d^3p$$
$$= \frac{8\pi}{(2\pi\hbar/L)^3}\int_0^\infty p^2 \epsilon_p n(\epsilon_p) dp. \tag{19}$$

Changing $\epsilon = p^2/2m$ as the integration variable we get

$$\frac{E}{V} = \frac{\sqrt{2m^3}}{\pi^2\hbar^3}\int_0^\infty \epsilon^{3/2} n(\epsilon) d\epsilon$$
$$= \int_0^\infty g(\epsilon)\epsilon\, n(\epsilon) d\epsilon. \tag{20}$$

In the second line we have expressed the same result by defining a density of states $g(\epsilon) = m\sqrt{2m\epsilon}/\pi^2\hbar^3$.

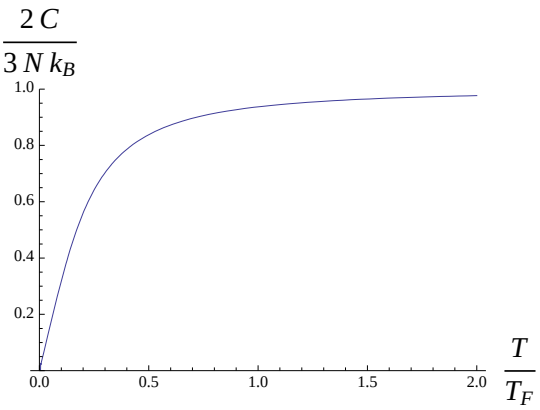
The specific heat is now obtained as the derivative of energy

$$C = \frac{\partial E(T,V,N)}{\partial T}. \tag{21}$$

In order eliminate μ appearing in the distribution function (17) one has to simultaneously satisfy

$$\frac{N}{V} = \int_0^\infty g(\epsilon) n(\epsilon) d\epsilon. \tag{22}$$

The result calculated with Mathematica is shown below.



At high temperatures $T \gg T_F = \epsilon_F/k_B$ the Fermi gas behaves like classical gas, where the average energy per particle is $3k_BT/2$, known as the equipartition theorem, and this gives the specific heat $3k_B/2$ per particle. We see that at lower temperatures $T < T_F$ the specific heat is reduced. This can be understood that only the particles with energies close to the Fermi surface can be excited. Those further than energy k_BT from the Fermi surface cannot be excited, and thus do not contribute to specific heat. At temperatures $T \ll T_F$ the specific heat is linear in T:

$$C = \frac{\pi^2}{3} g(\epsilon_F) k_B^2 T + O(T^2). \tag{23}$$

We see that the linear term is determined by the density of states at the Fermi surface

$$g(\epsilon_F) = \frac{m p_F}{\pi^2\hbar^3}. \tag{24}$$

(For detailed derivation see Ashcroft-Mermin, Solid state physics.)

Liquid ^{3}He

Helium has two stable isotopes, ^{4}He and ^{3}He. The former is by far more common in naturally occurring helium. It is a boson since in the ground state both the two electrons have total spin zero, and also the nuclear spin is zero. It has a lot of interesting properties that could be discussed, but here we concentrate on the other isotope. ^{3}He is a fermion because the nuclear spin is one half, $s = 1/2$. Studies of ^{3}He were started as it became available in larger quantities in the nuclear age after world war II as the decay product of tritium. The two isotopes of helium are the only substances that remain liquid even at the absolute zero of temperature.

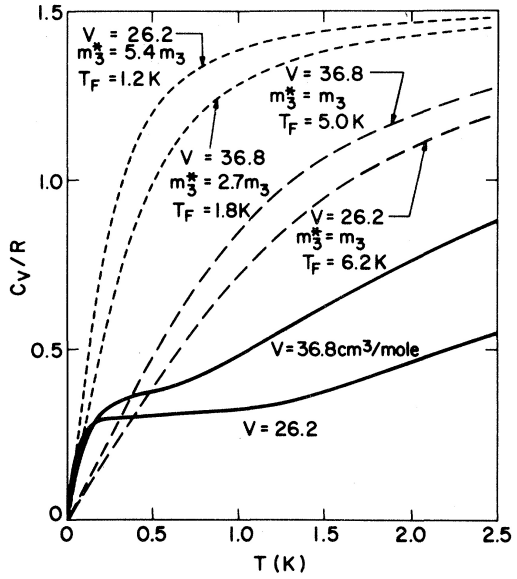
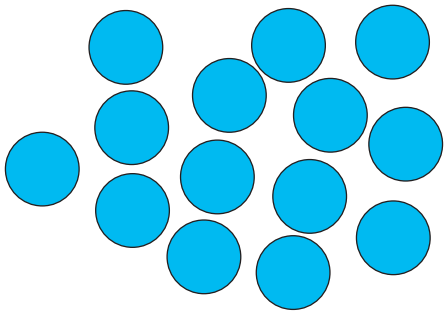


Figure: the specific heat of liquid ^{3}He at two different densities [D.S. Greywall, Phys. Rev. B **27**, 2747 (1983)].

We see that at low temperatures, the specific heat is linear in temperature. This resembles the ideal gas discussed above, but is quite puzzling since the atoms in a liquid are more like hard balls continuously touching each other!



Also, quantitative comparison of the slope with the ideal Fermi gas gives that the measured slope is by factor 2.7 larger.

3. Construction of the theory

Landau's idea

The experiment above raises the following idea. Could it be possible that low temperature liquid ^{3}He would effectively be like an ideal gas? This was the problem Landau started thinking. He had to answer the following questions

- How could dense helium atoms behave like an ideal gas?
- If there is explanation to the first question, how one can understand the difference by 2.7 in the density of states?
- If previous questions have positive answers, are any other modifications needed compared to the ideal gas?

An obvious problem with the ideal gas wave function (8) is that there is too little correlation between the locations $\boldsymbol{r}_k$ of the particles. The Pauli principle prohibits for two particles with the same spin to occupy the same location, but there is no such restriction for particles with opposite spins. Thus it is equally likely to find two opposite-spin particles just at the same place than at any other places in the ideal gas wave function (8).

