Prediction of ESTSP Competition Time Series by Unscented Kalman Filter and RTS Smoother

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- State space model with state $\mathbf{x}_k$ and measurements $\mathbf{y}_k$:

$$\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}, \mathbf{q}_{k-1})$$
$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k, \mathbf{r}_k),$$

  where $\mathbf{q}_{k-1}$ is the process noise and $\mathbf{r}_k$ is the measurement noise.
- The state $\mathbf{x}_k$ is the hidden internal dynamic state of the system on the time step k
- The measurements $\mathbf{y}_k$ model the output of the system
- We want to estimate the state from the measurements and use it for model based prediction of the time series

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Optimal Filters and Predictors

- Given measurements $\mathbf{y}_1, \ldots, \mathbf{y}_T$ optimal filter produces MMSE optimal online estimate:

$$\hat{\mathbf{x}}(t_k) = E(\mathbf{x}(t_k) \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_k).$$

for each $k = 1, \ldots, T$.

- Can be also used for computing the optimal predictions:

$$\hat{\mathbf{x}}(t) = E(\mathbf{x}(t) \,|\, \mathbf{y}_1, \ldots, \mathbf{y}_k).$$

for $t > t_k$.

- If the state space model is linear (i.e., Gaussian process), then Kalman filter provides the optimal solution
- If it is non-linear, then unscented Kalman filter can be used for approximating the optimal solution

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- If the dynamic model is linear, then Rauch-Tung-Striebel (RTS) smoother provides the optimal solution
- Approximate smoothers for nonlinear problems exists also
- In this article, the dynamic model is linear (i.e., Gaussian process) given the parameters and thus the linear RTS smoother suffices

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- Optimal smoother produces the optimal batch estimate:

$$\hat{\mathbf{x}}(t_k) = E(\mathbf{x}(t_k) \mid \mathbf{y}_1, \ldots, \mathbf{y}_T).$$

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Idea of the Approach

- **Model** the time series as consisting of periodic and bias components with a linear state space model
- **Model** the signal-residual dependence by including a non-linear correction term into the model
- **Model** the remaining residual autocorrelation with an autoregressive (AR) model
- **Estimate** the parameters and **predict the time series** with the model using the unscented Kalman filter (UKF) and Rauch-Tung-Striebel (RTS) smoother as the numerical methods

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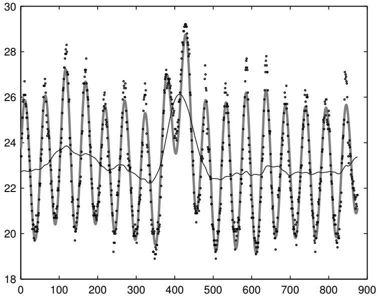
Bias and Periodic Components 1/3



- **Signal model**:

$$y_k = x_b(k) + x_r(k) + r_k$$

- **Bias** component $x_b(t)$ is modeled as integral of a white noise process $w_b(t)$

$$\frac{\mathrm{d}x_b}{\mathrm{d}t} = w_b(t)$$

- **Periodic** component $x_r(t)$ is modeled as a white noise driven stochastic resonator

$$\frac{\mathrm{d}^2 x_r}{\mathrm{d}t^2} = -\omega^2 x_r + w_r(t)$$

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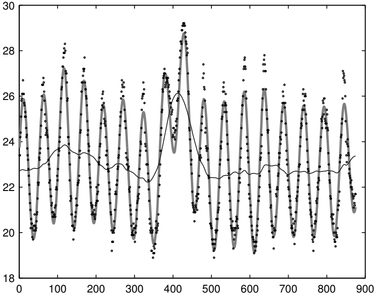

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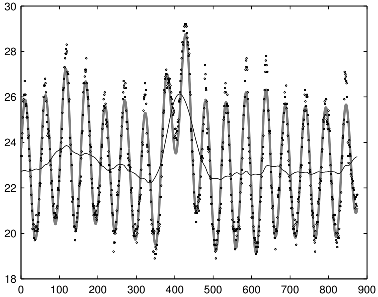

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Bias and Periodic Components 2/3

- Can be written as discretely measured continuous-time vector process $\mathbf{x}(t) = (x_b(t)\ x_r(t)\ dx_r(t)/dt)^T$ as follows:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{L}\,\mathbf{w}(t)$$

- Linear system theory $\Rightarrow$ equivalent discrete time model:

$$\mathbf{x}_k = \mathbf{A}\,\mathbf{x}_{k-1} + \mathbf{q}_{k-1}$$

- Measurement model is of the form

$$y_k = \mathbf{H}\,\mathbf{x}_k + r_k$$

- Linear state space model $\Rightarrow$ **Kalman filter** can be applied.

