Lecture 1: Pragmatic Introduction to Stochastic Differential Equations

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What is a stochastic differential equation (SDE)?

- At first, we have an ordinary differential equation (ODE):
  \[
  \frac{dx}{dt} = f(x, t).
  \]

- Then we add white noise to the right hand side:
  \[
  \frac{dx}{dt} = f(x, t) + w(t).
  \]

- Generalize a bit by adding a multiplier matrix on the right:
  \[
  \frac{dx}{dt} = f(x, t) + L(x, t)w(t).
  \]

- Now we have a stochastic differential equation (SDE).
- \(f(x, t)\) is the drift function and \(L(x, t)\) is the dispersion matrix.
White noise

1. \( w(t_1) \) and \( w(t_2) \) are independent if \( t_1 \neq t_2 \).
2. \( t \mapsto w(t) \) is a Gaussian process with the mean and covariance:
   \[
   E[w(t)] = 0 \\
   E[w(t)w^T(s)] = \delta(t - s)Q.
   \]

- \( Q \) is the spectral density of the process.
- The sample path \( t \mapsto w(t) \) is discontinuous almost everywhere.
- White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.
What does a solution of SDE look like?

- **Left**: Path of a Brownian motion which is solution to stochastic differential equation

\[
\frac{dx}{dt} = w(t)
\]

- **Right**: Evolution of probability density of Brownian motion.
What does a solution of SDE look like? (cont.)

Paths of stochastic spring model

\[
d^2x(t) + \gamma \frac{dx(t)}{dt} + \nu^2 x(t) = w(t).
\]
Einstein’s construction of Brownian motion
Langevin’s construction of Brownian motion

Random force from collisions

Movement is slowed down by friction
Noisy RC-circuit
Noisy Phase Locked Loop (PLL)

\[ w(t) = K \int_{0}^{t} \phi(t) - \theta_2(t) \]

\[ A \sin(\cdot) \]

\[ \theta_1(t) \]

\[ \phi(t) \]

\[ \theta_2(t) \]

\[ \int_{0}^{t} \]

Loop filter
Car model for navigation
Noisy pendulum model
Solutions of LTI SDEs

- Linear time-invariant stochastic differential equation (LTI SDE):

\[
\frac{dx(t)}{dt} = Fx(t) + Lw(t), \quad x(t_0) \sim N(m_0, P_0).
\]

- We can now take a “leap of faith” and solve this as if it was a deterministic ODE:
  1. Move \(Fx(t)\) to left and multiply by integrating factor \(\exp(-Ft)\):

\[
\exp(-Ft) \frac{dx(t)}{dt} - \exp(-Ft) Fx(t) = \exp(-Ft) Lw(t).
\]
  2. Rewrite this as

\[
\frac{d}{dt} [\exp(-Ft)x(t)] = \exp(-Ft) Lw(t).
\]
  3. Integrate from \(t_0\) to \(t\):

\[
\exp(-Ft)x(t) - \exp(-Ft_0)x(t_0) = \int_{t_0}^{t} \exp(-F\tau)Lw(\tau) \, d\tau.
\]
Rearranging then gives the solution:

\[ x(t) = \exp(F(t - t_0)) x(t_0) + \int_{t_0}^{t} \exp(F(t - \tau)) L w(\tau) \, d\tau. \]

- We have assumed that \( w(t) \) is an ordinary function, which it is not.
- Here we are lucky, because for linear SDEs we get the right solution, but generally not.
- The source of the problem is the integral of a non-integrable function on the right hand side.
Mean and covariance of LTI SDEs

- The mean can be computed by taking expectations:

\[
\mathbb{E}[\mathbf{x}(t)] = \mathbb{E}[\exp(\mathbf{F}(t - t_0))\mathbf{x}(t_0)] + \mathbb{E}\left[\int_{t_0}^{t} \exp(\mathbf{F}(t - \tau)) \mathbf{L} \mathbf{w}(\tau) \, d\tau\right]
\]

- Recalling that \(\mathbb{E}[\mathbf{x}(t_0)] = \mathbf{m}_0\) and \(\mathbb{E}[\mathbf{w}(t)] = 0\) then gives the mean

\[
\mathbf{m}(t) = \exp(\mathbf{F}(t - t_0)) \mathbf{m}_0.
\]