Weak interactions

As a first attempt to answer the questions, consider point-like particles (instead of real ^{3}He atoms). In an ideal gas the particles fly straight trajectories without ever colliding. If we now allow some small size for the particles, they will collide with each other.

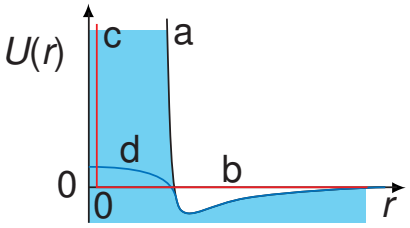


Figure: illustration of various particle-particle potentials $U(r)$: (a) the potential between two ^{3}He atoms, (b) ideal gas potential $U \equiv 0$, (c) potential used in scattering approach, (d) potential used in perturbation theory.

We have to consider two particles with momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ colliding and leaving with momenta $\boldsymbol{p}_1'$ and $\boldsymbol{p}_2'$. In such a process the momentum and energy has to be conserved,

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2' \tag{25}$$

$$\epsilon_1 + \epsilon_2 = \epsilon_1' + \epsilon_2'. \tag{26}$$

At least qualitatively the collision rate can be calculated using the golden rule

$$\Gamma = \frac{2\pi}{\hbar} \sum_f |\langle f|H_{\rm int}|i\rangle|^2 \delta(E_f - E_i). \tag{27}$$

We see that the rate is proportional to the number of available final states f. Consider specifically the case of filled Fermi sphere plus one particle at energy $\epsilon_1 > \epsilon_F$. We wish to estimate the allowed final states when particle 1 collides with any particle inside the Fermi sphere, $\epsilon_2 < \epsilon_F$. The final state has to have two particles outside the Fermi sphere ($\epsilon_1' > \epsilon_F$, $\epsilon_2' > \epsilon_F$) since the Pauli principle forbids all states inside. We see that the number final states gets very small when the initial particle is close to the Fermi energy, namely both ϵ_2 and ϵ_1' have to be chosen in an energy shell of thickness $\propto \epsilon_1 - \epsilon_F$. This means that the final states are limited by factor $\propto (\epsilon_1 - \epsilon_F)^2$. Thus the scattering of low energy particles is indeed suppressed and thus resembles the one in an ideal gas.

But ^{3}He atoms are not point particles, rather they touch each other continuously. Thus for one particle to move, the others must give the way. This is one of the hardest problems in many body theory even today, but one can get some idea of what happens with a model: instead of true ^{3}He-^{3}He interaction potential, one assumes a weak potential, whose effect can be calculated using quantum-mechanical perturbation theory. We will skip this calculation here (see Landau-Lifshitz). The result is that the excitation spectrum remains qualitatively similar as in free Fermi gas but there is a shift in energies. Consider specifically the case, already mentioned above, of a filled Fermi sphere plus one particle at momentum $\boldsymbol{p}$, with $p > p_F$. This excited state of the ideal gas corresponds to the excitation energy

$$\epsilon_p - \epsilon_F = \frac{p^2}{2m} - \epsilon_F \approx \frac{p_F}{m}(p - p_F), \tag{28}$$

where the approximation is good if p is not far from the Fermi surface ($p - p_F \ll p_F$). The effect of the weak interactions is now that the excitation energy still is linear in $p - p_F$, but the coefficient is no more p_F/m. It is customary to write the new excitation energy in the form

$$\epsilon_p - \epsilon_F = \frac{p_F}{m^*}(p - p_F), \tag{29}$$

where we have defined the *effective mass* m^*. Note that the Fermi momentum p_F is not changed, equation (16) still remains valid. With the new dispersion relation (29) we get a new density of states

$$g(\epsilon_F) = \frac{m^* p_F}{\pi^2 \hbar^3}. \tag{30}$$

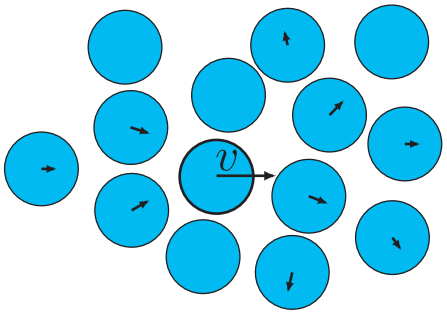
This is determined by the effective mass m^*, not the bare particle mass m as for ideal gas (24). We now see that weak interactions can explain that the specific heat coefficient (23) differs from its ideal gas value. However, the theory is valid for small perturbations, say 10%, and thus is insufficient to explain the factor 2.7.

Quasiparticles

Landau now made the following assumptions. i) Even for strong interactions, the excitation spectrum remains as in (29). Such excitations are called *quasiparticles*: they develop continuously from single-particle excitations when the interactions are "turned on", but they consist of correlated motion of the whole liquid. ii) The quasiparticles have long life time at low energy, like in the scattering approximation above.

It should be noticed that Landau's theory is phenomenological. At this stage it has one parameter, m^*, whose value is unknown theoretically, but can be obtained from experiments.

Although the detailed structure of the quasiparticle remained undetermined, we can develop a qualitative picture with a model. Consider a spherical object moving in otherwise stationary liquid. The details of this model are discussed in the appendix. The main result is that associated with the moving object, there is momentum in the fluid in the same direction. In the literature this is sometimes called "back flow", but I find this name misleading. Rather it should be called "forward flow"or that the moving object drags with itself part of the surrounding fluid. Thinking now that the total momentum of the quasiparticle is fixed, this means that switching on the interactions slows the original fermion down, since part of the momentum goes into the surrounding fluid and less is left for the original fermion.



