Supplemental Material for Review of "On The L^4 Convergence of Particle Filters with General Importance Distributions"

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Abstract

This supplemental material contains the details for the proof of the convergence theorem. This document is provided to reviewers to understand the steps of the proof, which could be time consuming based on the manuscript alone.

1 Definitions and notation

Suppose ν is a measure on $\mathbb{R}^n$, and ϕ is a function $\mathbb{R}^n \mapsto \mathbb{R}$ then we define

$$(\nu, \phi) = \int \phi(x)\, d\nu(x). \tag{1}$$

Let A be an operator and let's denote the corresponding kernel as $A(y, x)$, that is,

$$A\,\phi = \int A(y,x)\,\phi(y)\,dy. \tag{2}$$

We define a special norm for a function ϕ as

$$||\phi||_{t,4} = \max\Big\{1, \quad (\pi_{s|s}, |\phi|^4)^{1/4}, \quad s = 0, 1, \ldots, t\Big\}, \tag{3}$$

We will also use the more conventional supremum norm

$$||h|| = \sup_x |h(x)| \tag{4}$$

as well as the corresponding operator norm

$$||A|| = \sup_h \{\, ||A\,h|| : ||h|| = 1\}. \tag{5}$$

Some properties of the operator norm are

$$\begin{aligned} ||A\,h|| &\le ||A||\,||h|| \\ ||A\,B|| &\le ||A||\,||B|| \\ |(\nu, Ah)| &\le ||A||\,|(\nu, h)|. \end{aligned}$$

In the proof of the theorem, some basic inequalities that are needed are [1]:

- Minkowski inequality: Let X and Y be random variables and $p \geq 1$ then

$$\left(\mathbb{E}\left[\left|X+Y\right|^p\right]\right)^{1/p} \leq \left(\mathbb{E}\left[\left|X\right|^p\right]\right)^{1/p} + \left(\mathbb{E}\left[\left|Y\right|^p\right]\right)^{1/p} \tag{6}$$

- Hölder's inequality: Let X and Y be random variables and $p, q \geq 1$ with $1/p + 1/q = 1$ then

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq (\mathbb{E}[|X|^p])^{1/p}(\mathbb{E}[|X|^q])^{1/q} \tag{7}$$

- Jensen's inequality: Suppose $\psi(.)$ is a convex function and X and $\psi(X)$ have finite expectation then

$$\psi(\mathbb{E}[X]) \leq \mathbb{E}[\psi(X)] \tag{8}$$

  A straightforward applications of Jensen's inequality is

$$(\mathbb{E}[|X|])^p \leq \mathbb{E}[|X|^p]$$

- Markov inequality: For any $a > 0$, $P(|X| \leq a) \leq \mathbb{E}[X]/a$.

2 Conditions

We state the conditions required for the perticle filter and the theorem to hold.

(1). H0: For any given $y_{1:s}$ we have $(\pi_{s|s-1}, g) > \gamma_s > 0$, where $s = 1, \ldots, t$.

(2). H1: The dynamic model density $f(x_t \mid x_{t-1})$, measurement likelihood $g(y_t \mid x_t)$, and the importance weight

$$\rho(x_t, x_{t-1}) = \frac{g(y_t \mid x_t)\, f(x_t \mid x_{t-1})}{q(x_t \mid x_{t-1}, y_t)}$$

are bounded. That is, there exist constants C_f, C_g, and C_ρ such that $||f|| \leq C_f$, $||g|| \leq C_g$, and $||\rho|| \leq C_\rho$, where the first norm is a operator norm and the second two are function norms w.r.t. x_t.

(3). H2: The function of interest $\phi(\cdot)$ satisfies $\sup_{x_s} |\phi(x_s)|^4 g(y_s \mid x_s) < C(y_{1:s})$.

Theorem 2.1 (The convergence theorem)**.** *Consider the general modified particle filter algorithm and suppose that the conditions H0, H1, and H2 above hold. Then*

i. For sufficiently large N, the algorithm will not run into an infinite loop in steps 2-3.

ii. Let $L_t^4(g)$ be the class of functions satisfying H2, condition (3). For any $\phi \in L_t^4(g)$, there exists a constant $C_{t|t}$, independent of N such that

$$\mathbb{E}\left[\left|(\pi_{t|t}^N, \phi) - (\pi_{t|t}, \phi)\right|^4\right] \leq C_{t|t}\frac{||\phi||_{t,4}^4}{N^2}, \tag{9}$$

3 Auxiliary Lemmas

The following Lemmas are used in proving the convergence of the particle filter. We don't prove the lemmas here, for proofs see [2].

Lemma 3.1. *Let $\{\xi_i, \quad i = 1, \ldots, N\}$ be conditionally independent random variables given σ-algebra $\mathcal{G}$ such that $\mathbb{E}[\xi_i|\mathcal{G}] = 0$ and $\mathbb{E}[|\xi_i|^4|\mathcal{G}] < \infty$. Then*

$$\mathbb{E}\left[\left|\sum_{i=1}^N \xi_i\right|^4 \,\middle|\, \mathcal{G}\right] \leq \sum_{i=1}^N \mathbb{E}[|\xi_i|^4 \mid \mathcal{G}] + \left(\sum_{i=1}^N \mathbb{E}[|\xi_i|^2 \mid \mathcal{G}]\right)^2$$

Lemma 3.2. *If $\mathbb{E}[|\xi|^p] < \infty$, then*

$$\mathbb{E}\left[\left|\xi - \mathbb{E}[\xi]\right|^p\right] \le 2^p \mathbb{E}[|\xi|^p], \quad for \quad p \ge 1$$

Lemma 3.3. *If $1 \le r_1 \le r_2$ and $\mathbb{E}[|\xi|^{r_2}] < \infty$, then*

$$(\mathbb{E}[|\xi|^{r_1}])^{1/r_1} \le (\mathbb{E}[|\xi|^{r_2}])^{1/r_2}$$

Lemma 3.4. *Let $\{\xi_i, \quad i = 1, \dots, N\}$ be conditionally independent random variables given σ-algebra $\mathcal{G}$ such that $\mathbb{E}[\xi_i|\mathcal{G}] = 0$ and $\mathbb{E}[|\xi_i|^4|\mathcal{G}] < \infty$. Then*

$$\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\xi_i\right|^4 |\mathcal{G}\right] \le \frac{2 \quad \max_{1\le i\le N}\mathbb{E}[|\xi_i|^4|\mathcal{G}]}{N^2}$$

