Lecture 3: Probability Distributions and Statistics of SDEs

Simo Särkkä

Aalto University

Contents

Statistics of SDEs

- Consider the stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\,\mathrm{d}t + \mathbf{L}(\mathbf{x}, t)\,\mathrm{d}\beta.$$

- Each $\mathbf{x}(t)$ is random variable, and we denote its probability density with $p(\mathbf{x}, t)$ – or sometimes with $p(\mathbf{x}(t))$.
- The probability density is solution to a *partial differential equation* called Fokker–Planck–Kolmogorov equation.
- The mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$ are solutions of certain ordinary differential equations (with a catch. . .).
- For linear SDEs we get quite explicit results.

Fokker-Planck-Kolmogorov PDE: Derivation [1/5]

- Let $\phi(\mathbf{x})$ be an arbitrary twice differentiable function.
- The Itô differential of $\phi(\mathbf{x}(t))$ is, by the Itô formula, given as follows:

$$
\begin{aligned}
\mathrm{d}\phi &= \sum_i \frac{\partial \phi}{\partial x_i} f_i(\mathbf{x}, t)\,\mathrm{d}t + \sum_i \frac{\partial \phi}{\partial x_i} [\mathbf{L}(\mathbf{x}, t)\,\mathrm{d}\beta]_i \\
&\quad + \frac{1}{2} \sum_{ij} \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) [\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\,\mathrm{d}t.
\end{aligned}
$$

- Taking expectations and formally dividing by $\mathrm{d}t$ gives the following equation, which we will transform into FPK:

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{E}[\phi]}{\mathrm{d}t} &= \sum_i \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} f_i(\mathbf{x}, t) \right] \\
&\quad + \frac{1}{2} \sum_{ij} \mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) [\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij} \right].
\end{aligned}
$$

Fokker-Planck-Kolmogorov PDE: Derivation [2/5]

- The left hand side can now be written as follows:

$$\begin{aligned}\frac{\mathrm{d}\,\mathsf{E}[\phi]}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int \phi(\mathbf{x})\,p(\mathbf{x},t)\,\mathrm{d}\mathbf{x} \\ &= \int \phi(\mathbf{x})\,\frac{\partial p(\mathbf{x},t)}{\partial t}\,\mathrm{d}\mathbf{x}.\end{aligned}$$

- Recall the multidimensional integration by parts formula

$$\int_C \frac{\partial u(\mathbf{x})}{\partial x_i}\,v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial C} u(\mathbf{x})\,v(\mathbf{x})\,n_i\,\mathrm{d}S - \int_C u(\mathbf{x})\,\frac{\partial v(\mathbf{x})}{\partial x_i}\,\mathrm{d}\mathbf{x}.$$

- In this case, the boundary terms vanish and thus we have

$$\int \frac{\partial u(\mathbf{x})}{\partial x_i}\,v(\mathbf{x})\,\mathrm{d}\mathbf{x} = -\int u(\mathbf{x})\,\frac{\partial v(\mathbf{x})}{\partial x_i}\,\mathrm{d}\mathbf{x}.$$

Fokker-Planck-Kolmogorov PDE: Derivation [3/5]

- Currently, our equation looks like this:

$$\underbrace{\int \phi(\mathbf{x}) \frac{\partial p(x,t)}{\partial t} \, \mathrm{d}\mathbf{x}}_{\frac{\mathrm{d}\,\mathrm{E}[\phi]}{\mathrm{d}t}} = \sum_i \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} f_i(\mathbf{x},t)\right]$$

$$+ \frac{1}{2} \sum_{ij} \mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\right].$$

- For the first term on the right, we get via integration by parts:

$$\begin{aligned}
\mathrm{E}\left[\frac{\partial \phi}{\partial x_i} f_i(\mathbf{x},t)\right] &= \int \frac{\partial \phi}{\partial x_i} f_i(\mathbf{x},t)\, p(\mathbf{x},t) \, \mathrm{d}\mathbf{x} \\
&= -\int \phi(\mathbf{x}) \frac{\partial}{\partial x_i} [f_i(\mathbf{x},t)\, p(\mathbf{x},t)] \, \mathrm{d}\mathbf{x}
\end{aligned}$$

- We now have only one term to go.

