Lecture 3: Probability Distributions and Statistics of SDEs

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Statistics of SDEs

- Consider the stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\ \mathrm{d}t + \mathbf{L}(\mathbf{x}, t)\ \mathrm{d}\boldsymbol{\beta}.$$

- Each $\mathbf{x}(t)$ is random variable, and we denote its probability density with $p(\mathbf{x}, t)$.
- The probability density is solution to a *partial differential equation* called Fokker–Planck–Kolmogorov equation.
- The mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$ are solutions of certain ordinary differential equations.
- For LTI SDEs we can also compute the covariance function of the solution $\mathbf{C}(\tau) = \mathrm{E}[\mathbf{x}(t)\,\mathbf{x}(t+\tau)]$.

Fokker-Planck-Kolmogorov PDE

Fokker–Planck–Kolmogorov equation

The probability density $p(\mathbf{x}, t)$ of the solution of the SDE

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\,\mathrm{d}t + \mathbf{L}(\mathbf{x}, t)\,\mathrm{d}\boldsymbol{\beta},$$

solves the Fokker–Planck–Kolmogorov partial differential equation

$$\begin{aligned}\frac{\partial p(\mathbf{x}, t)}{\partial t} &= -\sum_i \frac{\partial}{\partial x_i}[f_i(x, t)\, p(\mathbf{x}, t)] \\ &\quad + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} \left\{ [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, p(\mathbf{x}, t) \right\}.\end{aligned}$$

- In physics literature it is called the Fokker–Planck equation.
- In stochastics it is the forward Kolmogorov equation.

Fokker-Planck-Kolmogorov PDE: Example 1

FPK Example: Diffusion equation

Brownian motion can be defined as solution to the SDE

$$\mathrm{d}x = \mathrm{d}\beta.$$

If we set the diffusion constant of the Brownian motion to be $q = 2\,D$, then the FPK reduces to the diffusion equation

$$\frac{\partial p}{\partial t} = D\,\frac{\partial^2 p}{\partial x^2}$$

Fokker-Planck-Kolmogorov PDE: Derivation [1/5]

- Let $\phi(\mathbf{x})$ be an arbitrary twice differentiable function.
- The Itô differential of $\phi(\mathbf{x}(t))$ is, by the Itô formula, given as follows:

$$
\begin{aligned}
\mathrm{d}\phi &= \sum_i \frac{\partial \phi}{\partial x_i} f_i(\mathbf{x}, t)\, \mathrm{d}t + \sum_i \frac{\partial \phi}{\partial x_i} [\mathbf{L}(\mathbf{x}, t)\, \mathrm{d}\beta]_i \\
&\quad + \frac{1}{2} \sum_{ij} \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij}\, \mathrm{d}t.
\end{aligned}
$$

- Taking expectations and formally dividing by $\mathrm{d}t$ gives the following equation, which we will transform into FPK:

$$
\begin{aligned}
\frac{\mathrm{d}\,\mathrm{E}[\phi]}{\mathrm{d}t} &= \sum_i \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} f_i(\mathbf{x}, t) \right] \\
&\quad + \frac{1}{2} \sum_{ij} \mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij} \right].
\end{aligned}
$$

Fokker-Planck-Kolmogorov PDE: Derivation [2/5]

- The left hand side can now be written as follows:

$$\frac{\mathrm{d}E[\phi]}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int \phi(\mathbf{x})\, p(\mathbf{x}, t) \,\mathrm{d}\mathbf{x} = \int \phi(\mathbf{x}) \frac{\partial p(x, t)}{\partial t} \,\mathrm{d}\mathbf{x}.$$

- Recall the multidimensional integration by parts formula

$$\int_C \frac{\partial u(\mathbf{x})}{\partial x_i} v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = \int_{\partial C} u(\mathbf{x})\, v(\mathbf{x})\, n_i \,\mathrm{d}S - \int_C u(\mathbf{x}) \frac{\partial v(\mathbf{x})}{\partial x_i} \,\mathrm{d}\mathbf{x}.$$