The same picture is obtained by quantum mechanical analysis. In order to get the velocity of the quasiparticle, we have to form a localized wave packet. This travels with the group velocity. Based on the dispersion relation (29) the group velocity is

$$v_{\text{group}} = \frac{dE_p}{dp} = \frac{p_F}{m^*}. \tag{31}$$

This means that the momentum of the original fermion in the interacting system is $mv_{\text{group}}\hat{\boldsymbol{p}} = (m/m^*)\boldsymbol{p}$, i.e. the original fermion contributes fraction m/m^* of the momentum $\boldsymbol{p}$ and the fraction $1 - m/m^*$ is contributed by other fermions surrounding the original one. The velocity of the quasiparticle (31) is known as the *Fermi velocity*

$$v_F = \frac{p_F}{m^*}. \tag{32}$$

Quasiparticle interactions

Thus far we have arrived at the picture that the low energy properties of a Fermi liquid can be understood as an ideal gas with the difference that the effective mass m^*

appears instead of the particle mass m. In the following we show that this cannot be the whole story, and one more ingredient has to be added in order to arrive at a consistent theory.

A general requirement of any physical theory is that the predictions of the theory should be independent of the coordinate system chosen. In the present case, one has to pay attention to Galilean invariance. That means that the physics should be the same in two coordinate frames that move at constant velocity with respect to each other. To be specific consider a coordinate system O, and a second coordinate system O$'$ that moves with velocity $\boldsymbol{u}$ as seen in the frame O. We assume to study a system of N particles (interacting or not) of mass m. If the total momentum of this system in O$'$ is $\boldsymbol{P}'$, then the momentum seen in frame O has to be $\boldsymbol{P} = \boldsymbol{P}' + Nm\boldsymbol{u}$. Now the Galilean invariance requires that if one determines the state of the system in O$'$ at fixed total momentum $\boldsymbol{P}'$, it is the same as one would do in O with momentum $\boldsymbol{P}$.

The ideal gas obviously satisfies Galilean invariance. However, when we replace the particle mass m by m^* in (29), the Galilean invariance is broken. The cure for this problem is that we have to allow interactions between the quasiparticles. Thus we rewrite (29) into the form

$$\epsilon_{\boldsymbol{p}} - \epsilon_F = \frac{p_F}{m^*}(p - p_F) + \delta\epsilon_{\boldsymbol{p}}, \tag{33}$$

$$\delta\epsilon_{\boldsymbol{p}} = \frac{1}{V}\sum_\sigma \sum_{\boldsymbol{p}'} f(\boldsymbol{p},\boldsymbol{p}')(n_{\boldsymbol{p}'} - n^{(0)}_{\boldsymbol{p}'}). \tag{34}$$

Here $n_{\boldsymbol{p}}$ is the distribution of the quasiparticles, $n^{(0)}_{\boldsymbol{p}} = \Theta(p_F - p)$ is the distribution function in the ground state, where $\Theta(x)$ is the step function ($\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$). The function $f(\boldsymbol{p},\boldsymbol{p}')$ describes the interaction energy between two quasiparticles having momenta $\boldsymbol{p}$ and $\boldsymbol{p}'$.

The first thing to notice is that when only one or a few quasiparticles are excited, the interaction term in (34) is negligible since $n_{\boldsymbol{p}} - n^{(0)}_{\boldsymbol{p}} \approx 0$. In this case the excitation energy $\epsilon_{\boldsymbol{p}}$ (34) reduces to its the previous expression (29).

Consider next an uniformly displaced Fermi sphere, $n_{\boldsymbol{p}} = n^{(0)}_{\boldsymbol{p}-m\boldsymbol{u}} = \Theta(p_F - |\boldsymbol{p} - m\boldsymbol{u}|)$. This is the stationary ground state in the O$'$ frame. In order to the theory to be Galilean invariant, the excitation energy must be changed from (29) to the ideal gas value (28) at the displaced Fermi surface. As a formula

$$\frac{p_F}{m}m\hat{\boldsymbol{p}}\cdot\boldsymbol{u} = \frac{p_F}{m^*}m\hat{\boldsymbol{p}}\cdot\boldsymbol{u} + \frac{1}{V}\sum_\sigma\sum_{\boldsymbol{p}'} f(\boldsymbol{p},\boldsymbol{p}')(n^{(0)}_{\boldsymbol{p}'-m\boldsymbol{u}} - n^{(0)}_{\boldsymbol{p}'}). \tag{35}$$

We see that this could not be satisfied without the interaction term. In order to work (35) further, we need to study $f(\boldsymbol{p},\boldsymbol{p}')$. Because of spherical symmetry, it can depend only on the relative directions of $\boldsymbol{p}$ and $\boldsymbol{p}'$, i.e. $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p, p')$. Futher since we are interested only in quasiparticles close to the Fermi surface, we approximate $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p, p') \approx f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}', p_F, p_F)$. This means that f depends only on the angle between $\boldsymbol{p}$ and $\boldsymbol{p}'$, i.e. $f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')$. In addition, it is conventional to define $F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = g(\epsilon_F)f(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}')$. Such a function can be expanded in Legendre polynomials

$$F(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}') = \sum_{l=0}^{\infty} F^s_l P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}'), \tag{36}$$

where $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, etc. Using these we can now reduce the requirement (35) to

$$\frac{m^*}{m} = 1 + \frac{F^s_1}{3}. \tag{37}$$

We have arrived at the result that in order to have $m^* \neq m$, we also should include an interaction between the quasiparticles of the form of the F^s_1 term in (36). The other interaction terms with coefficients F^s_l are not required for internal consistency of the theory but some of them appear as a result of perturbation theory. In order to make general phenomenology, they all should be retained.