Lemma 3.5. *Let the probability density function for the random variable η be $p(x)$ and let the probability density function for the random variable ξ be:*

$$\frac{p(x)I_A}{\int p(y)I_A \mathrm{d}y}$$

where I_A is the indicator function for a set A, such that:

$$P[\eta \in \Omega - A] \le \epsilon < 1.$$

Let ψ be a measurable function satisfying $\mathbb{E}[\psi^2(\eta)] < \infty$. Then, we have

$$|\mathbb{E}[\psi(\xi)] - \mathbb{E}[\psi(\eta)]| \le \frac{2\sqrt{\mathbb{E}[\psi^2(\eta)]}}{1-\epsilon}\sqrt{\epsilon}.$$

In the case $\mathbb{E}[\psi(\eta)] < \infty$

$$\mathbb{E}[|\psi(\xi)|] \le \frac{\mathbb{E}[|\psi(\eta)|]}{1-\epsilon}$$

4 Bayesian Filtering Equations

In [2] the Bayesian filter is formulated such that for any function $\phi(x)$ it computes the measures $\pi_{t|t-1}$ and $\pi_{t|t}$ solving the equations

$$\begin{aligned}(\pi_{t|t-1}, \phi) &= (\pi_{t-1|t-1}, f\,\phi)\\ (\pi_{t|t}, \phi) &= \frac{(\pi_{t|t-1}, \phi\, g)}{(\pi_{t|t-1}, g)}.\end{aligned} \tag{10}$$

where we have used the notation defined in (1) and (2). Let's now see where this comes from. Recall that in terms of probability densities we have the Bayesian filter

$$\begin{aligned}p(x_t \mid y_{1:t-1}) &= \int f(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\\ p(x_t \mid y_{1:t}) &= \frac{g(y_t \mid x_t)\, p(x_t \mid y_{1:t-1})}{\int g(y_t \mid x_t)\, p(x_t \mid y_{1:t-1})\, dx_t}\end{aligned} \tag{11}$$

If we multiply by $\phi(x_t)$ and integrate over x_t, we get

$$\begin{aligned}\int \phi(x_t)\, p(x_t \mid y_{1:t-1})\, dx_t &= \int \phi(x_t) \int f(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t\\ &= \int \left[\int f(x_t \mid x_{t-1})\, \phi(x_t)\, dx_t\right] p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\\ \int \phi(x_t)\, p(x_t \mid y_{1:t})\, dx_t &= \frac{\int \phi(x_t)\, g(y_t \mid x_t)\, p(x_t \mid y_{1:t-1})\, dx_t}{\int g(y_t \mid x_t)\, p(x_t \mid y_{1:t-1})\, dx_t}\end{aligned} \tag{12}$$

If we interpret f as a kernel of a linear operator, we can rewrite

$$\int \phi(x_t)\, p(x_t \mid y_{1:t-1})\, dx_t = \int f\, \phi\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1} \tag{13}$$

If we now write expectation of ϕ over ν as (ν, ϕ), as in Hu et al., and denote $\pi_{t|t-1} \triangleq p(x_t \mid y_{1:t-1})$ and $\pi_{t|t} \triangleq p(x_t \mid y_{1:t})$, these can be nicely written as

$$\begin{aligned}(\pi_{t|t-1}, \phi) &= (\pi_{t-1|t-1}, f\, \phi) \\ (\pi_{t|t}, \phi) &= \frac{(\pi_{t|t-1}, \phi\, g)}{(\pi_{t|t-1}, g)}.\end{aligned} \tag{14}$$

We need to remember meanings of all the notations, because for example, replacing $\phi\, g$ with $g\, \phi$ would be ambiquous (it would imply applying operator g to ϕ).

The boostrap filter can be seen as direct Monte Carlo implementation of these equations, because we sample from the dynamic model – which corresponds to the first equation – and then weight using the second equation. However, if we have some other importance distribution than the dynamic model, we need to modify the equations a bit to have such a similar direct Monte Carlo interpretation.

Let's now substitute the prediction step of the Bayesian filter to the update step and work out q into the equations:

$$\begin{aligned}p(x_t \mid y_{1:t}) &= \frac{g(y_t \mid x_t) \int f(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}}{\int g(y_t \mid x_t) \int f(x_t \mid x_{t-1})\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t} \\ &= \frac{\int \frac{g(y_t|x_t)\, f(x_t|x_{t-1})}{q(x_t|x_{t-1}, y_t)}\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}}{\int \int \frac{g(y_t|x_t)\, f(x_t|x_{t-1})}{q(x_t|x_{t-1}, y_t)}\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t}\end{aligned} \tag{15}$$

Let's now define

$$\rho(x_t, x_{t-1}) = \frac{g(y_t \mid x_t)\, f(x_t \mid x_{t-1})}{q(x_t \mid x_{t-1}, y_t)}$$

which thus gives

$$p(x_t \mid y_{1:t}) = \frac{\int \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}}{\int \int \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t} \tag{16}$$

If we now multiply by $\phi(x_t)$ and integrate we get

$$\int \phi(x_t)\, p(x_t \mid y_{1:t})\, dx_t = \frac{\int \int \phi(x_t)\, \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t}{\int \int \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t} \tag{17}$$

We can now write

$$\begin{aligned}&\int \int \phi(x_t)\, \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1}\, dx_t \\ &= \int \phi(x_t) \left[\int \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1} \right] dx_t \\ &= \int \phi(x_t)\, \hat{p}(x_t)\, dx_t\end{aligned} \tag{18}$$

where we have defined

$$\hat{p}(x_t) = \int \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, dx_{t-1} \tag{19}$$

In notation in terms of measures we have

$$\hat{\pi}_{t|t} = (\pi_{t-1|t-1}, \rho\, q), \tag{20}$$

where ρ is not interpreted as a linear operator kernel. This is now another (unnormalized) measure and we thus can write the expectation of ϕ over it as $(\hat{\pi}_{t|t}, \phi)$. This leads to the equation

$$(\pi_{t|t}, \phi) = \frac{((\pi_{t-1|t-1}, \rho\, q), \phi)}{((\pi_{t-1|t-1}, \rho\, q), 1)} \tag{21}$$

which thus now replaces the previous Bayesian filter formulation (14).

Note that we can also conveniently write

$$\begin{aligned} ((\pi_{t-1|t-1}, \rho\, q), \phi) &= ((\pi_{t-1|t-1}, q\,(g\, f/q)), \phi) \\ &= ((\pi_{t-1|t-1}, g\, f), \phi) \\ &= (\pi_{t-1|t-1}, f\, \phi\, g) \end{aligned} \tag{22}$$

and

$$\begin{aligned} ((\pi_{t-1|t-1}, \rho\, q), 1) &= ((\pi_{t-1|t-1}, q\,(g\, f/q)), 1) \\ &= ((\pi_{t-1|t-1}, g\, f), 1) \\ &= (\pi_{t-1|t-1}, f\, g) \end{aligned} \tag{23}$$

where f is interpreted in linear operator sense.