- In this case, the boundary terms vanish and thus we have

$$\int \frac{\partial u(\mathbf{x})}{\partial x_i} v(\mathbf{x}) \,\mathrm{d}\mathbf{x} = -\int u(\mathbf{x}) \frac{\partial v(\mathbf{x})}{\partial x_i} \,\mathrm{d}\mathbf{x}.$$

Fokker-Planck-Kolmogorov PDE: Derivation [3/5]

- Currently, our equation looks like this:

$$\int \phi(\mathbf{x}) \, \frac{\partial p(x,t)}{\partial t} \, \mathrm{d}\mathbf{x} = \sum_i \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} \, f_i(\mathbf{x},t)\right] + \frac{1}{2} \sum_{ij} \mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x},t) \, \mathbf{Q} \, \mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\right].$$

- For the first term on the right, we get via integration by parts:

$$\begin{aligned} \mathrm{E}\left[\frac{\partial \phi}{\partial x_i} \, f_i(\mathbf{x},t)\right] &= \int \frac{\partial \phi}{\partial x_i} \, f_i(\mathbf{x},t) \, p(\mathbf{x},t) \, \mathrm{d}\mathbf{x} \\ &= -\int \phi(\mathbf{x}) \, \frac{\partial}{\partial x_i}[f_i(\mathbf{x},t) \, p(\mathbf{x},t)] \, \mathrm{d}\mathbf{x} \end{aligned}$$

- We now have only one term left.

- For the remaining term we use integration by parts twice, which gives

$$
\begin{aligned}
&\mathrm{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\right] \\
&= \int \left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\, p(\mathbf{x},t)\; \mathrm{d}\mathbf{x} \\
&= -\int \left(\frac{\partial \phi}{\partial x_j}\right) \frac{\partial}{\partial x_i} \left\{[\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\, p(\mathbf{x},t)\right\} \mathrm{d}\mathbf{x} \\
&= \int \phi(\mathbf{x}) \frac{\partial^2}{\partial x_i\, \partial x_j} \left\{[\mathbf{L}(\mathbf{x},t)\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}(\mathbf{x},t)]_{ij}\, p(\mathbf{x},t)\right\} \mathrm{d}\mathbf{x}
\end{aligned}
$$

Fokker-Planck-Kolmogorov PDE: Derivation [5/5]

- Our equation now looks like this:

$$\int \phi(\mathbf{x}) \frac{\partial p(\mathbf{x}, t)}{\partial t} \, \mathrm{d}\mathbf{x} = -\sum_i \int \phi(\mathbf{x}) \frac{\partial}{\partial x_i} [f_i(\mathbf{x}, t) \, p(\mathbf{x}, t)] \, \mathrm{d}\mathbf{x}$$

$$+ \frac{1}{2} \sum_{ij} \int \phi(\mathbf{x}) \frac{\partial^2}{\partial x_i \, \partial x_j} \{ [\mathbf{L}(\mathbf{x}, t) \, \mathbf{Q} \, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij} \, p(\mathbf{x}, t) \} \, \mathrm{d}\mathbf{x}$$

- This can also be written as

$$\int \phi(\mathbf{x}) \Big[\frac{\partial p(\mathbf{x}, t)}{\partial t} + \sum_i \frac{\partial}{\partial x_i} [f_i(\mathbf{x}, t) \, p(\mathbf{x}, t)]$$

$$- \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} \{ [\mathbf{L}(\mathbf{x}, t) \, \mathbf{Q} \, \mathbf{L}^\mathsf{T}(\mathbf{x}, t)]_{ij} \, p(\mathbf{x}, t) \} \Big] \, \mathrm{d}\mathbf{x} = 0.$$

- But the function is $\phi(\mathbf{x})$ arbitrary and thus the term in the brackets must vanish $\Rightarrow$ Fokker–Planck–Kolmogorov equation.