Until now we have not considered the fermion spin except that they produced factors of two. In magnetic field the system becomes spin polarized, and this is no more sufficient. In the general case we have to consider the spin as a fully quantum object. This means that we have to replace $n_{\boldsymbol{p}}$ by a 2×2 density matrix in spin space. All the previous analysis can be generalized to include the spin dependence. This means that also the quasiparticle energy $\epsilon_{\boldsymbol{p}}$ becomes a 2×2 matrix. Equation (33) has to be generalized to

$$\epsilon_{\eta\nu}(\boldsymbol{p}) - \epsilon_F\delta_{\eta\nu} = \frac{p_F}{m^*}(p - p_F)\delta_{\eta\nu} + \frac{1}{V}\sum_{\boldsymbol{p}'}\sum_{\alpha,\beta} f_{\eta\alpha,\nu\beta}(\boldsymbol{p},\boldsymbol{p}')[n_{\alpha\beta}(\boldsymbol{p}') - n^{(0)}(\boldsymbol{p}')\delta_{\alpha\beta}], \tag{38}$$

where

$$\frac{f_{\eta\alpha,\nu\beta}(\boldsymbol{p},\boldsymbol{p}')}{g(\epsilon_F)} = \sum_{l=0}^{\infty}(F^s_l\delta_{\eta\alpha}\delta_{\nu\beta} + F^a_l\boldsymbol{\sigma}_{\eta\alpha}\cdot\boldsymbol{\sigma}_{\nu\beta})P_l(\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}'). \tag{39}$$

We see that including the spin dependence the theory has two sets of parameters, F^s_l and F^a_l with $l = 0, 1, \ldots$. (Here s denotes symmetric and a antisymmetric.) One of these parameters F^s_1 is related to m^* by (37).

The expression for $\epsilon_{\boldsymbol{p}} - \epsilon_F$ (33) can be seen as the constant and the first order term in the expansion of $\epsilon_{\boldsymbol{p}}$ in terms of distribution $n_{\boldsymbol{p}}$. In applications these two terms generally are of the same order of magnitude even though the deviation from ground state distribution is assumed small. The reason that they are of equal magnitude is that the first term in (33) vanishes at the Fermi surface $p = p_F$ whereas the latter term remains finite. The situation resembles the one in another theory developed by Landau: the Ginzburg-Landau theory. There the free energy is expanded in the order parameter as $F = \alpha|\psi|^2 + \beta|\psi|^4$. Although the latter term is small because of the higher power in the small $|\psi|$, the two terms are comparable because the coefficient α vanishes at the critical temperature whereas β remains finite.

4. Equation of motion

The central quantity in the theory is the quasiparticle distribution function $n_{\boldsymbol{p}}(\boldsymbol{r},t)$. Its equation of motion is derived similarly as the Boltzmann equation, by calculating the total time derivative

$$\begin{aligned}\frac{dn_{\boldsymbol{p}}}{dt} &= \frac{\partial n_{\boldsymbol{p}}}{\partial t} + \dot{\boldsymbol{r}}\cdot\boldsymbol{\nabla} n_{\boldsymbol{p}} + \dot{\boldsymbol{p}}\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} \\ &= \frac{\partial n_{\boldsymbol{p}}}{\partial t} + \frac{\partial \epsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}}\cdot\boldsymbol{\nabla} n_{\boldsymbol{p}} - \boldsymbol{\nabla}\epsilon_{\boldsymbol{p}}\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}},\end{aligned} \tag{40}$$

where $\dot{\boldsymbol{r}} \equiv d\boldsymbol{r}/dt = \partial\epsilon_{\boldsymbol{p}}/\partial\boldsymbol{p}$ and $\dot{\boldsymbol{p}} \equiv d\boldsymbol{p}/dt = -\boldsymbol{\nabla}\epsilon_{\boldsymbol{p}}$. Equating this with the rate of change caused by collisions $I_{\boldsymbol{p}}$ gives the Landau-Boltzmann equation

$$\frac{\partial n_{\boldsymbol{p}}}{\partial t} + \frac{\partial \epsilon_{\boldsymbol{p}}}{\partial \boldsymbol{p}}\cdot\boldsymbol{\nabla} n_{\boldsymbol{p}} - \boldsymbol{\nabla}\epsilon_{\boldsymbol{p}}\cdot\frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}} = I_{\boldsymbol{p}}. \tag{41}$$

This differs from the ordinary Boltzmann equation that the energy $\epsilon_{\boldsymbol{p}}(\boldsymbol{r},t)$ contains, in addition to external potentials, the interaction energy $\delta\epsilon_{\boldsymbol{p}}$ (34). The collisions have to conserve the fermion number, momentum and energy. This implies on the collision term the conditions

$$\sum_{\boldsymbol{p}} I_{\boldsymbol{p}} = 0, \quad \sum_{\boldsymbol{p}} \boldsymbol{p} I_{\boldsymbol{p}} = 0, \quad \sum_{\boldsymbol{p}} \epsilon_{\boldsymbol{p}} I_{\boldsymbol{p}} = 0. \tag{42}$$

The kinetic equation equation (41) is a nonlinear equation. For most, if not all, applications a linearized form of it is sufficient. In equilibrium both $\boldsymbol{\nabla} n_{\boldsymbol{p}}$ and $\boldsymbol{\nabla}\epsilon_{\boldsymbol{p}}$ vanish, and thus in a linearized theory their multipliers in (41) can be evaluated in the equilibrium state. Thus linearization of the left hand side gives

$$\frac{\partial n_{\boldsymbol{p}}}{\partial t} + v_{\rm F}\hat{\boldsymbol{p}}\cdot\boldsymbol{\nabla}\left(n_{\boldsymbol{p}} - \frac{dn_p^{(0)}}{d\epsilon_p}\delta\epsilon_{\hat{\boldsymbol{p}}}\right) = I_{\boldsymbol{p}}, \tag{43}$$

where $dn_p^{(0)}/d\epsilon_p$ denotes the derivative of the equilibrium distribution function (17) evaluated at the unperturbed energy (29).

Once we have solved for the distribution function $n_{\boldsymbol{p}}(\boldsymbol{r},t)$, we can calculate measurable quantities. For example, the mass density and momentum density of the fluid are given by

$$\rho = \rho_0 + m\frac{2}{V}\sum_{\boldsymbol{p}}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)}), \tag{44}$$

$$\boldsymbol{J} = \frac{2}{V}\sum_{\boldsymbol{p}}\boldsymbol{p}\,(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)}). \tag{45}$$

Here the factors of 2 come from spin. The momentum density is the same as the mass current density. Because of conservation of mass and momentum, these have to obey conservation laws

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0, \tag{46}$$

$$\frac{\partial \boldsymbol{J}}{\partial t} + \boldsymbol{\nabla}\cdot\Pi = 0, \tag{47}$$

where Π is the momentum flux tensor. The expressions for such fluxes can be obtained as follows. One takes the time derivative of the corresponding density, for example ρ (44), one applies the kinetic equation (43) to the partial time derivative. Because of conservation (42), the contribution from collision integral vanishes. Thus one is left with the divergence term in (43), which allows to identify the expression for the flux, $\boldsymbol{J}$ in case of ρ. In the case of ρ, we can verify the consistency of equations (44)-(46).

