Notes on Quaternions

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1 Basic Properties

Definition 1.1 (Quaternion). Quaternion *is a* $\mathbb{R}^4$ *vector of the form*

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}. \tag{1}$$

what makes quaternions different from ordinary 4-dimensional vectors are the algebraic properties to be defined in this section. In order to write the definitions of the basic algebraic operations of quaternions in compact form the components are often divided to scalar part s *and vector part* $\mathbf{v} \in \mathbb{R}^3$ *as follows:*

$$\mathbf{q} = \begin{pmatrix} s \\ \mathbf{v} \end{pmatrix}. \tag{2}$$

The quaternion $\mathbf{q}$ *can be also considered as a 4-component extended complex number of the form*

$$\mathbf{q} = q_1 + q_2\,\mathrm{i} + q_3\,\mathrm{j} + q_4\,\mathrm{k}, \tag{3}$$

where the imaginary components $\mathrm{i}, \mathrm{j}, \mathrm{k}$ *have the computation rules*

$$\mathrm{i} * \mathrm{i} = -1 \quad \mathrm{i} * \mathrm{j} = \mathrm{k} \quad \mathrm{j} * \mathrm{i} = -\mathrm{k} \quad \textit{etc.} \tag{4}$$

However, in this document the vector presentation is preferred to the representation as extended complex number.

Definition 1.2 (Quaternion conjugate). *The conjugate* $\mathbf{q}^*$ *of quaternion* $\mathbf{q}$ *is defined as*

$$\mathbf{q}^* = \begin{pmatrix} q_1 \\ -q_2 \\ -q_3 \\ -q_4 \end{pmatrix} \tag{5}$$

Definition 1.3 (Quaternion sum). *The sum of quaternions is the same as the vector sum:*

$$\mathbf{q} + \mathbf{p} = \begin{pmatrix} q_1 + p_1 \\ q_2 + p_2 \\ q_3 + p_3 \\ q_4 + p_4. \end{pmatrix} \tag{6}$$

Definition 1.4 (Quaternion product). *The product of two quaternions*

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \tag{7}$$

is defined as

$$\mathbf{q} * \mathbf{p} = \begin{pmatrix} q_1\,p_1 - q_2\,p_2 - q_3\,p_3 - q_4\,p_4 \\ q_2\,p_1 + q_1\,p_2 - q_4\,p_3 + q_3\,p_4 \\ q_3\,p_1 + q_4\,p_2 + q_1\,p_3 - q_2\,p_4 \\ q_4\,p_1 - q_3\,p_2 + q_2\,p_3 + q_1\,p_4 \end{pmatrix}. \tag{8}$$

This can be derived by writing the quaternions in extended complex form and expanding the product. The quaternion product is not commutative $\mathbf{q} * \mathbf{p} \neq \mathbf{p} * \mathbf{q}$.

Remark 1.1 (Partitioned quaternion product). *The product of two quaternions*

$$\mathbf{q} = \begin{pmatrix} s_q \\ \mathbf{v}_q \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} s_p \\ \mathbf{v}_p \end{pmatrix} \tag{9}$$

can be also computed as

$$\mathbf{q} * \mathbf{p} = \begin{pmatrix} s_q\,s_p - < \mathbf{v}_q, \mathbf{v}_p > \\ s_q\,\mathbf{v}_p + s_p\,\mathbf{v}_q + \mathbf{v}_q \times \mathbf{v}_p \end{pmatrix} \tag{10}$$

where $< \mathbf{v}_q, \mathbf{v}_p >$ *denotes the dot product and* $\mathbf{v}_q \times \mathbf{v}_p$ *denotes the cross product of vectors* $\mathbf{v}_q$ *and* $\mathbf{v}_p$.

