Exercise Round 3 (15.11.2012).

Exercise 1. (Fokker–Planck–Kolmogorov (FKP) equation)

A) Write down the FKP for

$$\mathrm{d}x = \tanh(x)\ \mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{1}$$

where $\beta(t)$ is a standard Brownian motion, and check that the following solves it:

$$p(x,t) = \frac{1}{\sqrt{2\pi\, t}}\ \cosh(x)\ \exp\left(-\frac{1}{2}\, t\right)\ \exp\left(-\frac{1}{2t}\, x^2\right).$$

B) Plot the evolution of the probability density at times $t \in [0, 5]$.

C) Simulate 1000 trajectories from the SDE using Euler-Maruyama method and check that the histogram matches the correct density at time $t = 5$.

Exercise 2. (Numerical solution of FPK)

Use finite-differences method to solve the FPK for the Equation (1). For simplicity, you can select a finite range $x \in [-L, L]$ and use the Diriclet boundary conditions $p(-L, t) = p(L, t) = 0$.

A) Divide the range to n grid points and let $h = 1/(n + 1)$. On the grid, approximate the partial derivatives of p(x,t) via

$$\begin{aligned}\frac{\partial p(x,t)}{\partial x} &\approx \frac{p(x+h,t) - p(x-h,t)}{2h} \\ \frac{\partial^2 p(x,t)}{\partial x^2} &\approx \frac{p(x+h,t) - 2\,p(x,t) + p(x-h,t)}{h^2}.\end{aligned} \tag{2}$$

B) Let $\mathbf{p}(t) = (p(h,t)\ p(2h,t)\ \cdots\ p(nh,t))^\mathsf{T}$ and from the above, form an equation of the form

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{p}. \tag{3}$$

C) Solve the above equation using (1) backward Euler (2) by numerical computation of $\exp(\mathbf{F}\,t)$ and by (3) forward Euler. Check that the results match the solution in the previous exercise.

Exercise 3. (Langevin's physical Brownian motion) Consider the Langevin's model of Brownian motion

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -c \, \frac{\mathrm{d}x}{\mathrm{d}t} + w, \qquad x(0) = (\mathrm{d}x/\,\mathrm{d}t)(0) = 0, \tag{4}$$

where $c = 6 \, \pi \, \eta \, r$ and the white noise $w(t)$ has some spectral density q.

A) Interpret the above model as Itô SDE and write it as a two-dimensional state-space form SDE.

B) Write down the differential equations for the elements of the mean $\mathbf{m}(t)$ and covariance $\mathbf{P}(t)$. Conclude that the mean is zero and find the closed form solutions for the elements $P_{11}(t)$, $P_{12}(t)$, $P_{21}(t)$, and $P_{22}(t)$ of the covariance matrix $\mathbf{P}(t)$. *Hint:* start by solving $P_{22}(t)$, then use it to find the solutions for $P_{12}(t) = P_{21}(t)$, and finally solve $P_{11}(t)$.

C) Find the limiting solution $P_{22}(t)$ when $t \to \infty$ and use the following to determine the diffusion coefficient (spectral density) q:

$$m \; \mathrm{E}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\right] = \frac{R\,T}{N}. \tag{5}$$

D) Plot the solution $P_{11}(t)$ and conclude that asympthotically approaches a straight line. Compute the asymphotic solution $P_{11}(t)$ when $t \to \infty$, and conclude that it approximately gives Langevin's result.