Above we have considered the case of no spin-dependence. The general case is more complicated since one has to use quantum equation of motion (von Neumann equation) for the spin density matrix. This allows to obtain a kinetic equation and spin dependent observables. We give as examples the spin density and the spin current density tensor and the conservation law for spin,

$$\boldsymbol{S} = \frac{\hbar}{2}\frac{1}{V}\sum_{\boldsymbol{p}}{\rm Tr}[\boldsymbol{\sigma}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)})], \tag{48}$$

$$J^{\rm spin} = \frac{\hbar}{2m^*}\left(1 + \frac{1}{3}F_1^a\right)\frac{1}{V}\sum_{\boldsymbol{p}}\boldsymbol{p}\,{\rm Tr}[\boldsymbol{\sigma}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)})], \tag{49}$$

$$\frac{\partial \boldsymbol{S}}{\partial t} + \boldsymbol{\nabla}\cdot J^{\rm spin} = 0. \tag{50}$$

Here ${\rm Tr}(\dots)$ denotes the trace of the 2×2 matrix and $\boldsymbol{\sigma} = \hat{\boldsymbol{x}}\sigma_x + \hat{\boldsymbol{y}}\sigma_y + \hat{\boldsymbol{z}}\sigma_z$ is the vector of Pauli matrices. In the spin-dependent case we have to generalize (44) and (45) to

$$\rho = \rho_0 + m\frac{1}{V}\sum_{\boldsymbol{p}}{\rm Tr}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)}), \tag{51}$$

$$\boldsymbol{J} = \frac{1}{V}\sum_{\boldsymbol{p}}\boldsymbol{p}\,{\rm Tr}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)}). \tag{52}$$

Properties of a quasiparticle

In order to develop intuition into the formulas above, let us consider a single localized quasiparticle. It could be defined by distribution $n_{\boldsymbol{p}}(\boldsymbol{r},t)$ that differs from the ground state distribution $n_{\boldsymbol{p}}^{(0)}$ only for momentum values $\boldsymbol{p}$ around some $\boldsymbol{p}_0$, where $p_0 \approx p_F$. We normalize the distribution by

$$\int d^3r\frac{1}{V}\sum_{\boldsymbol{p}}{\rm Tr}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)}) = 1, \tag{53}$$

$$\int d^3r\frac{1}{V}\sum_{\boldsymbol{p}}{\rm Tr}[\boldsymbol{\sigma}(n_{\boldsymbol{p}} - n_{\boldsymbol{p}}^{(0)})] = \hat{\boldsymbol{z}}, \tag{54}$$

where we have chosen the spin polarization in the z direction. Applying these to equations (51) and (48) gives that the quasiparticle has mass m and spin $\frac{1}{2}\hbar\hat{\boldsymbol{z}}$. This is the same as for an additional particle in a noninteracting system. The quasiparticle propagates at the Fermi velocity (32), $\boldsymbol{v} = v_F\hat{\boldsymbol{p}}_0$. From (52) we get that the momentum of the excitation is $\boldsymbol{p}_0$. As discussed above, the additional fermion now contributes only mv_F to the momentum and the rest of the momentum $(m^* - m)v_F = \frac{1}{3}F_1^s mv_F$ comes from *forward flow* of other fermions. Now we can state also the spin-content of this cloud. Namely, equations (52) and

(49) together imply that the effective number of spin up and spin down particles in the quasiparticle are

$$n_\uparrow = 1 + \frac{1}{6}(F_1^s + F_1^a), \quad n_\downarrow = \frac{1}{6}(F_1^s - F_1^a). \tag{55}$$

That is, the total mass current in the quasiparticle carried by spin-up particles is $n_\uparrow m v_F$ and correspondingly $n_\downarrow m v_F$ is carried by spin-down particles.

5. Applications

The Fermi-liquid theory can be applied to calculate several measurable properties of a Fermi liquid. Below we list some of them. The standard applications are only briefly mentioned as they can be found in most reviews of the Fermi-liquid theory [see references mentioned in the introduction].

Specific heat. For a Fermi gas this was calculated in Eq. (23):

$$C = \frac{\pi^2}{3} g(\epsilon_F) k_B^2 T + O(T^2). \tag{56}$$

The effect of interactions is that the density of states is given by the effective mass m^* in (30).

Magnetic susceptibility. In magnetic field the Fermi liquid is spin-polarized. The magnetic susceptibility is

$$\chi = \mu_0 \mu_m^2 \frac{g(\epsilon_F)}{1 + F_0^a}, \tag{57}$$

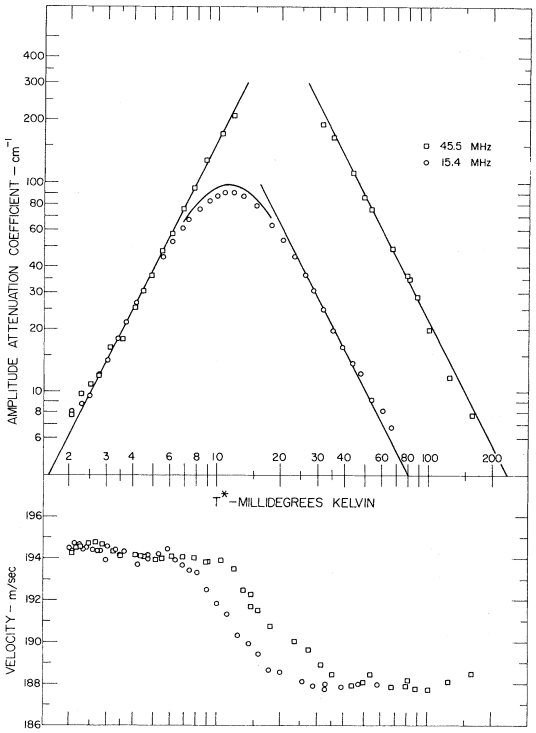
where μ_m is the magnetic moment of a particle. This result differs from a Fermi gas by the presence of F_0^a.