5 Proof of Theorem

Using the Lemmas and the the conditions stated in the preceding sub-section, we are now ready to give the proof of Theorem. The proofs for *initialization and resampling steps* are the same as in [2]. Therefore, here, we only prove the convergence of the (combined) *prediction and update steps.* Thus, we want to prove the convergence of

$$(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi) = \frac{((\pi^N_{t-1|t-1}, \rho\, q^N), \phi)}{((\pi^N_{t-1|t-1}, \rho\, q^N), 1)} - \frac{((\pi_{t-1|t-1}, \rho\, q), \phi)}{((\pi_{t-1|t-1}, \rho\, q), 1)}, \tag{24}$$

where – if the probability densities exists – we have

$$\begin{aligned} ((\pi^N_{t-1|t-1}, \rho\, q^N), \phi) &= \int \int \phi(x_t)\, \rho(x_t, x_{t-1})\, q^N(x_t \mid x_{t-1}, y_t)\, p^N(x_{t-1} \mid y_{1:t-1})\, \mathrm{d}x_{t-1}\, \mathrm{d}x_t, \\ ((\pi_{t-1|t-1}, \rho\, q), \phi) &= \int \int \phi(x_t)\, \rho(x_t, x_{t-1})\, q(x_t \mid x_{t-1}, y_t)\, p(x_{t-1} \mid y_{1:t-1})\, \mathrm{d}x_{t-1}\, \mathrm{d}x_t. \end{aligned}$$

If we define $\hat{\pi}^N_{t|t} = (\pi^N_{t-1|t-1}, \rho\, q^N)$ and $\hat{\pi}_{t|t} = (\pi_{t-1|t-1}, \rho\, q)$, then Equation (24) can be written as

$$(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi) = \frac{(\hat{\pi}^N_{t|t}, \phi)}{(\hat{\pi}^N_{t|t}, 1)} - \frac{(\hat{\pi}_{t|t}, \phi)}{(\hat{\pi}_{t|t}, 1)}, \tag{25}$$

As in [2] we will attempt to find appropriate bounds for the following terms:

$$\mathbb{E}\left[\left|(\pi^N_{t|t}, \phi) - (\pi_{t|t}, \phi)\right|^4\right] \quad \text{and} \quad \mathbb{E}[(\pi^N_{t|t}, |\phi|^4)]. \tag{26}$$

Let's do an induction assumption that there exist constants $C_{t-1|t-1}$ and $M_{t-1|t-1}$ such that

$$\mathbb{E}\left[\left|(\pi^N_{t-1|t-1}, \phi) - (\pi_{t-1|t-1}, \phi)\right|^4\right] \leq C_{t-1|t-1} \frac{||\phi||^4_{t-1,4}}{N^2}, \tag{27}$$

and

$$\mathbb{E}\left[\left|(\pi^N_{t-1|t-1}, |\phi|^4)\right|\right] \leq M_{t-1|t-1}||\phi||^4_{t-1,4}. \tag{28}$$

Let's start by considering the bounds for the following expectations

$$\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t}, \phi) - (\hat{\pi}_{t|t}, \phi)\right|^4\right] \quad \text{and} \quad \mathbb{E}\left[(\hat{\pi}^N_{t|t}, |\phi|^4)\right]. \tag{29}$$

We first start to study the boundedness of $\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t}, \phi) - (\hat{\pi}_{t|t}, \phi)\right|^4\right]$ and then we will do the same for $\mathbb{E}\left[(\hat{\pi}^N_{t|t}, |\phi|^4)\right]$. Let $\mathcal{F}_{t-1}$ be the σ-algebra generated by x^i_{t-1}. Then we can write $(\hat{\pi}^N_{t|t}, \phi) - (\hat{\pi}_{t|t}, \phi) = \Pi_1 + \Pi_2 + \Pi_3$, where

$$\Pi_1 = (\hat{\pi}^N_{t|t}, \phi) - \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[\phi(\tilde{x}^i_t)\, \rho(\tilde{x}^i_t, x_{t-1}) \mid \mathcal{F}_{t-1}], \tag{30}$$

$$\Pi_2 = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[\phi(\tilde{x}^i_t)\, \rho(\tilde{x}^i_t, x_{t-1}) \mid \mathcal{F}_{t-1}] - \frac{1}{N}\sum_{i=1}^{N}(\pi^{N,\alpha_i}_{t-1|t-1}, f\,\phi\, g), \tag{31}$$

$$\Pi_3 = \frac{1}{N}\sum_{i=1}^{N}(\pi^{N,\alpha_i}_{t-1|t-1}, f\,\phi\, g) - (\hat{\pi}_{t|t}, \phi). \tag{32}$$

Let $\bar{x}^i_t \sim (\pi^{N,\alpha_i}_{t-1|t-1}, q)$, then

$$\mathbb{E}[\phi(\bar{x}^i_t)\, \rho(\bar{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}] = (\pi^{N,\alpha_i}_{t-1|t-1}, f\,\phi\, g), \tag{33}$$

$$(\pi^N_{t-1|t-1}, f\,\phi\, g) = \frac{1}{N}\sum_{i=1}^{N}(\pi^{N,\alpha_i}_{t-1|t-1}, f\,\phi\, g). \tag{34}$$

Analogously to [2] we start by analyzing the probability of the thresholding which in the current setting corresponds to analysis of the event

$$\begin{aligned} A_t &= \{((\pi^N_{t-1|t-1}, \rho\, q), 1) \geq \gamma_t\} \\ &= \{(\pi^N_{t-1|t-1}, f\, g) \geq \gamma_t\} \end{aligned} \tag{35}$$

By (33) we have

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} \rho(\bar{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\right] = (\pi^{N,\alpha_i}_{t-1|t-1}, f\, g) \tag{36}$$

