Exercise Round 2 (8.11.2012).

Exercise 1. (Usage of Itô formula)

A) Compute the Itô differential of
\[ \phi(\beta) = t + \exp(\beta) \]
where \( \beta(t) \) is a Brownian motion with diffusion constant \( q \).

B) Compute the Itô differential of
\[ \phi(x) = x^2, \]
where \( x \) solves the scalar SDE
\[ dx = f(x) \, dt + \sigma \, d\beta, \]
\( \sigma \) is a constant, and \( \beta(t) \) is a standard Brownian motion (\( q = 1 \)).

C) Compute the Itô differential of
\[ \phi(x) = x^T x \]
where
\[ dx = F x \, dt + d\beta \]
where \( F \) is a constant matrix and the joint diffusion matrix of \( \beta \) is \( Q \).

Exercise 2. (Stochastic Differential Equations)

A) Check that
\[ x(t) = \exp(\beta(t)) \]
solves the SDE
\[ dx = \frac{1}{2} x \, dt + x \, d\beta, \]
where \( \beta(t) \) is a standard Brownian motion (\( q = 1 \)).

B) Solve the following SDE by changing the variable to \( y = \ln x \):
\[ dx = -c x \, d\beta \]
c > 0 is a constant, and \( \beta(t) \) is a standard Brownian motion.

C) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:
\[ dx_1 = -x_2 \circ d\beta_1 \]
\[ dx_2 = x_1 \circ d\beta_2 \]
where \( \beta_1 \) and \( \beta_2 \) are independent standard Brownian motions.
Exercise 3. (Mean and variance differential equations)

Derive the mean and covariance equations for the scalar SDE

\[ \text{d}x = f(x) \text{d}t + \sigma(x) \text{d}\beta, \]

where \( \beta \) has the diffusion coefficient \( q \), as follows:

A) Conclude from the definition of Itô integral that

\[ E \left[ \int_u^v \sigma(x(t)) \text{d}\beta(t) \right] = 0 \]

for any \( u \) and \( v \).

B) Take expectations from both sides of the SDE (1) and formally divide by \( \text{d}t \) to get the differential equation for the mean \( m(t) \).

C) Apply Itô formula to \( \phi(x, t) = (x - m(t))^2 \) and take expectation of the resulting equation to derive the differential equation for the variance.

D) Write down the mean and covariance differential equations for the scalar SDE

\[ \text{d}x = -\lambda x \text{d}t + \text{d}\beta, \]

where \( \lambda > 0 \) and solve them with \( x(0) = x_0 \).