Fokker-Planck-Kolmogorov PDE: Derivation [4/5]

- For the remaining term we use integration by parts twice, which gives

$$
\begin{aligned}
&\mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\right] \\
&= \int \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, p(\mathbf{x}, t)\; \mathrm{d}\mathbf{x} \\
&= -\int \left(\frac{\partial \phi}{\partial x_j}\right) \frac{\partial}{\partial x_i} \left\{[\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, p(\mathbf{x}, t)\right\}\; \mathrm{d}\mathbf{x} \\
&= \int \phi(\mathbf{x}) \frac{\partial^2}{\partial x_i\, \partial x_j} \left\{[\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, p(\mathbf{x}, t)\right\}\; \mathrm{d}\mathbf{x}
\end{aligned}
$$

Fokker-Planck-Kolmogorov PDE: Derivation [5/5]

- Our equation now looks like this:

$$\int \phi(\mathbf{x}) \frac{\partial p(\mathbf{x},t)}{\partial t} \, \mathrm{d}\mathbf{x} = -\sum_i \int \phi(\mathbf{x}) \frac{\partial}{\partial x_i}[f_i(\mathbf{x},t)\, p(\mathbf{x},t)] \, \mathrm{d}\mathbf{x}$$
$$+ \frac{1}{2} \sum_{ij} \int \phi(\mathbf{x}) \frac{\partial^2}{\partial x_i \, \partial x_j} \{[\mathbf{L}(\mathbf{x},t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\, p(\mathbf{x},t)\} \, \mathrm{d}\mathbf{x}$$

- This can also be written as

$$\int \phi(\mathbf{x}) \Big[\frac{\partial p(\mathbf{x},t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i}[f_i(\mathbf{x},t)\, p(\mathbf{x},t)]$$
$$- \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} \{[\mathbf{L}(\mathbf{x},t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\, p(\mathbf{x},t)\} \Big] \, \mathrm{d}\mathbf{x} = 0.$$

- But the function is $\phi(\mathbf{x})$ arbitrary and thus the term in the brackets must vanish $\Rightarrow$ Fokker–Planck–Kolmogorov equation.

Fokker-Planck-Kolmogorov PDE

Fokker–Planck–Kolmogorov equation

The probability density $p(\mathbf{x}, t)$ of the solution of the SDE

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\, dt + \mathbf{L}(\mathbf{x}, t)\, d\boldsymbol{\beta},$$

solves the Fokker–Planck–Kolmogorov partial differential equation

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[f_i(x, t)\, p(\mathbf{x}, t)] + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} \left\{ [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, p(\mathbf{x}, t) \right\}.$$

- In physics literature it is called the Fokker–Planck equation.
- In stochastics it is the forward Kolmogorov equation.

Fokker-Planck-Kolmogorov PDE: Example 1

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[f_i(x,t)\,p(\mathbf{x},t)] + \frac{1}{2}\sum_{ij}\frac{\partial^2}{\partial x_i\,\partial x_j}\left\{[\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}(\mathbf{x},t)]_{ij}\,p(\mathbf{x},t)\right\}.$$

FPK Example: Diffusion equation

Brownian motion can be defined as solution to the SDE

$$\mathrm{d}x = \mathrm{d}\beta.$$

If we set the diffusion constant of the Brownian motion to be $q = 2\,D$, then the FPK reduces to the diffusion equation

$$\frac{\partial p}{\partial t} = D\,\frac{\partial^2 p}{\partial x^2}$$

Fokker-Planck-Kolmogorov PDE: Example 2

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[f_i(x,t)\,p(\mathbf{x},t)] + \frac{1}{2}\sum_{ij} \frac{\partial^2}{\partial x_i\,\partial x_j}\left\{[\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\,p(\mathbf{x},t)\right\}.$$

FPK Example: Benes SDE

The FPK for the SDE

$$\mathrm{d}x = \tanh(x)\;\mathrm{d}t + \mathrm{d}\beta$$

can be written as

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left(\tanh(x)\,p(x,t)\right) + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2} \\ &= (\tanh^2(x) - 1)\,p(x,t) - \tanh(x)\,\frac{\partial p(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}.\end{aligned}$$

Mean and Covariance of SDE [1/2]

- Using Itô formula for $\phi(\mathbf{x}, t)$, taking expectations and dividing by $\mathrm{d}t$ gives

$$\frac{\mathrm{d}\,\mathrm{E}[\phi]}{\mathrm{d}t} = \mathrm{E}\left[\frac{\partial \phi}{\partial t}\right] + \sum_i \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} f_i(x, t)\right] + \frac{1}{2}\sum_{ij} \mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x}, t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(x, t)]_{ij}\right]$$