Remark 1.2 (Matrix representation of quaternion product). *The product of two quaternions*

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \qquad \mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} \tag{11}$$

can be also written as matrix product:

$$\mathbf{q} * \mathbf{p} = \mathbf{Q}(\mathbf{q})\,\mathbf{p} = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \tag{12}$$

or equivalently as

$$\mathbf{q} * \mathbf{p} = \mathbf{P}(\mathbf{p})\,\mathbf{q} = \begin{pmatrix} p_1 & -p_2 & -p_3 & -p_4 \\ p_2 & p_1 & p_4 & -p_3 \\ p_3 & -p_4 & p_1 & p_2 \\ p_4 & p_3 & -p_2 & p_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}. \tag{13}$$

Definition 1.5 (Quaternion length). *The length of a quaternion is the same as the norm of the corresponding vector:*

$$|\mathbf{q}| = \sqrt{\mathbf{q} * \mathbf{q}^*} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}. \tag{14}$$

Definition 1.6 (Quaternion inverse). *The inverse of quaternion* $\mathbf{q}$ *with respect to the quaternion product is given as*

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{|\mathbf{q}|^2}. \tag{15}$$

Theorem 1.1 (Quaternion exponential). *The exponential of quaternion*

$$\mathbf{q} = \begin{pmatrix} s \\ \mathbf{v} \end{pmatrix}. \tag{16}$$

is given as

$$\exp(\mathbf{q}) = \exp(s) \begin{pmatrix} \cos(|\mathbf{v}|) \\ \frac{\mathbf{v}}{|\mathbf{v}|}\sin(|\mathbf{v}|) \end{pmatrix} \tag{17}$$

Theorem 1.2 (Quaternion logarithm). *The logarithm of quaternion*

$$\mathbf{q} = \begin{pmatrix} s \\ \mathbf{v} \end{pmatrix}. \tag{18}$$

is given as

$$\ln(\mathbf{q}) = \begin{pmatrix} \ln(|\mathbf{q}|) \\ \frac{\mathbf{v}}{|\mathbf{v}|}\arccos\left(\frac{s}{|\mathbf{q}|}\right). \end{pmatrix} \tag{19}$$

Theorem 1.3 (Quaternion power). *Quaternion power can be defined as*

$$\mathbf{q}^{\mathbf{p}} = \exp(\ln(\mathbf{q}) * \mathbf{p}). \tag{20}$$

Note that if $\mathbf{p}$ *is in fact scalar, then the power is*

$$\mathbf{q}^{\mathbf{p}} = \exp(\ln(\mathbf{q})\,p). \tag{21}$$

2 Quaternion Representations of Rotations

Definition 2.1 (Unit quaternion). *If quaternion is of the form*

$$\mathbf{q} = \begin{pmatrix} \cos(\theta/2) \\ \mathbf{u}\,\sin(\theta/2) \end{pmatrix}, \tag{22}$$

where

- *θ is a rotation angle,*
- *$\mathbf{u}$ is a 3-dimensional unit vector.*

then $\mathbf{q}$ is a unit quaternion. *Unit quaternions can be used for representing a rotation of angle θ around the axis $\mathbf{u}$. The unit quaternion also has unit length $|\mathbf{q}| = 1$.*

Definition 2.2 (Augmented point). *In order to clean up the notation we shall define an augmented point, which is a quaternion $\overline{\mathbf{r}}$ formed from point $\mathbf{r} \in \mathbb{R}^3$ as follows:*

$$\overline{\mathbf{r}} = \begin{pmatrix} 0 \\ \mathbf{r} \end{pmatrix} \tag{23}$$

This augmentation is always denoted by the line over the vector.

Theorem 2.1 (Rotation). *Rotation of vector $\mathbf{r} \in \mathbb{R}^3$ an angle θ around unit vector $\mathbf{u} \in \mathbb{R}^3$ can be computed as*

$$\overline{\mathbf{r}}' = \mathbf{q} * \overline{\mathbf{r}} * \mathbf{q}^*, \tag{24}$$

where $\overline{\mathbf{r}}$ is the augmented original vector, $\overline{\mathbf{r}}'$ denotes the aumented rotated vector and $\mathbf{q}$ is a unit quaternion defined as in (22)*.*

Remark 2.1 (Inverse rotation). *The inverse rotation can be obtained by conjugating the rotation quaternion, that is:*

$$\overline{\mathbf{r}} = \mathbf{q}^* * \overline{\mathbf{r}}' * \mathbf{q}, \tag{25}$$