Fokker-Planck-Kolmogorov PDE: Example 2

FPK Example: Benes SDE

The FPK for the SDE

$$\mathrm{d}x = \tanh(x)\,\mathrm{d}t + \mathrm{d}\beta$$

can be written as

$$\begin{aligned}
\frac{\partial p(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left(\tanh(x)\,p(x,t)\right) + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2} \\
&= (\tanh^2(x) - 1)\,p(x,t) - \tanh(x)\,\frac{\partial p(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}.
\end{aligned}$$

Mean and Covariance of SDE [1/2]

- Using Itô formula for $\phi(\mathbf{x}, t)$, taking expectations and dividing by $\mathrm{d}t$ gives

$$\frac{\mathrm{d}\,\mathsf{E}[\phi]}{\mathrm{d}t} = \mathsf{E}\left[\frac{\partial \phi}{\partial t}\right] + \sum_i \mathsf{E}\left[\frac{\partial \phi}{\partial x_i} f_i(x, t)\right] + \frac{1}{2} \sum_{ij} \mathsf{E}\left[\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^\mathsf{T}(x, t)]_{ij}\right]$$

- If we select the function as $\phi(\mathbf{x}, t) = x_u$, then we get

$$\frac{\mathrm{d}\,\mathsf{E}[x_u]}{\mathrm{d}t} = \mathsf{E}\left[f_u(\mathbf{x}, t)\right]$$

- In vector form this gives the differential equation for the mean:

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = \mathsf{E}\left[\mathbf{f}(\mathbf{x}, t)\right]$$

Mean and Covariance of SDE [2/2]

- If we select $\phi(\mathbf{x}, t) = x_u\, x_v - m_u(t)\, m_v(t)$, then we get differential equation for the components of covariance:

$$
\begin{aligned}
&\frac{\mathrm{d}\, \mathrm{E}[x_u\, x_v - m_u(t)\, m_v(t)]}{\mathrm{d}t} \\
&= \mathrm{E}\left[(x_v - m_v(t))\, f_u(x, t)\right] + \mathrm{E}\left[(x_u - m_u(v))\, f_v(x, t)\right] \\
&\quad + [\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^{\mathsf{T}}(\mathbf{x}, t)]_{uv}.
\end{aligned}
$$

- The final mean and covariance differential equations are

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} &= \mathrm{E}\left[\mathbf{f}(\mathbf{x}, t)\right] \\
\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} &= \mathrm{E}\left[\mathbf{f}(\mathbf{x}, t)\, (\mathbf{x} - \mathbf{m})^{\mathsf{T}}\right] + \mathrm{E}\left[(\mathbf{x} - \mathbf{m})\, \mathbf{f}^{\mathsf{T}}(\mathbf{x}, t)\right] \\
&\quad + \mathrm{E}\left[\mathbf{L}(\mathbf{x}, t)\, \mathbf{Q}\, \mathbf{L}^{\mathsf{T}}(\mathbf{x}, t)\right]
\end{aligned}
$$

- Note that the expectations are w.r.t. $p(\mathbf{x}, t)$!

Mean and Covariance of SDE: Notes

- To solve the equations, we need to know $p(\mathbf{x}, t)$, the solution to the FPK.
- In linear-Gaussian case the first two moments indeed characterize the solution.
- Useful starting point for Gaussian approximations of SDEs.

Mean and Covariance of SDE: Example

$$dx(t) = \tanh(x(t))\, dt + d\beta(t), \quad x(0) = 0,$$

Higher Order Moments

- It is also possible to derive differential equations for the higher order moments of SDEs.
- But with state dimension n, we have n^3 third order moments, n^4 fourth order moments and so on.
- Recall that a given scalar function $\phi(x)$ satisfies

$$\frac{\mathrm{d}\,\mathsf{E}[\phi(x)]}{\mathrm{d}t} = \mathsf{E}\left[\frac{\partial \phi(x)}{\partial x}\, f(x)\right] + \frac{q}{2}\,\mathsf{E}\left[\frac{\partial^2 \phi(x)}{\partial x^2}\, L^2(x)\right].$$

- If we apply this to $\phi(x) = x^n$:

$$\frac{\mathrm{d}\,\mathsf{E}[x^n]}{\mathrm{d}t} = n\,\mathsf{E}[x^{n-1}\, f(x,t)] + \frac{q}{2}\, n\,(n-1)\,\mathsf{E}[x^{n-2}\, L^2(x)]$$

- This, in principle, is an equation for higher order moments.
- To actually use this, we need to use moment closure methods.