Analogously to (56) in [2] we thus get

$$
\begin{aligned}
&P\left[\frac{1}{N}\sum_{i=1}^{N}\rho(\bar{x}_s^i, x_{s-1}^i) < \gamma_t \middle| \mathcal{F}_{t-1}\right] \\
&= P\left[(\pi_{t-1|t-1}^N, f\,g) < \gamma_t\right] \\
&= P\left[(\pi_{t-1|t-1}^N, f\,g) - (\pi_{t-1|t-1}, f\,g) < \gamma_t - (\pi_{t-1|t-1}, f\,g)\right] \\
&\leq P\left[\left|(\pi_{t-1|t-1}^N, f\,g) - (\pi_{t-1|t-1}, f\,g)\right| > \left|\gamma_t - (\pi_{t-1|t-1}, f\,g)\right|\right] \\
&= P\left[\left|(\pi_{t-1|t-1}^N, f\,g) - (\pi_{t-1|t-1}, f\,g)\right|^4 > \left|\gamma_t - (\pi_{t-1|t-1}, f\,g)\right|^4\right] \\
&\leq \frac{\mathbb{E}\left[\left|(\pi_{t-1|t-1}^N, f\,g) - (\pi_{t-1|t-1}, f\,g)\right|^4\right]}{\left|\gamma_t - (\pi_{t-1|t-1}, f\,g)\right|^4} \quad \text{Markov's inequality} \\
&\leq ||f||^4 ||g||^4 \frac{\mathbb{E}\left[\left|(\pi_{t-1|t-1}^N, 1) - (\pi_{t-1|t-1}, 1)\right|^4\right]}{\left|\gamma_t - (\pi_{t-1|t-1}, f\,g)\right|^4} \\
&= ||f||^4 ||g||^4 \frac{\mathbb{E}\left[\left|(\pi_{t-1|t-1}^N, 1) - (\pi_{t-1|t-1}, 1)\right|^4\right]}{\left|\gamma_t - (\pi_{t|t-1}, g)\right|^4} \\
&\leq \frac{C_{t-1|t-1}||f||^4||g||^4}{N^2\left|\gamma_t - (\pi_{t|t-1}, g)\right|^4} \quad \text{By Equation (27)} \\
&= \frac{\tilde{C}_{\gamma_t}}{N^2} = \epsilon.
\end{aligned} \tag{37}
$$

provided that $||f||$ and $||g||$ are bounded and H0 is true.

From Equation (30), we have

$$
\begin{aligned}
&\mathbb{E}[|\Pi_1|^4|\mathcal{F}_{t-1}] \\
&= \mathbb{E}\left[\left|(\hat{\pi}^N_{t|t}, \phi) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^{N}\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\Big(\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\Big)\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&= \frac{1}{N^4}\mathbb{E}\left[\left|\sum_{i=1}^{N}\Big(\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\Big)\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&\leq \frac{1}{N^4}\sum_{i=1}^{N}\mathbb{E}\left[\left|\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right|^4 \mid \mathcal{F}_{t-1}\right] \\
&+ \frac{1}{N^4}\left(\sum_{i=1}^{N}\mathbb{E}\left[\left|\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right|^2 \mid \mathcal{F}_{t-1}\right]\right)^2 \quad \text{by lemma 3.1} \\
&\leq \frac{2^4}{N^4}\left[\sum_{i=1}^{N}\mathbb{E}\left[\left|\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1})\right|^4 \mid \mathcal{F}_{t-1}\right] + \left(\sum_{i=1}^{N}\mathbb{E}\left[\left|\phi(\tilde{x}^i_t)\,\rho(\tilde{x}^i_t, x^i_{t-1})\right|^2 \mid \mathcal{F}_{t-1}\right]\right)^2\right] \quad \text{by lemma 3.2} \\
&\leq \frac{2^4}{N^4}\left[\sum_{i=1}^{N}\frac{\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^4|\mathcal{F}_{t-1}\right]}{1-\epsilon} + \left(\sum_{i=1}^{N}\frac{\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^2|\mathcal{F}_{t-1}\right]}{1-\epsilon}\right)^2\right] \quad \text{by lemma 3.5} \\
&= \frac{2^4}{N^4}\left[\frac{1}{1-\epsilon}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^4|\mathcal{F}_{t-1}\right] + \frac{1}{(1-\epsilon)^2}\left(\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^2|\mathcal{F}_{t-1}\right]\right)^2\right] \\
&\leq \frac{2^4}{N^4}\left[\frac{1}{(1-\epsilon)^2}\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^4|\mathcal{F}_{t-1}\right] + \frac{1}{(1-\epsilon)^2}\left(\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^2|\mathcal{F}_{t-1}\right]\right)^2\right] \\
&= \frac{2^4}{N^4(1-\epsilon)^2}\left[\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^4|\mathcal{F}_{t-1}\right] + \left(\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}^i_t)\,\rho(\bar{x}^i_t, x^i_{t-1})|^2|\mathcal{F}_{t-1}\right]\right)^2\right]
\end{aligned}
$$

Provided that $\rho \leq C_\rho$, and $||f||$ and $||g||$ are bounded by some constants, we get

$$
\begin{aligned}
&\mathbb{E}[|\Pi_1|^4|\mathcal{F}_{t-1}] \\
&\leq \frac{2^4}{N^4(1-\epsilon)^2}\left[\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}^i)|^4|\mathcal{F}_{t-1}\right] + \left(\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}^i)|^2|\mathcal{F}_{t-1}\right]\right)^2\right] \\
&\leq \frac{2^4}{N^4(1-\epsilon)^2}\left[C_\rho^3\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i, x_{t-1}^i)|\mathcal{F}_{t-1}\right] + C_\rho^2\left(\sum_{i=1}^{N}\mathbb{E}\left[|\phi(\bar{x}_t^i)|^2\,\rho(\bar{x}_t^i, x_{t-1}^i)|\mathcal{F}_{t-1}\right]\right)^2\right] \\
&= \frac{2^4}{N^4(1-\epsilon)^2}\left[C_\rho^3\sum_{i=1}^{N}\left(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^4\,g\right) + C_\rho^2\left(\sum_{i=1}^{N}\left(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^2\,g\right)\right)^2\right] \quad \text{By Equation (33)} \\
&= \frac{2^4}{N^4(1-\epsilon)^2}\left[C_\rho^3 N\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g\right) + C_\rho^2\left(N\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^2\,g\right)\right)^2\right] \\
&= \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^3\frac{\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g\right)}{N^3} + C_\rho^2\frac{\left(\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^2\,g\right)\right)^2}{N^2}\right] \\
&\leq \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^3\frac{\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g\right)}{N^2} + C_\rho^2\frac{\left(\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^2\,g\right)\right)^2}{N^2}\right] \\
&\leq \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^3||f||||g||\frac{\left(\pi_{t-1|t-1}^{N}, |\phi|^4\right)}{N^2} + C_\rho^2||f||^2||g||^2\frac{\left(\left(\pi_{t-1|t-1}^{N}, |\phi|^2\right)\right)^2}{N^2}\right] \\
&\leq \frac{2^4}{(1-\epsilon)^2}\left[C_\rho^3||f||||g||\frac{\left(\pi_{t-1|t-1}^{N}, |\phi|^4\right)}{N^2} + C_\rho^2||f||^2||g||^2\frac{\left(\pi_{t-1|t-1}^{N}, |\phi|^4\right)}{N^2}\right] \quad \text{by lemma 3.3} \\
&\leq \frac{2^5\tilde{C}_\rho}{(1-\epsilon)^2}\left[\frac{\left(\pi_{t-1|t-1}^{N}, |\phi|^4\right)}{N^2}\right] \\
&\leq \frac{2^5\tilde{C}_\rho}{(1-\epsilon)^2}\left[\frac{M_{t-1|t-1}||\phi||_{t-1,4}^4}{N^2}\right] \quad \text{By Equation (28)} \\
&= \tilde{C}_{\Pi_1}\frac{||\phi||_{t-1,4}^4}{N^2}
\end{aligned} \tag{38}
$$