- We also get the following covariance (see the exercises...):

\[
\mathbf{P}(t) = \mathbb{E}\left[ (\mathbf{x}(t) - \mathbf{m}(t)) (\mathbf{x}(t) - \mathbf{m})^T \right] \\
= \exp(\mathbf{F}t) \mathbf{P}_0 \exp(\mathbf{F}t)^T \\
+ \int_{0}^{t} \exp(\mathbf{F}(t - \tau)) \mathbf{L} \mathbf{Q} \mathbf{L}^T \exp(\mathbf{F}(t - \tau))^T \, d\tau.
\]
By differentiating the mean and covariance expression we can derive the following differential equations for the mean and covariance:

\[
\frac{d\mathbf{m}(t)}{dt} = \mathbf{F} \mathbf{m}(t)
\]
\[
\frac{d\mathbf{P}(t)}{dt} = \mathbf{F} \mathbf{P}(t) + \mathbf{P}(t) \mathbf{F}^T + \mathbf{L} \mathbf{Q} \mathbf{L}^T.
\]

For example, let’s consider the spring model:

\[
\begin{pmatrix}
\frac{dx_1(t)}{dt} \\
\frac{dx_2(t)}{dt} \\
\frac{d\mathbf{x}(t)}{dt}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-\nu^2 & -\gamma
\end{pmatrix} \begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} + \begin{pmatrix}
0 \\
1
\end{pmatrix} \mathbf{w}(t).
\]
The mean and covariance equations:

\[
\begin{align*}
\frac{\text{d}m_1}{\text{d}t} &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\
\frac{\text{d}P_{11}}{\text{d}t} &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\
&+ \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix}^T \\
&+ \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}
\end{align*}
\]
We can also attempt to derive mean and covariance equations directly from

\[
\frac{dx(t)}{dt} = Fx(t) + LW(t), \quad x(t_0) \sim N(m_0, P_0).
\]

By taking expectations from both sides gives

\[
E \left[ \frac{dx(t)}{dt} \right] = \frac{dE[x(t)]}{dt} = E[Fx(t) + LW(t)] = F \ E[x(t)].
\]

This thus gives the correct mean differential equation

\[
\frac{dm(t)}{dt} = Fm(t)
\]
For the covariance we use
\[
\frac{d}{dt} \left[ (\mathbf{x} - \mathbf{m}) (\mathbf{x} - \mathbf{m})^T \right] = \left( \frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right) (\mathbf{x} - \mathbf{m})^T
+ (\mathbf{x} - \mathbf{m}) \left( \frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right)^T
\]

Substitute \( \frac{d\mathbf{x}(t)}{dt} = \mathbf{F} \mathbf{x}(t) + \mathbf{L} \mathbf{w}(t) \) and take expectation:
\[
\frac{d}{dt} \mathbb{E} \left[ (\mathbf{x} - \mathbf{m}) (\mathbf{x} - \mathbf{m})^T \right] = \mathbf{F} \mathbb{E} \left[ (\mathbf{x}(t) - \mathbf{m}(t)) (\mathbf{x}(t) - \mathbf{m}(t))^T \right]
+ \mathbb{E} \left[ (\mathbf{x}(t) - \mathbf{m}(t)) (\mathbf{x}(t) - \mathbf{m}(t))^T \right] \mathbf{F}^T
\]

This implies the covariance differential equation
\[
\frac{d\mathbf{P}(t)}{dt} = \mathbf{F} \mathbf{P}(t) + \mathbf{P}(t) \mathbf{F}^T.
\]

But this solution is wrong!
Our mistake was to assume

\[
\frac{d}{dt} \left[ (x - m) (x - m)^T \right] = \left( \frac{dx}{dt} - \frac{dm}{dt} \right) (x - m)^T \\
+ (x - m) \left( \frac{dx}{dt} - \frac{dm}{dt} \right)^T
\]

However, this result from basic calculus is not valid when \( x(t) \) is stochastic.

The mean equation was ok, because its derivation did not involve the usage of chain rule (or product rule) above.

But which results are right and which wrong?

We need to develop a whole new calculus to deal with this...
Fourier domain solution of SDE

- Consider the scalar SDE (Ornstein–Uhlenbeck process):
  \[
  \frac{dx(t)}{dt} = -\lambda x(t) + w(t)
  \]

- Let’s take a formal Fourier transform (Warning: \(w(t)\) is not a square-integrable function!):
  \[
  (i\omega)X(i\omega) = -\lambda X(i\omega) + W(i\omega)
  \]

- Solving for \(X(i\omega)\) gives
  \[
  X(i\omega) = \frac{W(i\omega)}{(i\omega) + \lambda}
  \]

- This can be seen to have the transfer function form
  \[
  X(i\omega) = H(i\omega)W(i\omega)
  \]

  where the transfer function is
  \[
  H(i\omega) = \frac{1}{(i\omega) + \lambda}
  \]
By direct calculation we get

\[ h(t) = \mathcal{F}^{-1}[H(i \omega)] = \exp(-\lambda t) u(t), \]

where \( u(t) \) is the Heaviside step function.