Mean and covariance of linear SDEs

- Consider a linear stochastic differential equation

$$\mathrm{d}\mathbf{x} = \mathbf{F}(t)\,\mathbf{x}(t)\;\mathrm{d}t + \mathbf{u}(t)\;\mathrm{d}t + \mathbf{L}(t)\;\mathrm{d}\beta(t), \quad \mathbf{x}(t_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{P}_0).$$

- The mean and covariance equations are now given as

$$\begin{aligned}\frac{\mathrm{d}\mathbf{m}(t)}{\mathrm{d}t} &= \mathbf{F}(t)\,\mathbf{m}(t) + \mathbf{u}(t)\\ \frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &= \mathbf{F}(t)\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^{\mathsf{T}}(t) + \mathbf{L}(t)\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}(t),\end{aligned}$$

- The general solutions are given as

$$\begin{aligned}\mathbf{m}(t) &= \boldsymbol{\Psi}(t,t_0)\,\mathbf{m}(t_0) + \int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\,\mathbf{u}(\tau)\;\mathrm{d}\tau\\ \mathbf{P}(t) &= \boldsymbol{\Psi}(t,t_0)\,\mathbf{P}(t_0)\,\boldsymbol{\Psi}^{\mathsf{T}}(t,t_0)\\ &\quad + \int_{t_0}^{t} \boldsymbol{\Psi}(t,\tau)\,\mathbf{L}(\tau)\,\mathbf{Q}(\tau)\,\mathbf{L}^{\mathsf{T}}(\tau)\,\boldsymbol{\Psi}^{\mathsf{T}}(t,\tau)\;\mathrm{d}\tau\end{aligned}$$

Mean and covariance of LTI SDEs

- In LTI SDE case

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}(t)\,\mathrm{d}t + \mathbf{L}\,\mathrm{d}\beta(t),$$

  we have similarly

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{m}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{m}(t) \\
\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^\mathsf{T} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}
\end{aligned}$$

- The explicit solutions are

$$\begin{aligned}
\mathbf{m}(t) &= \exp(\mathbf{F}\,(t - t_0))\,\mathbf{m}(t_0) \\
\mathbf{P}(t) &= \exp(\mathbf{F}\,(t - t_0))\,\mathbf{P}(t_0)\,\exp(\mathbf{F}\,(t - t_0))^\mathsf{T} \\
&\quad + \int_{t_0}^{t} \exp(\mathbf{F}\,(t - \tau))\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp(\mathbf{F}\,(t - \tau))^\mathsf{T}\,\mathrm{d}\tau.
\end{aligned}$$

LTI SDEs: Matrix fractions

- Let the matrices $\mathbf{C}(t)$ and $\mathbf{D}(t)$ solve the LTI differential equation

$$\begin{pmatrix} \mathrm{d}\mathbf{C}(t)/\,\mathrm{d}t \\ \mathrm{d}\mathbf{D}(t)/\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} \mathbf{F} & \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \\ \mathbf{0} & -\mathbf{F}^\mathsf{T} \end{pmatrix} \begin{pmatrix} \mathbf{C}(t) \\ \mathbf{D}(t) \end{pmatrix}$$

- Then $\mathbf{P}(t) = \mathbf{C}(t)\,\mathbf{D}^{-1}(t)$ solves the differential equation

$$\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} = \mathbf{F}\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^\mathsf{T} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}$$

- Thus we can solve the covariance with matrix exponential as well:

$$\begin{pmatrix} \mathbf{C}(t) \\ \mathbf{D}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \mathbf{F} & \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T} \\ \mathbf{0} & -\mathbf{F}^\mathsf{T} \end{pmatrix} t \right\} \begin{pmatrix} \mathbf{C}(t_0) \\ \mathbf{D}(t_0) \end{pmatrix}.$$

Steady State Solutions of Linear SDEs [1/4]

- Let's now consider steady state solution of LTI SDEs

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}\;\mathrm{d}t + \mathbf{L}\;\mathrm{d}\beta$$

- At the steady state, the time derivatives of mean and covariance should be zero:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{m}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{m}(t) = \mathbf{0} \\
\frac{\mathrm{d}\mathbf{P}(t)}{\mathrm{d}t} &= \mathbf{F}\,\mathbf{P}(t) + \mathbf{P}(t)\,\mathbf{F}^{\mathsf{T}} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}} = \mathbf{0}.
\end{aligned}$$

