

Exercise Round 4

The deadline of this exercise round is **November 25, 2014**. The solutions will be gone through during the exercise session in room F255, F-building starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1 (Milstein's method)

Consider the following scalar SDE:

$$dx = -c x dt + g x d\beta, \quad x(0) = x_0, \quad (1)$$

where a , g and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

(a) Check using the Itô formula that the solution to this equation is

$$x(t) = x_0 \exp [(-c - g^2/2)t + g\beta(t)] \quad (2)$$

Hint: $\phi(\beta(t), t) = x_0 \exp [(-c - g^2/2)t + g\beta(t)]$.

(b) Simulate trajectories from the equation using Milstein's method with parameters $x_0 = 1$, $c = 1/10$, $g = 1/10$, and check that the histogram at $t = 1$ looks the same as obtained by sampling from the above exact solution.

Exercise 2 (Strong and weak approximations)

Consider the following scalar SDE:

$$dx = \tanh(x) dt + d\beta, \quad x(0) = 0, \quad (3)$$

where $\beta(t)$ is a standard Brownian motion. Recall that it has the exact solution

$$p(x, t) = \frac{1}{\sqrt{2\pi t}} \cosh(x) \exp\left(-\frac{1}{2}t\right) \exp\left(-\frac{1}{2t}x^2\right).$$

Use the following numerical methods for simulating from the model:

- (a) Simulate 1000 trajectories from the SDE with the order strong 1.5 Itô–Taylor series based method presented in the lecture notes. Compare the histogram to the exact solution at $t = 5$.
- (b) Simulate 1000 trajectories from the SDE with the weak order 2.0 Itô–Taylor series based method presented in the lecture notes using both (i) Gaussian increments, (ii) the three-point distributed random variables. Compare the histograms to the exact solution at $t = 5$.
- (c) What can you say about the simulated trajectories when $t \in [0, 5]$ for the different methods?

Exercise 3 (Gaussian approximation of SDEs)

- (a) Form a Gaussian assumed density approximation to the SDE in Equation (3) in the time interval $t \in [0, 5]$, and compare it to the exact solution. Compute the Gaussian integrals numerically on a uniform grid.
- (b) Form a Gaussian assumed density approximation to Equation (1) and numerically compare it to the histogram obtained in 1(b).