- If we select the function as $\phi(\mathbf{x}, t) = x_u$, then we get

$$\frac{\mathrm{d}\,\mathrm{E}[x_u]}{\mathrm{d}t} = \mathrm{E}\left[f_u(\mathbf{x}, t)\right]$$

- In vector form this gives the differential equation for the mean:

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \mathrm{E}\left[\mathbf{f}(\mathbf{x}, t)\right]$$

Mean and Covariance of SDE [2/2]

- If we select $\phi(\mathbf{x}, t) = x_u \, x_v - m_u(t) \, m_v(t)$, then we get differential equation for the components of covariance:

$$
\begin{aligned}
&\frac{\mathrm{d}\,\mathrm{E}[x_u \, x_v - m_u(t) \, m_v(t)]}{\mathrm{d}t} \\
&= \mathrm{E}\left[(x_v - m_v(t)) \, f_u(x, t)\right] + \mathrm{E}\left[(x_u - m_u(t)) \, f_v(x, t)\right] \\
&\quad + [\mathbf{L}(\mathbf{x}, t) \, \mathbf{Q} \, \mathbf{L}^{\mathsf{T}}(\mathbf{x}, t)]_{uv}.
\end{aligned}
$$

- The final mean and covariance differential equations are

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} &= \mathrm{E}\left[\mathbf{f}(\mathbf{x}, t)\right] \\
\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} &= \mathrm{E}\left[\mathbf{f}(\mathbf{x}, t) \, (\mathbf{x} - \mathbf{m})^{\mathsf{T}}\right] + \mathrm{E}\left[(\mathbf{x} - \mathbf{m}) \, \mathbf{f}^{\mathsf{T}}(\mathbf{x}, t)\right] \\
&\quad + \mathrm{E}\left[\mathbf{L}(\mathbf{x}, t) \, \mathbf{Q} \, \mathbf{L}^{\mathsf{T}}(\mathbf{x}, t)\right]
\end{aligned}
$$

- Note that the expectations are w.r.t. $p(\mathbf{x}, t)$!

Mean and Covariance of SDE: Notes

- To solve the equations, we need to know $p(\mathbf{x}, t)$, the solution to the FPK.
- In linear-Gaussian case the first two moments indeed characterize the solution.
- Useful starting point for Gaussian approximations of SDEs.

Mean and Covariance of SDE: Example

Example (Moments of an Ornstein–Uhlenbeck process)

The Ornstein–Uhlenbeck process is

$$\mathrm{d}x = -\lambda\, x \,\mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = x_0,$$

We have $f(x) = -\lambda\, x$ and thus the differential equations for the mean and variance are thus given as

$$\begin{aligned}\frac{\mathrm{d}m}{\mathrm{d}t} &= -\lambda\, m \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= -2\lambda\, P + q,\end{aligned}$$

with $m(0) = x_0$, $P(0) = 0$. The whole state distribution is

$$p(x,t) \triangleq p(x(t)) = \mathrm{N}(x(t) \mid m(t), P(t)).$$

Higher Order Moments

- It is also possible to derive differential equations for the higher order moments of SDEs.
- But with state dimension n, we have n^3 third order moments, n^4 fourth order moments and so on.
- Recall that a given scalar function $\phi(x)$ satisfies

$$\frac{\mathrm{d}\,\mathsf{E}[\phi(x)]}{\mathrm{d}t} = \mathsf{E}\left[\frac{\partial \phi(x)}{\partial x}\, f(x)\right] + \frac{q}{2}\,\mathsf{E}\left[\frac{\partial^2 \phi(x)}{\partial x^2}\, L^2(x)\right].$$

- If we apply this to $\phi(x) = x^n$:

$$\frac{\mathrm{d}\,\mathsf{E}[x^n]}{\mathrm{d}t} = n\,\mathsf{E}[x^{n-1}\, f(x,t)] + \frac{q}{2}\, n\,(n-1)\;\mathsf{E}[x^{n-2}\, L^2(x)]$$

- This, in principle, is an equation for higher order moments.
- To actually use this, we need to use moment closure methods.