For Π_2 we get

$$
\begin{aligned}
\left|\Pi_2\right|^4 &= \left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\phi(\tilde{x}_t^i)\,\rho(\tilde{x}_t^i, x_{t-1}) \mid \mathcal{F}_t] - \frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t}^{N,\alpha_i}, f\,\phi\,g)\right|^4 \\
&= \left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\phi(\tilde{x}_t^i)\,\rho(\tilde{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}] - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}]\right|^4 \quad \text{By Equation (33)} \\
&= \left|\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\left[\phi(\tilde{x}_t^i)\,\rho(\tilde{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}\right] - \mathbb{E}\left[\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1}) \mid \mathcal{F}_{t-1}\right]\right)\right|^4 \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^4}\left(\sum_{i=1}^{N}\mathbb{E}\left[\left(\phi(\bar{x}_t^i)\,\rho(\bar{x}_t^i, x_{t-1})\right)^2 |\mathcal{F}_{t-1}\right]\right)^2 \quad \text{by lemma 3.5} \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^4}C_\rho^2\left(\sum_{i=1}^{N}\mathbb{E}\left[\left|\phi(\bar{x}_t^i)\right|^2\rho(\bar{x}_t^i, x_{t-1})\right)|\mathcal{F}_{t-1}\right]\right)^2 \quad \text{by lemma 3.5} \\
&= \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^4}C_\rho^2\left(\sum_{i=1}^{N}\left(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^2\,g\right)\right)^2 \quad \text{By Equation (33)} \\
&= \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^4}C_\rho^2\left(N\left(\pi_{t-1|t-1}^{N}, f\,|\phi|^2\,g\right)\right)^2 \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^2}C_\rho^2||f||^2||g||^2\left(\pi_{t-1|t-1}^{N}, |\phi|^2\right)^2 \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^2}C_\rho^2||f||^2||g||^2\left(\pi_{t-1|t-1}^{N}, |\phi|^4\right) \\
&\leq \frac{2^4\epsilon^2}{(1-\epsilon)^4 N^2}C_\rho^2||f||^2||g||^2 M_{t-1|t-1}||\phi||_{t-1,4}^4 \quad \text{By Equation (28)} \\
&\leq \tilde{C}_{\Pi_2}\frac{||\phi||_{t-1,4}^4}{N^2}
\end{aligned} \tag{39}
$$

For Π_3 we get

$$
\begin{aligned}
\mathbb{E}\left[\left|\Pi_3\right|^4\right] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t-1}^{N,\alpha_i}, f\,\phi\,g) - (\hat{\pi}_{t|t}, \phi)\right|^4\right] \\
&= \mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, f\,\phi\,g) - (\hat{\pi}_{t|t}, \phi)\right|^4\right] \\
&= \mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, f\,\phi\,g) - (\pi_{t-1|t-1}, f\,\phi\,g)\right|^4\right] \\
&\leq ||f||^4||g||^4\mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, \phi) - (\pi_{t-1|t-1}, \phi)\right|^4\right] \\
&\leq \tilde{C}_{t-1|t-1}\frac{||f||^4||g||^4||\phi||_{t-1,4}^4}{N^2} \quad \text{by Equation (27)} \\
&= \tilde{C}_{\Pi_3}\frac{||\phi||_{t-1,4}^4}{N^2}
\end{aligned} \tag{40}
$$

Combining Equations (38), (39) and (40) using Minkowski's inequality, we have

$$
\begin{aligned}
\left(\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t},\phi)-(\hat{\pi}_{t|t},\phi)\right|^4\right]\right)^{1/4} &= \left(\mathbb{E}\left[\left|\Pi_1+\Pi_2+\Pi_3\right|^4\right]\right)^{1/4} \\
&\le (\mathbb{E}[|\Pi_1|^4])^{1/4}+(\mathbb{E}[|\Pi_2|^4])^{1/4}+(\mathbb{E}[|\Pi_3|^4])^{1/4} \\
&\le \left(\tilde{C}_{\Pi_1}\frac{||\phi||^4_{t-1,4}}{N^2}\right)^{1/4}+\left(\tilde{C}_{\Pi_2}\frac{||\phi||^4_{t-1|4}}{N^2}\right)^{1/4}+\left(\tilde{C}_{\Pi_3}\frac{||\phi||^4_{t-1,4}}{N^2}\right)^{1/4} \\
&= \left(\tilde{C}^{1/4}_{\Pi_1}+\tilde{C}^{1/4}_{\Pi_2}+\tilde{C}^{1/4}_{\Pi_3}\right)\frac{||\phi||_{t-1,4}}{N^{1/2}} \\
&= \hat{C}^{1/4}_{t|t}\frac{||\phi||_{t-1,4}}{N^{1/2}}
\end{aligned}
$$

Therefore

$$
\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t},\phi)-(\hat{\pi}_{t|t},\phi)\right|^4\right] \le \hat{C}_{t|t}\frac{||\phi||^4_{t-1,4}}{N^2}. \tag{41}
$$