Sound velocity. An ordinary (longitudinal) sound wave can propagate in a Fermi liquid. The velocity of the wave is

$$c = v_F \sqrt{\frac{1}{3}(1 + F_0^s)(1 + \frac{1}{3}F_1^s)}. \tag{58}$$

Once F_1^s is extracted from measurement of the specific heat, measuring the sound velocity can give the value of F_0^s. In ^{3}He F_0^s ranges from 9 to 90 depending on pressure. The large value reflects the strong repulsion of the atoms upon compression.

Zero sound. Ordinary sound depends on collisions that keep the fluid elements in their local equilibrium state. In a Fermi liquid the collisions get rare at low temperatures, which leads to strong damping of the ordinary sound. A new type of propagating wave, called zero sound, can appear at low temperatures. Depending on the interaction parameters, it can propagate in total absence of collisions. The velocity of the wave depends in a complicated way on the interaction parameters. In ^{3}He its velocity is rather close to the velocity of the ordinary sound because of the large value of F_0^s.

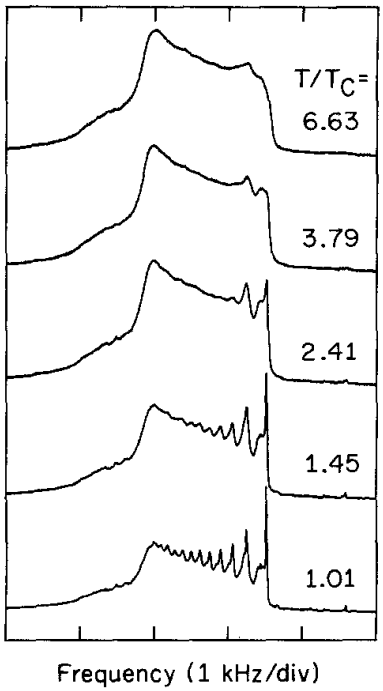


[W.R. Abel, A.C. Anderson and J.C. Wheatley, Phys. Rev. Lett. **17**, 74 (1966).]

Spin waves. Consider a local deviation of spin density from its equilibrium value. In a Fermi gas this relaxes by diffusion of the particles, called spin diffusion. In a Fermi liquid the interactions intervene so that instead of diffusion, propagating waves of spin density can be generated. One particularly interesting case is the following. Consider static magnetic field $\boldsymbol{H} = H\hat{\boldsymbol{z}}$ and the magnetization vector $\boldsymbol{M}$. For $M^+ = M_x + iM_y$ we can write the equation

$$iD\nabla^2 M^+ + \Omega M^+ = i\frac{\partial M^+}{\partial t}. \tag{59}$$

At high temperature D is real and this is a simple diffusion equation with an additional term $-i\Omega M^+$ coming from precession at the Larmor frequency $\Omega = \gamma H$. At low temperature D is imaginary, and Eq. (59) becomes the Schrödinger equation, where Ω is the potential. Thus as a function of temperature one changes continuously from a dissipative diffusion equation to a conservative wave equation. The standing waves in a potential well caused by inhomogeneous $\boldsymbol{H}(\boldsymbol{r})$ nicely show up in the experiment by Candela et al [J. Low Temp. Phys. **63**, 369 (1986)].



Kinetic coefficients: viscosity, thermal conductivity, spin diffusion. The general method to calculate the thermal conductivity, for example, is to solve the Landau-Boltzmann equation (41). For that one has to take a closer look at the collision term $I_{\boldsymbol{p}}$. A general form for it is

$$\begin{aligned} I_{\boldsymbol{p}} = -\sum_{\boldsymbol{p}_1,\boldsymbol{p}',\boldsymbol{p}'_1} & w(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}'_1) \\ & \times[n_{\boldsymbol{p}} n_{\boldsymbol{p}_1}(1-n_{\boldsymbol{p}'})(1-n_{\boldsymbol{p}'_1}) \\ & -(1-n_{\boldsymbol{p}})(1-n_{\boldsymbol{p}_1})n_{\boldsymbol{p}'}n_{\boldsymbol{p}'_1}]. \end{aligned} \tag{60}$$

Here $w(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}'_1)$ is the transition rate between the different quasiparticle states, and it has to conserve energy and momentum. The latter factor in Eq. (60) takes into account the effect of the occupation of the quasiparticle states. The problem is that $w(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}'_1)$ is not well known. In some simple cases one can still find an analytic solution of Landau-Boltzmann equation (41) with the collision term (60). This happens when calculating the kinetic coefficients, viscosity, thermal conductivity and spin diffusion coefficient. In the results, some averages over $w(\boldsymbol{p},\boldsymbol{p}_1;\boldsymbol{p}',\boldsymbol{p}'_1)$ appear [BP]. However, in most cases this is too complicated. A more practical but approximate approach is to use a relaxation time approximation. Its simplest form is

$$I(n_{\boldsymbol{p}}) = -\frac{n_{\boldsymbol{p}} - n^{\mathrm{l.e.}}_{\boldsymbol{p}}}{\tau}, \tag{61}$$

where τ is the relaxation time. Here $n^{\mathrm{l.e.}}_{\boldsymbol{p}}$ is the local equilibrium distribution, which equals the equilibrium distribution (17) in local rest frame of the fluid and corresponds to the same density and energy as the local $n_{\boldsymbol{p}}(\boldsymbol{r},t)$. In this model the calculation of the kinetic coefficients is simple. For example, viscosity is given by

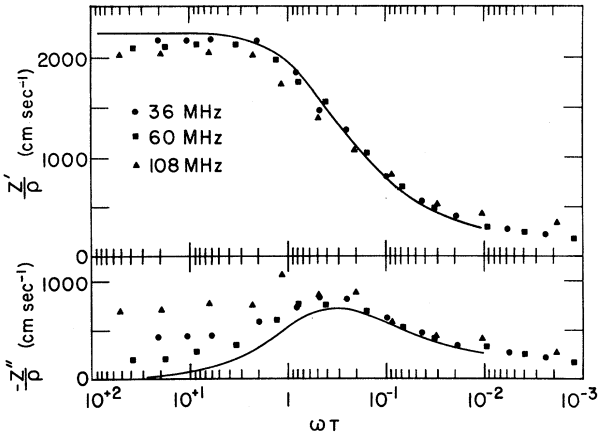
$$\eta = \frac{1}{5} n_F v_F p_F \tau \tag{62}$$

where $n_F = \rho/m$ is the number density of fermions.

