Exercise Round 1 (1.11.2012).

Exercise 1. (Mean and covariance equations)

A) Complete the missing steps in the derivation of the covariance (2.37).

B) Derive the mean and covariance differential equations (2.38) by differentiating the equations (2.36) and (2.37).

Exercise 2. (Solution of Ornstein–Uhlenbeck process)

A) Find the complete solution $x(t)$ as well as the mean $m(t)$ and variance $P(t)$ of the following scalar stochastic differential equation:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -\lambda\, x(t) + w(t), \quad x(0) = x_0, \tag{1}$$

where x_0 and $\lambda > 0$ are given constants and the white noise $w(t)$ has the spectral density q.

B) Compute the limit of the mean and variance when $t \to \infty$ (1) directly via $\lim_{t\to\infty} P(t)$ and (2) by solving the stationary state of the variance differential equation $\mathrm{d}P/\mathrm{d}t = 0$.

Exercise 3. (Euler–Maruyama solution of O–U process)

Simulate 1000 trajectories on the time interval $t \in [0, 1]$ from the Ornstein–Uhlenbeck process in the previous exercise using the Euler–Maruyama method with $\lambda = 1/2$, $q = 1$, $\Delta t = 1/100$, $x_0 = 1$ and check that the mean and covariance trajectories approximately agree with the theoretical values.