The next step is to study the boundedness of $\mathbb{E}\left[(\hat{\pi}^N_{t|t-1},|\phi|^4)\right]$. We use the same techniques and simplifications as above. Let $\mathcal{F}_{t-1}$ be the σ-algebra generated by x^i_{t-1}. Then we can write $(\hat{\pi}^N_{t|t},|\phi|^4)-(\hat{\pi}_{t|t},|\phi|^4)=\bar{\Pi}_1+\bar{\Pi}_2+\bar{\Pi}_3$, where

$$
\bar{\Pi}_1 = (\hat{\pi}^N_{t|t},|\phi|^4)-\frac{1}{N}\sum_{i=1}^N \mathbb{E}[|\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t,x_{t-1})\mid\mathcal{F}_{t-1}] \tag{42}
$$

$$
\bar{\Pi}_2 = \frac{1}{N}\sum_{i=1}^N \mathbb{E}[|\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t,x_{t-1})\mid\mathcal{F}_{t-1}]-\frac{1}{N}\sum_{i=1}^N(\pi^{N,\alpha_i}_{t-1|t-1},f\,|\phi|^4\,g) \tag{43}
$$

$$
\bar{\Pi}_3 = \frac{1}{N}\sum_{i=1}^N(\pi^{N,\alpha_i}_{t-1|t-1},f\,|\phi|^4\,g)-(\hat{\pi}_{t|t},|\phi|^4) \tag{44}
$$

From Equation (42), we have

$$
\begin{aligned}
\mathbb{E}[|\bar{\Pi}_1||\mathcal{F}_{t-1}] &= \mathbb{E}\left[\left|(\hat{\pi}^N_{t|t}, |\phi|^4) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[|\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right| \mid \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N |\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[|\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right| \mid \mathcal{F}_{t-1}\right] \\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N |\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1}) - \mathbb{E}\Big[\frac{1}{N}\sum_{i=1}^N |\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1}) \mid \mathcal{F}_{t-1}\Big]\right| \mid \mathcal{F}_{t-1}\right] \\
&\le 2\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N |\phi(\tilde{x}^i_t)|^4\,\rho(\tilde{x}^i_t, x^i_{t-1})\right| \mid \mathcal{F}_{t-1}\right] \quad \text{By Lemma 3.2} \\
&\le \frac{2}{1-\epsilon}\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N |\phi(\bar{x}^i_t)|^4\,\rho(\bar{x}^i_t, x^i_{t-1})\right| \mid \mathcal{F}_{t-1}\right] \quad \text{By Lemma 3.5} \\
&\le \frac{2}{1-\epsilon}\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left||\phi(\bar{x}^i_t)|^4\,\rho(\bar{x}^i_t, x^i_{t-1})\right| \mid \mathcal{F}_{t-1}\right] \\
&= \frac{2}{1-\epsilon}\frac{1}{N}\sum_{i=1}^N (\pi^{N,\alpha_i}_{t-1|t-1}\, f\,|\phi|^4\, g) = \frac{2}{1-\epsilon}(\pi^N_{t-1|t-1}\, f\,|\phi|^4\, g) \quad \text{By Equation (34)} \\
&\le \frac{2}{1-\epsilon}\|f\|\,\|g\|\,(\pi^N_{t-1|t-1}\ |\phi|^4) \\
&\le \frac{2}{1-\epsilon}\|f\|\,\|g\|\,M_{t-1|t-1}\,\|\phi\|^4_{t-1|4} \quad \text{By Equation (28)}
\end{aligned} \tag{45}
$$

From Equation (43), we have

$$
\begin{aligned}
&\mathbb{E}\left[\left|\bar{\Pi}_2\right|\right] \qquad (46)\\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}] - \frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^4\,g)\right|\right]\\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}] - \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[|\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}]\right|\right] \quad \text{By Equation (33)}\\
&= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\left[|\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right] - \mathbb{E}\left[|\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right]\right)\right|\right]\\
&\le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left|\left(\mathbb{E}\left[|\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right] - \mathbb{E}\left[|\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right]\right)\right|\right]\\
&\le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\left|\mathbb{E}\left[|\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right]\right| + \left|\mathbb{E}\left[|\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1}) \mid \mathcal{F}_{t-1}\right]\right|\right)\right]\\
&\le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\mathbb{E}\left[\left||\phi(\tilde{x}_t^i)|^4\,\rho(\tilde{x}_t^i,x_{t-1})\right| \mid \mathcal{F}_{t-1}\right] + \mathbb{E}\left[\left||\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1})\right| \mid \mathcal{F}_{t-1}\right]\right)\right]\\
&\le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{1-\epsilon}\mathbb{E}\left[\left||\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1})\right| \mid \mathcal{F}_{t-1}\right] + \mathbb{E}\left[\left||\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1})\right| \mid \mathcal{F}_{t-1}\right]\right)\right] \quad \text{by lemma 3.5}\\
&\le \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left(\frac{2-\epsilon}{1-\epsilon}\mathbb{E}\left[\left||\phi(\bar{x}_t^i)|^4\,\rho(\bar{x}_t^i,x_{t-1})\right| \mid \mathcal{F}_{t-1}\right]\right)\right]\\
&= \frac{2-\epsilon}{1-\epsilon}\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^4\,g)\right] \quad \text{By Equation (33)}\\
&= \frac{2-\epsilon}{1-\epsilon}\mathbb{E}\left[(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g)\right] \quad \text{By Equation (34)}\\
&\le \frac{2-\epsilon}{1-\epsilon}\|f\|\,\|g\|(\pi_{t-1|t-1}^{N}, |\phi|^4)\\
&\le \frac{2-\epsilon}{1-\epsilon}\|f\|\,\|g\|\,M_{t-1|t-1}\,\|\phi\|_{t-1|4}^4 \quad \text{By Equation (28)} \qquad (47)
\end{aligned}
$$

From Equation (44), we have

$$
\begin{aligned}
\mathbb{E}\left[\left|\bar{\Pi}_3\right|\right] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^{N}(\pi_{t-1|t-1}^{N,\alpha_i}, f\,|\phi|^4\,g) - (\hat{\pi}_{t|t}, |\phi|^4)\right]\\
&= \mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g) - (\hat{\pi}_{t|t}, |\phi|^4)\right|\right] \quad \text{By Equation (34)}\\
&= \mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, f\,|\phi|^4\,g) - (\pi_{t-1|t-1}, f\,|\phi|^4\,g)\right|\right]\\
&\le \|f\|\,\|g\|\mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, |\phi|^4) - (\pi_{t-1|t-1}, |\phi|^4)\right|\right]\\
&\le \|f\|\,\|g\|\mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, |\phi|^4)\right| + \left|(\pi_{t-1|t-1}, |\phi|^4)\right|\right]\\
&= \|f\|\,\|g\|\left(\mathbb{E}\left[\left|(\pi_{t-1|t-1}^{N}, |\phi|^4)\right|\right) + \mathbb{E}\left[\left|(\pi_{t-1|t-1}, |\phi|^4)\right|\right]\right)\\
&\le \|f\|\,\|g\|\left(M_{t-1|t-1}\,\|\phi\|_{t-1|4}^4 + \|\phi\|_{t-1|4}^4\right) = \|f\|\,\|g\|\left(M_{t-1|t-1}+1\right)\|\phi\|_{t-1|4}^4 \qquad (48)
\end{aligned}
$$

