Sigma Point and Particle Approximations of Stochastic Differential Equations in Optimal Filtering

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The *unscented transform* (UT) is a relatively recent method for approximating non-linear transformations of random variables. Instead of the classical Taylor series approximations, it is based on forming a set of *sigma points*, which are propagated through the non-linearity. The *unscented Kalman filter* (UKF) is an alternative to the extended Kalman filter (EKF), which utilizes the unscented transform in the filter computations. However, in its original form, the UKF is a discrete-time algorithm and it cannot be directly applied to estimation problems, where the state dynamics are modeled in continuous-time as stochastic differential equations.

It has been recently shown that by taking the formal continuous-time limit of the discrete-time UKF prediction equations, it is possible to derive sigma-point differential equations, which can be used for approximating the mean and covariance of a stochastic differential equation (SDE). By combining these differential equations with UKF update equation, we obtain the *continuous-discrete unscented Kalman filter*, which can be used for approximate recursive inference on discretely observed stochastic differential equations.

The solutions of stochastic differential equations can be also approximated by simulating random trajectories from the equation and by forming Monte Carlo or particle approximations from the simulated trajectories. A commonly used framework for statistical inference in this context is *sequential importance resampling*. In continuous-time setting the evaluation of importance weights is problematic, because the exact evaluation would require solving an instance of Kolmogorov forward partial differential equation, which is an intractable task in general.

One way of coping with this problem is to use the *Girsanov theorem* for evaluation of the likelihood ratios of stochastic differential equations and in turn importance weights by numerical simulation. The Girsanov theorem is a theorem from mathematical probability theory, which can be used for computing likelihood ratios of stochastic processes. It states that the likelihood ratio of a stochastic process and Brownian motion, that is, the Radon-Nikodym derivative of the measure of the stochastic process with respect to the measure of Brownian motion, can be represented as an exponential martingale which is the solution to a certain stochastic differential equation.

In the talk I will review the Taylor series, sigma-point (unscented) and particle approximations of stochastic differential equations in optimal (Bayesian) filtering context and present some applications of the methods in navigation systems and in monitoring of chemical processes.