Exercise Round 2

The deadline of this exercise round is **November 11, 2014**. The solutions will be gone through during the exercise session in room F255, F-building starting at 14:15.

The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

Exercise 1 (Usage of the Itō formula)

(a) Compute the Itō differential of

\[ \phi(\beta) = t + \exp(\beta), \]

where \( \beta(t) \) is a Brownian motion with diffusion constant \( q \).

(b) Compute the Itō differential of

\[ \phi(x) = x^2, \]

where \( x \) solves the scalar SDE

\[ dx = f(x) \, dt + \sigma \, d\beta, \]

\( \sigma \) is a constant, and \( \beta(t) \) is a standard Brownian motion \( (q = 1) \).

(c) Compute the Itō differential of

\[ \phi(x) = x^T x, \]

where

\[ dx = F \, x \, dt + d\beta, \]

where \( F \) is a constant matrix and the joint diffusion matrix of \( \beta \) is \( Q \).

Exercise 2 (Stochastic differential equations)

(a) Check that

\[ x(t) = \exp(\beta(t)) \]

solves the SDE

\[ dx = \frac{1}{2} x \, dt + x \, d\beta, \]

where \( \beta(t) \) is a standard Brownian motion \( (q = 1) \).
(b) Solve the following SDE by changing the variable to $y = \ln x$:

$$dx = -c x \, dB,$$

where $c > 0$ is a constant, and $B(t)$ is a standard Brownian motion.

(c) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:

$$dx_1 = -x_2 \circ dB,$$
$$dx_2 = x_1 \circ dB,$$

where $B(t)$ is a scalar Brownian motion.

**Exercise 3. (Mean and variance of differential equations)**

Derive the mean and covariance equations for the scalar SDE

$$dx = f(x) \, dt + \sigma(x) \, dB,$$  \hspace{1cm} (1)

where $B$ has the diffusion coefficient $q$, as follows:

(a) Conclude from the definition of the Itô integral that

$$E \left[ \int_u^v \sigma(x(t)) \, dB(t) \right] = 0$$

for any $u$ and $v$.

(b) Take expectations from both sides of the SDE (1) and formally divide by $dt$ to get the differential equation for the mean $m(t)$.

(c) Apply the Itô formula to $\phi(x,t) = (x - m(t))^2$ and take the expectation of the resulting equation to derive the differential equation for the variance.

(d) Write down the mean and covariance differential equations for the scalar SDE

$$dx = -\lambda x \, dt + dB,$$

where $\lambda > 0$ and solve them with $x(0) = x_0$. 

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