- The first equation implies that the stationary mean should be identically zero $\mathbf{m}_\infty = \mathbf{0}$.
- The second equation gives the Lyapunov equation, a special case of algebraic Riccati equations (AREs):

$$\mathbf{F}\,\mathbf{P}_\infty + \mathbf{P}_\infty\,\mathbf{F}^{\mathsf{T}} + \mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}} = \mathbf{0}.$$

Steady State Solutions of Linear SDEs [2/4]

- The general solution of LTI SDE is

$$\mathbf{x}(t) = \exp\left(\mathbf{F}\,(t - t_0)\right)\,\mathbf{x}(t_0) + \int_{t_0}^{t} \exp\left(\mathbf{F}\,(t - \tau)\right)\,\mathbf{L}\,\mathrm{d}\beta(\tau).$$

- If we let $t_0 \to -\infty$ then this becomes:

$$\mathbf{x}(t) = \int_{-\infty}^{t} \exp\left(\mathbf{F}\,(t - \tau)\right)\,\mathbf{L}\,\mathrm{d}\beta(\tau)$$

- The covariance function is now given as

$$\begin{aligned}
&\mathsf{E}[\mathbf{x}(t)\,\mathbf{x}^{\mathsf{T}}(t')] \\
&= \mathsf{E}\left\{\left[\int_{-\infty}^{t} \exp\left(\mathbf{F}(t - \tau)\right)\mathbf{L}\,\mathrm{d}\beta(\tau)\right]\left[\int_{-\infty}^{t'} \exp\left(\mathbf{F}(t' - \tau')\right)\mathbf{L}\,\mathrm{d}\beta(\tau')\right]^{\mathsf{T}}\right. \\
&= \int_{-\infty}^{\min(t',t)} \exp\left(\mathbf{F}\,(t - \tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,\exp\left(\mathbf{F}\,(t' - \tau)\right)^{\mathsf{T}}\,\mathrm{d}\tau.
\end{aligned}$$

Steady State Solutions of Linear SDEs [3/4]

- But we already know the following:

$$\mathbf{P}_\infty = \int_{-\infty}^{t} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau,$$

which, by definition, should be independent of t.

- If $t \le t'$, we have

$$\begin{aligned}
&\mathrm{E}[\mathbf{x}(t)\,\mathbf{x}^\mathsf{T}(t')] \\
&= \int_{-\infty}^{t} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t'-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t'-t+t-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{t} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau\,\exp\left(\mathbf{F}\,(t'-t)\right)^\mathsf{T} \\
&= \mathbf{P}_\infty\,\exp\left(\mathbf{F}\,(t'-t)\right)^\mathsf{T}.
\end{aligned}$$

Steady State Solutions of Linear SDEs [4/4]

- If $t > t'$, we get similarly

$$
\begin{aligned}
&\mathrm{E}[\mathbf{x}(t)\,\mathbf{x}^\mathsf{T}(t')] \\
&= \int_{-\infty}^{t'} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t'-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau \\
&= \exp\left(\mathbf{F}\,(t-t')\right)\int_{-\infty}^{t'} \exp\left(\mathbf{F}\,(t-\tau)\right)\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^\mathsf{T}\,\exp\left(\mathbf{F}\,(t'-\tau)\right)^\mathsf{T}\,\mathrm{d}\tau \\
&= \exp\left(\mathbf{F}\,(t-t')\right)\,\mathbf{P}_\infty.
\end{aligned}
$$

- Thus the covariance function of LTI SDE is simply

$$
\mathbf{C}(\tau) = \begin{cases} \mathbf{P}_\infty\,\exp\left(\mathbf{F}\,\tau\right)^\mathsf{T} & \text{if } \tau \geq 0 \\ \exp\left(-\mathbf{F}\,\tau\right)\,\mathbf{P}_\infty & \text{if } \tau < 0. \end{cases}
$$

Fourier Analysis of LTI SDE Revisited

- Let's reconsider Fourier domain solutions of LTI SDEs

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}(t)\,\mathrm{d}t + \mathbf{L}\,\mathrm{d}\beta(t)$$

- We already analyzed them in white noise formalism, which required computation of

$$W(i\,\omega) = \int_{-\infty}^{\infty} w(t)\,\exp(-i\,\omega\,t)\,\mathrm{d}t,$$

- Every stationary Gaussian process $x(t)$ has a representation of the form

$$x(t) = \int_{0}^{\infty} \exp(i\,\omega\,t)\,\mathrm{d}\zeta(i\,\omega),$$

- $\omega \mapsto \zeta(i\,\omega)$ is some complex valued Gaussian process with independent increments.

