Extra Exercise Round 5 (DL 16.12.2012).

- These are extra exercises which you can do if you otherwise will not reach the required number of 8 exercise points. You get one point per exercise. The deadline for the extra exercises is **December 16, 2012**.

The answers should be sent as email to the teacher (simo.sarkka@aalto.fi) in PDF form. When sending the email, please add "MAT55216" to the subject.

Exercise 1. (Application of Girsanov Theorem)

Solve the probability density of the SDE

$$\mathrm{d}x = \tanh(x)\ \mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{1}$$

where $\beta(t)$ is a standard Brownian motion, with the Girsanov theorem as follows:

1. Define $\tilde{x}(t) = \beta(t)$ and $\theta(t) = f(\tilde{x})$ and express the new Brownian motion $\tilde{\beta}(t)$ given by the Girsanov theorem in terms of $\tilde{x}(t)$.
2. Rearrange the terms and write the resulting stochastic equation in differential form such that you can conclude that the pair $(\tilde{x}, \tilde{\beta})$ forms a weak solution to the SDE.
3. Compute the corresponding $Z(t)$ and use the result of applying Itô formula to $\ln(\cosh(\beta))$ for evaluating the integrals in the exponent explicitly.
4. Form the probability density of $x(t)$ by multiplying the probability density of Brownian motion with $Z(t)$.

Exercise 2. (Stochastic Runge–Kutta)

Solve the SDE in Equation (1) with the strong 1.5 order stochastic Runge-Kutta method found in the book of Kloeden et al. 1994 and compare the histogram to the true one at t = 5. Plot the histogram against the exact solution, report the results and return the Matlab codes (or other codes) as well.

Exercise 3. (Karhunen–Loeve Expansion)

Solve the SDE in Equation (1) using the sinusoidal series expansion for the Brownian motion (see e.g. Luo 2006) and compare the histogram to the true one at t = 5 with different numbers of basis functions. Plot the histogram against the exact solution, report the results and return the Matlab codes (or other codes) as well.