# Exercise Round 6

The deadline of this exercise round is **December 9, 2014 at 23:59**. This is the last round of exercises, and the solutions (in PDF form, codes also as separate files) should be sent to arno.solin@aalto.fi. Include "[Becs-114.4202] Exercise 6" in the subject.

## Exercise 1 (Kalman filter and RTS smoother for OU)

Consider the Ornstein–Uhlenbeck model

$$\begin{aligned} \mathrm{d}x &= -\lambda\, x \,\mathrm{d}t + \mathrm{d}\beta, \\ y_k &= x(t_k) + \varepsilon_k, \end{aligned} \tag{1}$$

with $\lambda = 1/2$, $q = 1$, $x(0) \sim \mathrm{N}(0, P_\infty)$, $\varepsilon_k \sim \mathrm{N}(0,1)$, where $P_\infty$ is the stationary variance of the SDE.

(a) Simulate data from the model by using Euler–Maruyama with step size $\Delta t = 1/100$ over the time period $[0, 10]$, and generate noisy measurements only at the time steps $t_j = j$, for $j = 1, 2, \ldots, 10$.

(b) Implement a Kalman filter to the model. Plot the simulated data, the observed values, and the filter mean in the same figure.

(c) Implement an RTS smoother to the problem. Plot the simulated data, the observed values, and the smoother mean in the same figure.

(d) How would you compute the smoothing solution at an arbitrary $t$?

## Exercise 2 (Continuous-time filtering)

(a) Write down the Kushner–Stratonovich equation for the model

$$\begin{aligned} \mathrm{d}x &= -\lambda\, x \,\mathrm{d}t + \mathrm{d}\beta, \\ \mathrm{d}y &= x \,\mathrm{d}t + \mathrm{d}\eta, \end{aligned} \tag{2}$$

where $\beta$ and $\eta$ are independent standard Brownian motions.

(b) Write down the corresponding Zakai equation.

(c) Write down the Kalman–Bucy filter for the model.

(d) Show that the filters in (a), (b), and (c) are equivalent.

## Exercise 3 (Continuous-time approximate non-linear filtering)

Consider the model

$$\begin{aligned} \mathrm{d}x &= \tanh(x)\,\mathrm{d}t + \mathrm{d}\beta, \\ \mathrm{d}y &= \sin(x)\,\mathrm{d}t + \mathrm{d}\eta, \end{aligned} \tag{3}$$

where $\beta$ and $\eta$ are independent Brownian motions with diffusions $Q = 1$ and $R = 0.01$, respectively.

(a) Write down the extended Kalman–Bucy filter for this model.

(b) Simulate data from the model over a time span $[0, 5]$ with $\Delta t = 1/100$, and try implementing the filtering method numerically. How does it work?