Theorem 2.2 (Conversion to direction cosine matrix). *Rotations can be equivalently represented in terms of direction cosine matrix $\mathbf{C}$ as follows:*

$$\mathbf{r}' = \mathbf{C}\,\mathbf{r}. \tag{26}$$

A unit quaternion can be converted into equivalent direction cosine matrix as follows:

$$\mathbf{C} = \begin{pmatrix} (q_1^2 + q_2^2 - q_3^2 - q_4^2) & 2(q_2\,q_3 - q_1\,q_4) & 2(q_2\,q_4 + q_1\,q_3) \\ 2(q_2\,q_3 + q_1\,q_4) & (q_1^2 - q_2^2 + q_3^2 - q_4^2) & 2(q_3\,q_4 - q_1\,q_2) \\ 2(q_2\,q_4 - q_1\,q_3) & 2(q_3\,q_4 + q_1\,q_2) & (q_1^2 - q_2^2 - q_3^2 + q_4^2) \end{pmatrix}. \tag{27}$$

3 Quaternion differential equations

Theorem 3.1 (Time behavior of unit quaternion). *The time evolution of time varying quaternion with angular velocity* $\boldsymbol{\omega}(t) = \begin{pmatrix}\omega_1(t) & \omega_2(t) & \omega_3(t)\end{pmatrix}^T$ *is given by the differential equation*

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{1}{2}\mathbf{q} * \overline{\boldsymbol{\omega}}, \tag{28}$$

where $\overline{\boldsymbol{\omega}}$ *is the augmented angular velocity vector.*

Remark 3.1 (Matrix representations of time behavior). *The equation* (28) *can be also written in form*

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \mathbf{F}_q(\boldsymbol{\omega})\,\mathbf{q}, \tag{29}$$

where

$$\mathbf{F}_q(\boldsymbol{\omega}) = \frac{1}{2}\begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{30}$$

Theorem 3.2 (Solution with constant angular velocity). *If the angular velocity* $\boldsymbol{\omega}$ *is time independent constant, then the solution to the equation* (28) *with given initial conditions* $\mathbf{q}(t_0)$ *can be written as*

$$\mathbf{q}(t) = \boldsymbol{\Phi}_q(t_0, t; \boldsymbol{\omega})\,\mathbf{q}(t_0), \tag{31}$$

where the transition matrix $\boldsymbol{\Phi}_q(t_0, t; \boldsymbol{\omega})$ *is given as*

$$\begin{aligned}\boldsymbol{\Phi}_q(t_0, t; \boldsymbol{\omega}) &= \exp\left[\mathbf{F}_q(\boldsymbol{\omega})\,\Delta t\right] \\ &= \cos\left(|\boldsymbol{\omega}|\,\Delta t/2\right)\,\mathbf{I} + 2\frac{\sin\left(|\boldsymbol{\omega}|\,\Delta t/2\right)}{|\boldsymbol{\omega}|}\,\mathbf{F}_q(\boldsymbol{\omega}),\end{aligned} \tag{32}$$

where $\Delta t = t - t_0$.

Theorem 3.3 (Solution in quaternion form). *The solution* (31) *can be expressed as the quaternion product*

$$\mathbf{q}(t) = \mathbf{q}(t_0) * \boldsymbol{\phi}_q(t_0, t; \boldsymbol{\omega}), \tag{33}$$

where

$$\begin{aligned}\boldsymbol{\phi}_q(t_0, t; \boldsymbol{\omega}) &= \exp(\overline{\boldsymbol{\omega}}\,\Delta t/2) \\ &= \begin{pmatrix} \cos\left(|\boldsymbol{\omega}|\,\Delta t/2\right) \\ \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}\sin\left(|\boldsymbol{\omega}|\,\Delta t/2\right)\end{pmatrix}\end{aligned} \tag{34}$$

and $\Delta t = t - t_0$.

4 Quaternion interpolation

Definition 4.1 (SLERP).

$$\mathbf{p}(t) = (\mathbf{q}_1 * \mathbf{q}_0^{-1})^t * \mathbf{q}_0. \tag{35}$$

References

[Titterton and Weston, 1997] Titterton, D. H. and Weston, J. L. (1997). *Strapdown Inertial Navigation Technology*. Peter Pregrinus Ltd.