Combining Equations (45), (47) and (48) we have

$$
\begin{aligned}
&\left(\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t}, |\phi|^4) - (\hat{\pi}_{t|t}, |\phi|^4)\right|\right]\right) \\
&= \mathbb{E}\left[\left|\bar{\Pi}_1 + \bar{\Pi}_2 + \bar{\Pi}_3\right|\right] = \mathbb{E}\left[|\bar{\Pi}_1|\right] + \mathbb{E}\left[|\bar{\Pi}_2|\right] + \mathbb{E}\left[|\bar{\Pi}_3|\right] \\
&\leq \left(\frac{2}{1-\epsilon}\|f\|\,\|g\|\,M_{t-1|t-1} + \frac{2-\epsilon}{1-\epsilon}\|f\|\,\|g\|M_{t-1|t-1} + \|f\|\,\|g\|\left(M_{t-1|t-1}+1\right)\right)\|\phi\|^4_{t-1|4} \\
&= M_{t|t}||\phi||^4_{t-1,4},
\end{aligned}
$$

where $M_{t|t} = \frac{2}{1-\epsilon}\|f\|\,\|g\|\,M_{t-1|t-1} + \frac{2-\epsilon}{1-\epsilon}\|f\|\,\|g\|M_{t-1|t-1} + \|f\|\,\|g\|\left(M_{t-1|t-1}+1\right)$. Therefore

$$
\mathbb{E}[(\hat{\pi}^N_{t|t-1}, |\phi|^4) - (\hat{\pi}_{t|t-1}, |\phi|^4)] \leq M_{t|t}||\phi||^4_{t-1,4} \tag{49}
$$

Since Equations (41) and (49) show that the expectation of the differences of the numerators is bounded for any function ϕ, the expectation of the differences of the denominators is also bounded. That is for $\phi = 1$ we have

$$
\mathbb{E}\left[\left|(\hat{\pi}^N_{t|t-1}, 1) - (\hat{\pi}_{t|t-1}, 1)\right|^4\right] \leq \hat{C}_{t|t}\frac{||1||^4_{t-1,4}}{N^2} = \frac{\hat{C}_{t|t}}{N^2} \tag{50}
$$

$$
\mathbb{E}[(\hat{\pi}^N_{t|t}, |1|^4) - (\hat{\pi}_{t|t}, |1|^4)] \leq M_{t|t}||1||^4_{t-1,4} = M_{t|t} \tag{51}
$$

To complete the proof, we use Equations (41), (49), (50) and (51) to deduce that:

$$
\begin{aligned}
&\left(\mathbb{E}\left[\left|(\pi_{t|t}^N,\phi)-(\pi_{t|t},\phi)\right|^4\right]\right)^{1/4}\\
&=\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)}{(\hat{\pi}_{t|t}^N,1)}-\frac{(\hat{\pi}_{t|t},\phi)}{(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&=\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)(\hat{\pi}_{t|t},1)-(\hat{\pi}_{t|t},\phi)(\hat{\pi}_{t|t}^N,1)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&=\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)\left((\hat{\pi}_{t|t},1)-(\hat{\pi}_{t|t}^N,1)\right)+(\hat{\pi}_{t|t}^N,1)\left((\hat{\pi}_{t|t}^N,\phi)-(\hat{\pi}_{t|t},\phi)\right)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&=\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)\left((\hat{\pi}_{t|t},1)-(\hat{\pi}_{t|t}^N,1)\right)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}+\frac{(\hat{\pi}_{t|t}^N,1)\left((\hat{\pi}_{t|t}^N,\phi)-(\hat{\pi}_{t|t},\phi)\right)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&\leq\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)\left((\hat{\pi}_{t|t},1)-(\hat{\pi}_{t|t}^N,1)\right)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&+\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,1)\left((\hat{\pi}_{t|t}^N,\phi)-(\hat{\pi}_{t|t},\phi)\right)}{(\hat{\pi}_{t|t}^N,1)(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\qquad\text{By Minkowski's inequality}\\
&\leq\left(\mathbb{E}\left[\left|\frac{(\hat{\pi}_{t|t}^N,\phi)\left((\hat{\pi}_{t|t},1)-(\hat{\pi}_{t|t}^N,1)\right)}{\gamma_t(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&+\left(\mathbb{E}\left[\left|\frac{\left((\hat{\pi}_{t|t}^N,\phi)-(\hat{\pi}_{t|t},\phi)\right)}{(\hat{\pi}_{t|t},1)}\right|^4\right]\right)^{1/4}\\
&=\left|\frac{(\hat{\pi}_{t|t}^N,\phi)}{\gamma_t(\hat{\pi}_{t|t},1)}\right|\left(\mathbb{E}\left[\left|(\hat{\pi}_{t|t}^N,1)-(\hat{\pi}_{t|t},1)\right|^4\right]\right)^{1/4}\\
&+\left|\frac{1}{(\hat{\pi}_{t|t},1)}\right|\left(\mathbb{E}\left[\left|(\hat{\pi}_{t|t}^N,\phi)-(\hat{\pi}_{t|t},\phi)\right|^4\right]\right)^{1/4}\\
&\leq\left|\frac{(\hat{\pi}_{t|t}^N,\phi)}{\gamma_t(\hat{\pi}_{t|t},1)}\right|\left(\frac{\hat{C}_{t|t}}{N^2}\right)^{1/4}+\left|\frac{1}{(\hat{\pi}_{t|t},1)}\right|\left(\hat{C}_{t|t}\frac{||\phi||_{t-1,4}^4}{N^2}\right)^{1/4}
\end{aligned}
$$