Mean and covariance of linear SDEs

- Consider a linear stochastic differential equation

$$\mathrm{d}\mathbf{x} = \mathbf{F}(t)\,\mathbf{x}(t)\;\mathrm{d}t + \mathbf{u}(t)\;\mathrm{d}t + \mathbf{L}(t)\;\mathrm{d}\beta(t), \quad \mathbf{x}(t_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{P}_0).$$

- The mean and covariance equations are now given as

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{m}(t)}{\mathrm{d}t} &= \mathbf{F}(t)\,\mathbf{m}(t) + \mathbf{u}(t) \\
\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &= \mathbf{F}(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^{\mathsf{T}}(t) + \mathbf{L}(t)\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}(t),
\end{aligned}$$

- The general solutions are given as

$$\begin{aligned}
\mathbf{m}(t) &= \boldsymbol{\Psi}(t, t_0)\,\mathbf{m}(t_0) + \int_{t_0}^{t} \boldsymbol{\Psi}(t, \tau)\,\mathbf{u}(\tau)\;\mathrm{d}\tau \\
\mathbf{P}(t) &= \boldsymbol{\Psi}(t, t_0)\,\mathbf{P}(t_0)\,\boldsymbol{\Psi}^{\mathsf{T}}(t, t_0) \\
&\quad + \int_{t_0}^{t} \boldsymbol{\Psi}(t, \tau)\,\mathbf{L}(\tau)\,\mathbf{Q}(\tau)\,\mathbf{L}^{\mathsf{T}}(\tau)\,\boldsymbol{\Psi}^{\mathsf{T}}(t, \tau)\;\mathrm{d}\tau
\end{aligned}$$

Mean and covariance of LTI SDEs

- In LTI SDE case

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}(t)\,\mathrm{d}t + \mathbf{L}\,\mathrm{d}\beta(t),$$

  we have similarly

$$\begin{aligned}\frac{\mathrm{d}\mathbf{m}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{m}(t)\\ \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^{\mathsf{T}} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\end{aligned}$$

- The explicit solutions are

$$\begin{aligned}\mathbf{m}(t) &= \exp(\mathbf{F}\,(t - t_0))\,\mathbf{m}(t_0)\\ \mathbf{P}(t) &= \exp(\mathbf{F}\,(t - t_0))\,\mathbf{P}(t_0)\,\exp(\mathbf{F}\,(t - t_0))^{\mathsf{T}}\\ &\quad + \int_{t_0}^{t} \exp(\mathbf{F}\,(t - \tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,\exp(\mathbf{F}\,(t - \tau))^{\mathsf{T}}\,\mathrm{d}\tau.\end{aligned}$$

LTI SDEs: Matrix fractions

- Let the matrices $\mathbf{C}(t)$ and $\mathbf{D}(t)$ solve the LTI differential equation

$$\begin{pmatrix} \mathrm{d}\mathbf{C}(t)/\,\mathrm{d}t \\ \mathrm{d}\mathbf{D}(t)/\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} \mathbf{F} & \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \\ \mathbf{0} & -\mathbf{F}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{C}(t) \\ \mathbf{D}(t) \end{pmatrix}$$

- Then $\mathbf{P}(t) = \mathbf{C}(t)\,\mathbf{D}^{-1}(t)$ solves the differential equation

$$\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} = \mathbf{F}\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^\mathsf{T} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}$$

- Thus we can solve the covariance with matrix exponential as well:

$$\begin{pmatrix} \mathbf{C}(t) \\ \mathbf{D}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \mathbf{F} & \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \\ \mathbf{0} & -\mathbf{F}^\mathsf{T} \end{pmatrix} t \right\} \begin{pmatrix} \mathbf{C}(t_0) \\ \mathbf{D}(t_0) \end{pmatrix}.$$

Markov properties of SDEs

- Definition of a Markov process:

$$p(\mathbf{x}(t) \mid \mathcal{X}_s) = p(\mathbf{x}(t) \mid \mathbf{x}(s)), \qquad \text{for all } t \geq s.$$

  where

$$\mathscr{X}_t = \{\mathbf{x}(\tau) \ : \ 0 \leq \tau \leq t\}.$$

- All Itô processes are Markov processes.
- $p(\mathbf{x}(t) \mid \mathbf{x}(s))$ is the transition density of the process.
- The transition density is also a solution to the Fokker–Planck–Kolmogorov equation.
- Finite-dimensional distributions can be constructed as

$$p(\mathbf{x}(t_0), \mathbf{x}(t_1), \ldots, \mathbf{x}(t_n)) = p(\mathbf{x}(t_0)) \prod_{i=1}^{n} p(\mathbf{x}(t_i) \mid \mathbf{x}(t_{i-1})).$$