Fourier Analysis of LTI SDE Revisited (cont.)

- The mean squared difference $\mathrm{E}[|\zeta(\omega_{k+1}) - \zeta(\omega_k)|^2]$ corresponds to the mean power on the interval $[\omega_k, \omega_{k+1}]$.
- The spectral density then corresponds to a function $S(\omega)$ such that

$$\mathrm{E}[|\zeta(\omega_{k+1}) - \zeta(\omega_k)|^2] = \frac{1}{\pi} \int_{\omega_k}^{\omega_{k+1}} S(\omega)\, \mathrm{d}\omega,$$

- By using this kind of integrated Fourier transform the Fourier analysis can be made rigorous.
- For more information, see, for example, Van Trees (1968).

Fourier Analysis of LTI SDE Revisited II

- Another is to consider ODE with smooth Gaussian process $\mathbf{u}$:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{L}\,\mathbf{u}(t),$$

- We can take

$$\mathbf{C}_u(\tau;\Delta) = \mathbf{Q}\,\frac{1}{\sqrt{2\pi\,\Delta^2}}\,\exp\left(-\frac{1}{2\,\Delta^2}\tau^2\right)$$

  which in the limit $\Delta \to 0$ gives the white noise.

- Spectral density of the ODE solution is then

$$\mathbf{S}_{\mathbf{x}}(\omega;\Delta) = (\mathbf{F} - (i\,\omega)\,\mathbf{I})^{-1}\,\mathbf{L}\,\mathbf{Q}\,\exp\left(-\frac{\Delta^2}{2}\,\omega^2\right)\,\mathbf{L}^\mathsf{T}\,(\mathbf{F} + (i\,\omega)\,\mathbf{I})^{-\mathsf{T}}.$$

Fourier Analysis of LTI SDE Revisited II (cont.)

- In the limit $\Delta \to 0$ to get the spectral density corresponding to the white noise input:

$$\mathbf{S}_{\mathbf{x}}(\omega) = \lim_{\Delta \to 0} \mathbf{S}_{\mathbf{x}}(\omega; \Delta) = (\mathbf{F} - (i\,\omega)\,\mathbf{I})^{-1}\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,(\mathbf{F} + (i\,\omega)\,\mathbf{I})^{-\mathsf{T}},$$

- The limiting covariance function is then

$$\mathbf{C}_{\mathbf{x}}(\tau) = \mathscr{F}^{-1}[(\mathbf{F} - (i\,\omega)\,\mathbf{I})^{-1}\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,(\mathbf{F} + (i\,\omega)\,\mathbf{I})^{-\mathsf{T}}].$$

- Because $\mathbf{C}_{\mathbf{x}}(0) = \mathbf{P}_\infty$, we also get the following interesting identity:

$$\mathbf{P}_\infty = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathbf{F} - (i\,\omega)\,\mathbf{I})^{-1}\,\mathbf{L}\,\mathbf{Q}\,\mathbf{L}^{\mathsf{T}}\,(\mathbf{F} + (i\,\omega)\,\mathbf{I})^{-\mathsf{T}}]\,\mathrm{d}\omega$$

Summary

- The probability density of SDE solution $\mathbf{x}(t)$ solves the Fokker–Planck–Kolmogorov (FKP) *partial differential equation*.
- The mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$ of the solution solve a pair of *ordinary differential equations*.
- In non-linear case, the expectations in the mean and covariance equations cannot be solved without knowing the whole probability density.
- For higher moment moments we can derive (theoretical) differential equations as well—can be approximated with moment closure.
- In linear case, we can solve the probability density and all the moments.
- The covariance functions for LTI SDEs can be solved by considering stationary solutions to the equations.