The solution can be expressed as convolution, which thus gives

\[ x(t) = h(t) \ast w(t) \]

\[ = \int_{-\infty}^{\infty} \exp(-\lambda (t - \tau)) u(t - \tau) w(\tau) \, d\tau \]

\[ = \int_{0}^{t} \exp(-\lambda (t - \tau)) w(\tau) \, d\tau \]

provided that \( w(t) \) is assumed to be zero for \( t < 0 \).

Analogous derivation works for multidimensional LTI SDEs

\[ \frac{dx(t)}{dt} = F x(t) + L w(t) \]
A useful quantity is the **spectral density** which is defined as

\[ S_x(\omega) = |X(i\omega)|^2 = X(i\omega)X(-i\omega). \]

What makes it useful is that the stationary-state **covariance function** is its inverse Fourier transform:

\[ C_x(\tau) = \mathbb{E}[x(t)x(t+\tau)] = \mathcal{F}^{-1}[S_x(\omega)] \]

For the **Ornstein–Uhlenbeck** process we get

\[ S_x(\omega) = \frac{|W(i\omega)|^2}{|i\omega + \lambda|^2} = \frac{q}{\omega^2 + \lambda^2}, \]

and

\[ C(\tau) = \frac{q}{2\lambda} \exp(-\lambda |\tau|). \]
In multidimensional case we have (joint) spectral density matrix:

\[ S_x(\omega) = X(i\omega)X^T(-i\omega), \]

The joint covariance matrix is its inverse Fourier transform

\[ C_x(\tau) = \mathcal{F}^{-1}[S_x(\omega)]. \]

For general LTI SDEs

\[ \frac{dx(t)}{dt} = Fx(t) + Lw(t), \]

we get

\[ S_x(\omega) = (F - (i\omega)I)^{-1}LQL^T(F + (i\omega)I)^{-T} \]

\[ C_x(\tau) = \mathcal{F}^{-1}[(F - (i\omega)I)^{-1}LQL^T(F + (i\omega)I)^{-T}]. \]
We could now attempt to analyze \textit{non-linear SDEs} of the form

\[
\frac{dx}{dt} = f(x, t) + L(x, t) w(t)
\]

We cannot solve the deterministic case—no possibility for a \textit{“leap of faith”}. We don’t know how to derive the mean and covariance equations. What we can do is to simulate by using \textit{Euler–Maruyama}:

\[
\hat{x}(t_{k+1}) = \hat{x}(t_k) + f(\hat{x}(t_k), t_k) \Delta t + L(\hat{x}(t_k), t_k) \Delta \beta_k,
\]

where $\Delta \beta_k$ is a Gaussian random variable with distribution $N(0, Q \Delta t)$. Note that the variance is proportional to $\Delta t$, not the standard derivation.
Problem with general solutions (cont.)

- **Picard–Lindelöf** theorem can be useful for analyzing existence and uniqueness of ODE solutions. Let’s try that for

\[
\frac{dx}{dt} = f(x, t) + L(x, t)w(t)
\]

- The basic assumption in the theorem for the right hand side of the differential equation were:
  - Continuity in both arguments.
  - Lipschitz continuity in the first argument.

- But white noise is **discontinuous everywhere**!

- We need a new existence theory for SDE solutions as well...
Stochastic differential equation (SDE) is an ordinary differential equation (ODE) with a stochastic driving force. SDEs arise in various physics and engineering problems. Solutions for linear SDEs can be (heuristically) derived in the similar way as for deterministic ODEs. We can also compute the mean and covariance of the solutions of a linear SDE. Fourier transform solutions to linear time-invariant (LTI) SDEs lead to the useful concepts of spectral density and covariance function. The heuristic treatment only works for some analysis of linear SDEs, and for e.g. non-linear equations we need a new theory. One way to approximate solution of SDE is to simulate trajectories from it using the Euler–Maruyama method.
\[ \frac{dx(t)}{dt} = -\lambda x(t) + w(t), \quad x(0) = x_0, \]