Transition Densities of LTI SDEs

- The solution of LTI SDE starting from $\mathbf{x}(s) \sim \mathrm{N}(\mathbf{m}(s), \mathbf{P}(s))$:

$$
\begin{aligned}
\mathbf{m}(t) &= \exp(\mathbf{F}\,(t-s))\,\mathbf{m}(s) \\
\mathbf{P}(t) &= \exp(\mathbf{F}\,(t-s))\,\mathbf{P}(s)\,\exp(\mathbf{F}\,(t-s))^{\mathsf{T}} \\
&\quad + \int_s^t \exp(\mathbf{F}\,(t-\tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,\exp(\mathbf{F}\,(t-\tau))^{\mathsf{T}}\,\mathrm{d}\tau.
\end{aligned}
$$

- Starting exactly at $\mathbf{x}(s)$ corresponds to $\mathbf{m}(s) = \mathbf{x}(s)$, $\mathbf{P}(s) = \mathbf{0}$.
- Thus we have

$$
p(\mathbf{x}(t) \mid \mathbf{x}(s)) = \mathrm{N}(\mathbf{x}(t) \mid \mathbf{m}(t \mid s), \mathbf{P}(t \mid s)),
$$

where

$$
\begin{aligned}
\mathbf{m}(t \mid s) &= \exp(\mathbf{F}\,(t-s))\,\mathbf{x}(s) \\
\mathbf{P}(t \mid s) &= \int_s^t \exp(\mathbf{F}\,(t-\tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,\exp(\mathbf{F}\,(t-\tau))^{\mathsf{T}}\,\mathrm{d}\tau.
\end{aligned}
$$

Transition Densities of LTI SDEs (cont.)

- Let $p(\mathbf{x}(t_{k+1}) \mid \mathbf{x}(t_k))$ and $\Delta t_k = t_{k+1} - t_k$, which gives:

$$p(\mathbf{x}(t_{k+1}) \mid \mathbf{x}(t_k))$$
$$= \mathrm{N}\left(\mathbf{x}(t_{k+1}) \mid \exp(\mathbf{F}\,\Delta t_k)\,\mathbf{x}(t_k), \int_0^{\Delta t_k} \exp(\mathbf{F}\,(\Delta t_k - \tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \exp(\mathbf{F}\,(\Delta t_k - \tau))^\mathsf{T}\,\mathrm{d}\tau\right)$$

- This is now equivalent to a discrete-time system

$$\mathbf{x}(t_{k+1}) = \mathbf{A}_k\,\mathbf{x}(t_k) + \mathbf{q}_k, \qquad \mathbf{q}_k \sim \mathrm{N}(\mathbf{0}, \Sigma_k)$$

where

$$\mathbf{A}_k = \exp(\mathbf{F}\,\Delta t_k)$$
$$\Sigma_k = \int_0^{\Delta t_k} \exp(\mathbf{F}\,(\Delta t_k - \tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \exp(\mathbf{F}\,(\Delta t_k - \tau))^\mathsf{T}\,\mathrm{d}\tau.$$

Transition Densities of LTI SDEs: Example

Example (Discretized Wiener velocity model)

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = w(t).$$

In more rigorous Itô SDE form this model can be written as

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathrm{d}t + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{L}} \mathrm{d}\beta(t),$$

We now get (note that $\mathbf{F}^2 = 0$):

$$\mathbf{A} = \exp\left(\mathbf{F}\,\Delta t\right) = \mathbf{I} + \mathbf{F}\,\Delta = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix}$$

$$\Sigma = \int_0^{\Delta t} \begin{pmatrix} 1 & \Delta t - \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 1 & \Delta t - \tau \\ 0 & 1 \end{pmatrix}^{\mathsf{T}} \mathrm{d}\tau = \begin{pmatrix} \frac{1}{3}\Delta t^3 & \frac{1}{2}\Delta t^2 \\ \frac{1}{2}\Delta t^2 & \Delta t \end{pmatrix} q.$$

Summary

- The probability density of SDE solution $\mathbf{x}(t)$ solves the Fokker–Planck–Kolmogorov (FKP) *partial differential equation*.
- The mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$ of the solution solve a pair of *ordinary differential equations*.
- In non-linear case, the expectations in the mean and covariance equations cannot be solved without knowing the whole probability density.
- For higher moment moments we can derive (theoretical) differential equations as well—can be approximated with moment closure.
- SDEs are Markov processes and can be characterised via transition densities.
- For linear SDEs, we can solve the probability density, transition densities, and all the moments explicitly.