By Holder's and Jensen's inequalities we have $|(\pi,\phi)|^4 \leq |(\pi,|\phi|)|^4 \leq (\pi,|\phi|^4)$. Therefore

$$\begin{aligned}
\left(\mathbb{E}\left[\left|(\pi_{t|t}^N,\phi)-(\pi_{t|t},\phi)\right|^4\right]\right)^{1/4} &\leq \left|\frac{(\hat{\pi}_{t|t}^N,\phi)}{\gamma_t(\hat{\pi}_{t|t},1)}\right|\left(\frac{\hat{C}_{t|t}}{N^2}\right)^{1/4}+\left|\frac{1}{(\hat{\pi}_{t|t},1)}\right|\left(\hat{C}_{t|t}\frac{||\phi||_{t-1,4}^4}{N^2}\right)^{1/4} \\
&= \left|\frac{(\hat{\pi}_{t|t}^N,\phi)\hat{C}_{t|t}^{1/4}}{\gamma_t(\hat{\pi}_{t|t},1)}\right|\left(\frac{1}{N^2}\right)^{1/4}+\left|\frac{\hat{C}_{t|t}^{1/4}}{(\hat{\pi}_{t|t},1)}\right|\left(\frac{||\phi||_{t-1,4}^4}{N^2}\right)^{1/4} \\
&= C_{1_{t|t}}\left(\frac{1}{N^2}\right)^{1/4}+C_{2_{t|t}}\left(\frac{||\phi||_{t-1,4}^4}{N^2}\right)^{1/4} \\
&= \frac{1}{N^{1/2}}\left(C_{1_{t|t}}+C_{2_{t|t}}||\phi||_{t-1,4}\right) \\
&= \frac{1}{N^{1/2}}\left(\frac{C_{1_{t|t}}}{||\phi||_{t-1,4}}+C_{2_{t|t}}\right)||\phi||_{t-1,4} \\
&\leq \frac{1}{N^{1/2}}\left(C_{1_{t|t}}+C_{2_{t|t}}\right)||\phi||_{t-1,4} \quad \text{Because} \quad C_{1_{t|t}}\geq 0 \quad ||\phi||_{t-1,4}\geq 1 \\
&= \frac{1}{N^{1/2}}C_{3_{t|t}}||\phi||_{t-1,4}
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left|(\pi_{t|t}^N,\phi)-(\pi_{t|t},\phi)\right|^4\right] \leq C_{t|t}\frac{||\phi||_{t-1,4}^4}{N^2} \tag{52}$$

For $\mathbb{E}[(\pi_{t|t}^N, |\phi|^4)]$; we have

$$\begin{aligned}
\mathbb{E}\left[(\pi_{t|t}^N, |\phi|^4) - (\pi_{t|t}, |\phi|^4)\right] &= \mathbb{E}\left[\frac{(\hat{\pi}_{t|t}^N, |\phi|^4)}{(\hat{\pi}_{t|t}^N, 1)} - \frac{(\hat{\pi}_{t|t}, |\phi|^4)}{(\hat{\pi}_{t|t}, 1)}\right] \\
&= \mathbb{E}\left[\frac{(\hat{\pi}_{t|t}^N, |\phi|^4)(\hat{\pi}_{t|t}, 1) - (\hat{\pi}_{t|t}, |\phi|^4)(\hat{\pi}_{t|t}^N, 1)}{(\hat{\pi}_{t|t}^N, 1)(\hat{\pi}_{t|t}, 1)}\right] \\
&= \mathbb{E}\left[\frac{(\hat{\pi}_{t|t}^N, |\phi|^4)\Big((\hat{\pi}_{t|t}, 1) - (\hat{\pi}_{t|t}^N, 1)\Big) + (\hat{\pi}_{t|t}^N, 1)\Big((\hat{\pi}_{t|t}^N, |\phi|^4) - (\hat{\pi}_{t|t}, |\phi|^4)\Big)}{(\hat{\pi}_{t|t}^N, 1)(\hat{\pi}_{t|t}, 1)}\right] \\
&= \mathbb{E}\left[\frac{(\hat{\pi}_{t|t}^N, |\phi|^4)\Big((\hat{\pi}_{t|t}, 1) - (\hat{\pi}_{t|t}^N, 1)\Big)}{(\hat{\pi}_{t|t}^N, 1)(\hat{\pi}_{t|t}, 1)} + \frac{(\hat{\pi}_{t|t}^N, 1)\Big((\hat{\pi}_{t|t}^N, |\phi|^4) - (\hat{\pi}_{t|t}, |\phi|^4)\Big)}{(\hat{\pi}_{t|t}^N, 1)(\hat{\pi}_{t|t}, 1)}\right] \\
&\leq \mathbb{E}\left[\frac{(\hat{\pi}_{t|t}^N, |\phi|^4)\Big((\hat{\pi}_{t|t}, 1) - (\hat{\pi}_{t|t}^N, 1)\Big)}{\gamma_t(\hat{\pi}_{t|t}, 1)}\right] + \mathbb{E}\left[\frac{\Big((\hat{\pi}_{t|t}^N, |\phi|^4) - (\hat{\pi}_{t|t}, |\phi|^4)\Big)}{(\hat{\pi}_{t|t}, 1)}\right] \\
&= \frac{(\hat{\pi}_{t|t}^N, |\phi|^4)}{\gamma_t(\hat{\pi}_{t|t}, 1)}\mathbb{E}\left[(\hat{\pi}_{t|t}^N, 1) - (\hat{\pi}_{t|t}, 1)\right] + \frac{1}{(\hat{\pi}_{t|t}, 1)}\mathbb{E}\left[(\hat{\pi}_{t|t}^N, |\phi|^4) - (\hat{\pi}_{t|t}, |\phi|^4)\right] \\
&\leq \frac{(\hat{\pi}_{t|t}^N, |\phi|^4)}{\gamma_t(\hat{\pi}_{t|t}, 1)} M_{t|t} + \frac{1}{(\hat{\pi}_{t|t}, 1)} M_{t|t}\, \|\phi\|_{t-1,4}^4 \\
&= \tilde{M}_{1_{t|t}} + \tilde{M}_{2_{t|t}}\, \|\phi\|_{t-1,4}^4 \\
&= \left(\frac{\tilde{M}_{1_{t|t}}}{\|\phi\|_{t-1,4}^4} + \tilde{M}_{2_{t|t}}\right)\|\phi\|_{t-1,4}^4 \\
&\leq \left(\tilde{M}_{1_{t|t}} + \tilde{M}_{2_{t|t}}\right)\|\phi\|_{t-1,4}^4 \quad \text{Because } \|\phi\|_{t-1,4}^4 \geq 1 \\
&= \bar{M}_{t|t}\|\phi\|_{t-1,4}^4
\end{aligned}$$

References

[1] Kai Lai Chung, *A course in probability theory*, London: Academic Press, 3rd edition, 2001.

[2] Xiao-li Hu, Thomas B. Schön, and Lennart Ljung, "A basic convergence result for particle filtering," *IEEE Transactions on Signal Processing*, vol. 56, no. 4, pp. 1337–1348, 2008.