Acoustic impedance. The analytical methods to solve the Landau-Boltzmann equation were developed to high sophistication in 1960's. Besides the exact calculation of the kinetic coefficients, another such problem was the acoustic impedance first solved by I. L. Bekarevich and I. M. Khalatnikov [Sov. Phys. JETP **12**, 1187 (1961)] and further developed by E. G. Flowers and R. W. Richardson [Phys. Rev. B **17**, 1238 (1978)]. Consider a semi-infinite Fermi liquid bounded by a planar wall. Suppose small oscillations of the wall (either transverse or perpendicular to the wall). The acoustic impedance Z is defined as the ratio of the force on the liquid $\boldsymbol{F}$ divided by the velocity of the wall $\boldsymbol{u}$,

$$\boldsymbol{F} = Z\boldsymbol{u}, \tag{63}$$

With harmonic time dependence $\propto \exp(-i\omega t)$, the impedance is complex valued, $Z = Z' + iZ''$. Here Z' gives the dissipation and Z'' the mass of the fluid coupled to oscillation.

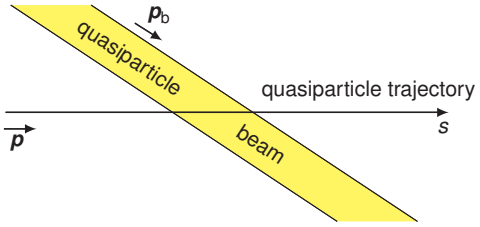


Experimental data from P.R. Roach and J.B. Ketterson [Phys. Rev. Lett. **36**, 736 (1976)]. Figure: E. G. Flowers, R. W. Richardson and S.J. Willamson [Phys. Rev. Lett **37**, 309 (1976)].

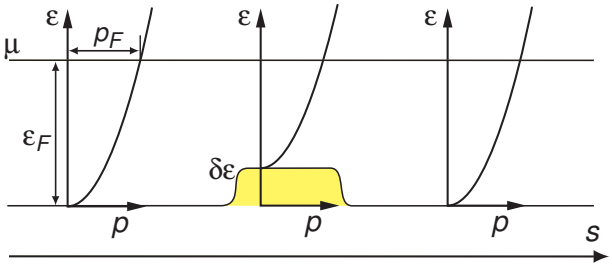
5.1 Mechanical forces

Besides the planar semi-infinite geometry, we can consider other oscillating bodies in a Fermi liquid. Special cases are vibrating wires, and two parallel plates at a finite distance. Let us consider generally how the forces arise in a Fermi liquid.

Let us first consider the case of liquid in a ground state except that there is a directed beam of quasiparticles at momentum $\boldsymbol{p}_0$ Let us study the effect of the beam on a quasiparticle trajectory that crosses the beam. We consider such a low intensity beam that the collisions can be neglected.



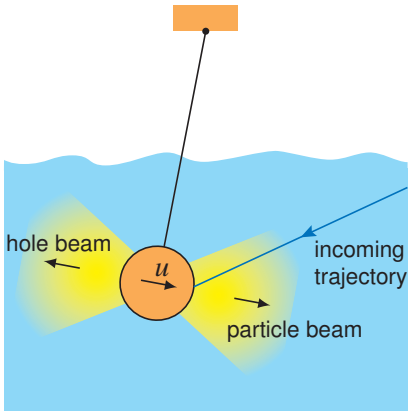
In a Fermi gas, a quasiparticle on the trajectory would not react to the beam. In a Fermi liquid it experiences the potential change $\delta\epsilon$ (34) caused by the beam. In the simple case that F_0^s is the only relevant Fermi-liquid parameter, the potential $\delta\epsilon = [F_0/2N(0)]\delta n$ is determined by the particle density in the beam.



A basic assumption of the Fermi-liquid theory is that the potential is small compared to the Fermi energy ϵ_F, i.e. $\delta\epsilon \ll \epsilon_F$. This means that a quasiparticle is slightly decelerated when it enters the beam and it is accelerated back when it leaves the crossing region. In the energy point of view (figure above), the quasiparticle flies at constant energy $\epsilon \approx \mu$ and the potential is effectively compensated by depletion of fermions with the same momentum direction in the crossing region.

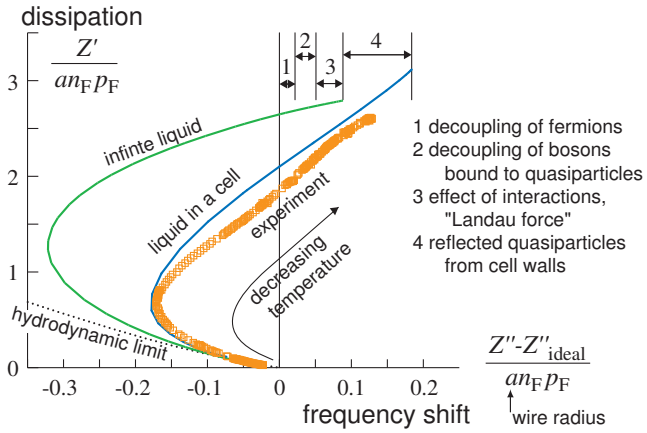
Let us now consider the case that the intensity of the quasiparticle beam is varying in time. This means that the potential seen on the crossing trajectory changes. This changes the number of particles stored in the crossing region, and thus leads to emission of particle or hole like quasiparticles from the crossing region. This takes place on all crossing trajectories.

