Applied Stochastic Differential Equations **Autumn 2012**

Exercise Round 2 (8.11.2012).

Exercise 1. (Usage of Itô formula)

A) Compute the Itô differential of

$$\phi(\beta) = t + \exp(\beta)$$

where $\beta(t)$ is a Brownian motion with diffusion constant q.

B) Compute the Itô differential of

$$\phi(x) = x^2,$$

where x solves the scalar SDE

$$\mathrm{d}x = f(x)\ \mathrm{d}t + \sigma\ \mathrm{d}\beta,$$

σ is a constant, and $\beta(t)$ is a standard Brownian motion ($q = 1$).

C) Compute the Itô differential of

$$\phi(\mathbf{x}) = \mathbf{x}^\mathsf{T}\,\mathbf{x}$$

where

$$\mathrm{d}\mathbf{x} = \mathbf{F}\,\mathbf{x}\ \mathrm{d}t + \mathrm{d}\boldsymbol{\beta}$$

where **F** is a constant matrix and the joint diffusion matrix of $\boldsymbol{\beta}$ is **Q**.

Exercise 2. (Stochastic Differential Equations)

A) Check that

$$x(t) = \exp(\beta(t))$$

solves the SDE

$$\mathrm{d}x = \frac{1}{2}\,x\ \mathrm{d}t + x\ \mathrm{d}\beta,$$

where $\beta(t)$ is a standard Brownian motion ($q = 1$).

B) Solve the following SDE by changing the variable to $y = \ln x$:

$$\mathrm{d}x = -c\,x\ \mathrm{d}\,\beta$$

$c > 0$ is a constant, and $\beta(t)$ is a standard Brownian motion.

C) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:

$$\begin{aligned}\mathrm{d}x_1 &= -x_2\ \circ \mathrm{d}\beta_1 \\ \mathrm{d}x_2 &= x_1\ \circ \mathrm{d}\beta_2\end{aligned}$$

where β_1 and β_2 are independent standard Brownian motions.

Exercise 3. (Mean and variance differential equations)

Derive the mean and covariance equations for the scalar SDE

$$\mathrm{d}x = f(x)\ \mathrm{d}t + \sigma(x)\ \mathrm{d}\beta, \tag{1}$$

where β has the diffusion coefficient q, as follows:

A) Conclude from the definition of Itô integral that

$$\mathrm{E}\left[\int_u^v \sigma(x(t))\ \mathrm{d}\beta(t)\right] = 0$$

for any u and v.

B) Take expectations from both sides of the SDE (1) and formally divide by $\mathrm{d}t$ to get the differential equation for the mean $m(t)$.

C) Apply Itô formula to $\phi(x,t) = (x - m(t))^2$ and take expectation of the resulting equation to derive the differential equation for the variance.

D) Write down the mean and covariance differential equations for the scalar SDE

$$\mathrm{d}x = -\lambda\, x\ \mathrm{d}t + \mathrm{d}\beta,$$

where $\lambda > 0$ and solve them with $x(0) = x_0$.