Exercise Round 3

The deadline of this exercise round is **November 18, 2014**. The solutions will be gone through during the exercise session in room F255, F-building starting at 14:15.

The problems should be solved before the exercise session, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1 (Fokker–Planck–Kolmogorov (FKP) equation)

(a) Write down the FKP for

\[ dx = \tanh(x) \ dt + \ d\beta, \quad x(0) = 0, \]

where \( \beta(t) \) is a standard Brownian motion, and check that the following solves it:

\[ p(x, t) = \frac{1}{\sqrt{2\pi t}} \cosh(x) \exp\left( -\frac{1}{2} t \right) \exp\left( -\frac{1}{2} t x^2 \right). \]

(b) Plot the evolution of the probability density when \( t \in [0, 5] \).

(c) Simulate 1000 trajectories from the SDE using the Euler–Maruyama method and check visually that the histogram matches the correct density at time \( t = 5 \).

Exercise 2 (Numerical solution of FPK)

Use a finite-differences method to solve the FPK for the Equation (1). For simplicity, you can select a finite range \( x \in [-L, L] \) and use the Diriclet boundary conditions \( p(-L, t) = p(L, t) = 0 \).

(a) Divide the range to \( n \) grid points and let \( h = 1/(n + 1) \). On the grid, approximate the partial derivatives of \( p(x, t) \) via

\[
\frac{\partial p(x, t)}{\partial x} \approx \frac{p(x + h, t) - p(x - h, t)}{2h}, \quad \frac{\partial^2 p(x, t)}{\partial x^2} \approx \frac{p(x + h, t) - 2p(x, t) + p(x - h, t)}{h^2}.
\]

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(b) Let \( \mathbf{p}(t) = (p(h, t) \; p(2h, t) \; \cdots \; p(nh, t))^T \) and from the above, form an equation of the form

\[
\frac{d \mathbf{p}}{dt} = \mathbf{F} \mathbf{p}.
\]  

(3)

(c) Solve the above equation using (i) backward Euler, (ii) by numerical computation of \( \exp(\mathbf{F} t) \), and (iii) by forward Euler. Check that the results match the solution in the previous exercise.

Exercise 3 (Langevin’s physical Brownian motion)

Consider the Langevin model of Brownian motion

\[
\frac{d^2 x}{dt^2} = -c \frac{dx}{dt} + w, \quad x(0) = (dx/dt)(0) = 0,
\]  

(4)

where \( c = 6 \pi \eta r \) and the white noise \( w(t) \) has some spectral density \( q \).

(a) Interpret the above model as an Itô SDE and write it as a two-dimensional state-space form SDE.

(b) Write down the differential equations for the elements of the mean \( \mathbf{m}(t) \) and covariance \( \mathbf{P}(t) \). Conclude that the mean is zero and find the closed-form solutions for the elements \( P_{11}(t) \), \( P_{12}(t) \), \( P_{21}(t) \), and \( P_{22}(t) \) of the covariance matrix \( \mathbf{P}(t) \). **Hint:** Start by solving \( P_{22}(t) \), then use it to find the solutions for \( P_{12}(t) = P_{21}(t) \), and finally solve \( P_{11}(t) \).

(c) Find the limiting solution \( P_{22}(t) \), when \( t \to \infty \) and use the following to determine the diffusion coefficient (spectral density) \( q \):

\[
m \mathbb{E} \left[ \left( \frac{dx}{dt} \right)^2 \right] = \frac{RT}{N}.
\]  

(5)

(d) Plot the solution \( P_{11}(t) \) and conclude that it asymptotically approaches a straight line. Compute the asymptotic solution \( P_{11}(t) \) when \( t \to \infty \), and conclude that it approximately gives Langevin’s result.