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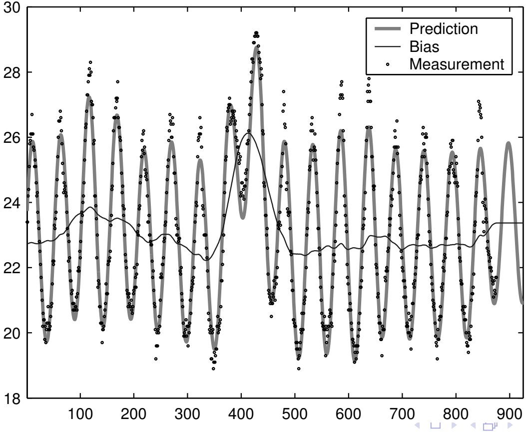
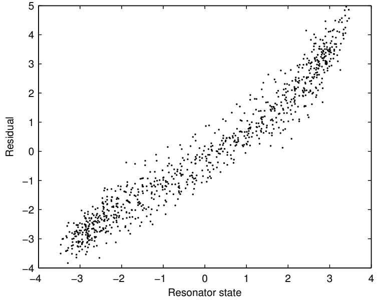
Bias and Periodic Components 3/3

Finding the parameters that minimize the 50 step prediction error, results in the following kind of prediction:



Non-linear Correction Term

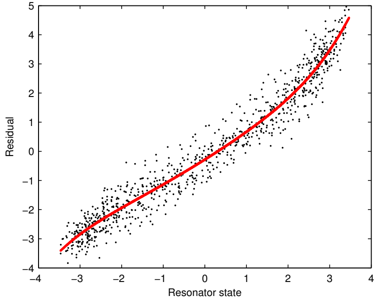


- Residual and periodic component still have a non-linear relationship
- 5th degree polynomial gives a suitable fit
- Measurement model becomes non-linear

$$y_k = x_b(r) + \sum_{i=0}^{5} c_i \left(x_r(k)\right)^i + r_k$$

- Use unscented Kalman filter (UKF) instead of Kalman filter

Non-linear Correction Term



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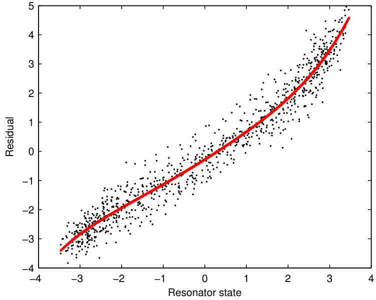

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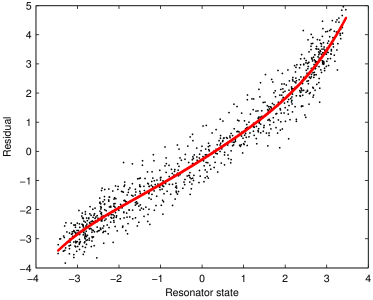

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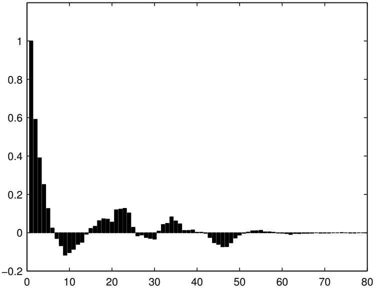

- Residual has non-delta autocorrelation, indicating a periodic component
- Can be modeled as second order AR-model:

$$e_k = \sum_{i=1}^{2} a_i \, e_{k-i} + r_k^{\mathrm{ar}}$$

- Can be estimated with Rauch-Tung-Striebel smoother

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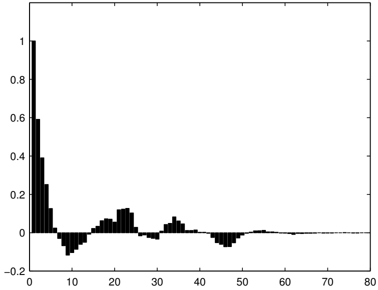

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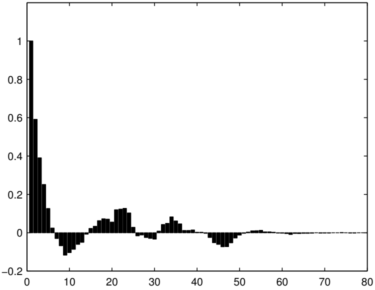

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Estimation of parameters

- The unknown parameters are the spectral densities of process noises and the angular velocity of the resonator
- Form a discrete grid of sensible parameter values
- Evaluate parameters by computing 50 step prediction errors in known parts of time series
- First find roughly the location of minimum and form denser grid on that area
- Find the final smoothed estimate of the time series and make final prediction with the parameter values giving the minimum error

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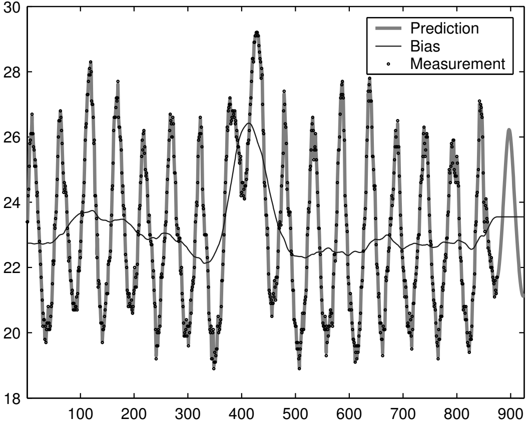
Prediction

The final estimate of the signal and the prediction result:



Summary

- First the bias and periodic components are **modeled** as linear state space model
- Non-linear dependence between residual and periodic component is **modeled** with 5th degree polynomial
- Remaining autocorrelation is **modeled** with second order autoregressive (AR) model
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