The effect is similar as sailing in an ocean. Usually one is not interested how deep the water is, as long as it is sufficiently deep. The situation is different if the ocean floor is changing in time: one creates a tsunami.



In the case of a vibrating wire, the motion of the wire

generates a beam of quasiparticles. This beam interacts with the quasiparticles that are coming to the wire. In case of negative F_0, which is the case in ^{3}He-^{4}He mixture, this increases the restoring force. Thus the oscillation frequency is increased by the effect of Fermi-liquid interactions. In order to determine this quantitatively, one has to solve the Landau-Boltzmann equation (43).



Experimental data: J. Martikainen, J. Tuoriniemi, T. Knuuttila, and G. Pickett, J. Low Temp. Phys. **126**, 139 (2002), theory: T. Virtanen and E.T., Phys. Rev. Lett. **106**, 055301 (2011).

Effect of Fermi-liquid interactions in semi-infinite fluid: see acoustic impedance (above). In Fermi liquid film: J. Kuorelahti and E.T., poster in QFS2015.

6. Generalizations

Landau's theory was originally proposed of a pure, non-superfluid Fermi liquid. Various generalizations are possible.

The theory can be generalized to the presence of condensed bosons [I. M. Khalatnikov, Sov. Phys. JETP **28**, 1014 (1969); E.V. Thuneberg and T.H. Virtanen, Phys. Rev. B **83**, 245137 (2011)]. The main application of this theory is to liquid mixtures of ^{4}He and ^{3}He. However, it can also clarify some conceptual points of Landau's theory. In particular, the relation (37) between m^* and F_1^s makes some formulas of the theory to appear as trivial. In fermion-boson mixtures (37) has to be generalized and thus may lead to better appreciation of the theory.

The Fermi-liquid theory can be generalized to the superfluid phase by taking into account the pairing of fermions to form Cooper pairs. The theoretical basis of the resulting *Fermi-liquid theory of superfluidity* has been discussed by J.W. Serene and D. Rainer [Phys. Rep. **101**, 221 (1983)]. This applies in particular to the superfluid phases of liquid ^{3}He. The theory and its results form a vast field, too large to be discussed here. We simply state that essentially all quantities of the superfluid state are affected by the Fermi-liquid interaction parameters. Let us mention here our recent work, where the Fermi-liquid interactions lead to a qualitative change of quasiparticle states in a vortex core [M. A. Silaev, E. V. Thuneberg and M. Fogelström, arXiv:1505.02136].

The Fermi-liquid theory can also be generalized other fermion systems than ^{3}He. The most notable case is the conduction electrons of metals. It applies to both the normal conducting and the superconducting phases. In metals the presence of the crystal lattice leads to complications and additional features compared to a translationally invariant Fermi system. Another important difference to uncharged system is the electric charge of the electron.

All the above considered three-dimensional Fermi liquid. Landau's theory can also be applied to interacting fermions limited to move in two dimensions. I one dimensions a different behavior is expected.

7. Derivation and limitations

Landau developed the Fermi-liquid theory intuitively as the low energy expansion of a Fermi system. What it means that the Fermi-liquid state will appear if nothing else happens. The possible alternative is that if the interactions are strong enough, the particles will be localized and the system becomes solid instead of a liquid state. An interesting approach to this is presented by R. Shankar [Rev. Mod. Phys. **66**, 129 (1994)]. He uses the renormalization-group method and finds that zooming into the neighborhood of the Fermi surface, the Fermi-liquid model appears as one alternative possibility.

The Fermi-liquid theory should appear as the low energy limit of a microscopic many-body theory. This means that one should be able to calculate the Fermi-liquid parameters starting from a microscopic theory. At the moment it seems that the quantum-fluids community is using experimental values of the parameters. My hope is that some day the many-body community would calculate more accurate values of the parameters.

8. Conclusion

In this introduction to the Landau's Fermi-liquid theory we have concentrated on its starting point. Especially we have tried to clarify the nature of the quasiparticle. Out of the many applications of the theory to ^{3}He and elsewhere we have only briefly mentioned a few. We hope that understanding the elements of the theory gives good starting point to study the known applications, or finding new ones.

Appendix: hydrodynamic model of a quasiparticle

The starting point is Euler's equation and the equation of continuity

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} &= -\frac{1}{\rho}\boldsymbol{\nabla}p \\ \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) &= 0, \end{aligned} \tag{64}$$

where p is now the pressure. We assume a small velocity so that the nonlinear term can be dropped and assume incompressible fluid, ρ = constant. The boundary condition on the surface of a sphere of radius a is $\boldsymbol{n}\cdot\boldsymbol{v} = \boldsymbol{n}\cdot\boldsymbol{u}$, where $\boldsymbol{u}$ is the velocity of the sphere and $\boldsymbol{n}$ the surface normal. We assume that far from the sphere $\boldsymbol{v} \to 0$. The velocity can be represented using the potential $\boldsymbol{v} = \nabla\chi$ where $\chi = -a^3\boldsymbol{u}\cdot\boldsymbol{r}/2r^3$. By the Euler equation the pressure $p = -\rho\dot{\chi}$, and the force exerted by the sphere on the fluid

$$\boldsymbol{F} = \int d\boldsymbol{a}\, p = \frac{2\pi a^3\rho}{3}\dot{\boldsymbol{u}} \tag{65}$$

is proportional to the acceleration $\dot{\boldsymbol{u}}$. Therefore, associated with a moving sphere, there is momentum in the fluid

$$\boldsymbol{p} = \frac{2\pi a^3\rho}{3}\boldsymbol{u} \tag{66}$$

corresponding to half of the fluid displaced by the sphere and moving in the same direction and at the same velocity as the sphere. Note that this may differ from the total momentum of the fluid, which is not essential here, and is undetermined in the present limit of unlimited fluid.