

Lecture 1: Pragmatic Introduction to Stochastic Differential Equations

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What is a stochastic differential equation (SDE)?

- At first, we have an **ordinary differential equation (ODE)**:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t).$$

- Then we add **white noise** to the right hand side:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t).$$

- Generalize a bit by adding a **multiplier matrix** on the right:

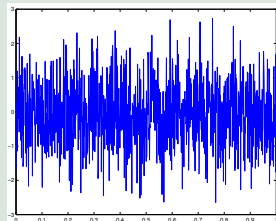
$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t) \mathbf{w}(t).$$

- Now we have a **stochastic differential equation (SDE)**.
- $\mathbf{f}(\mathbf{x}, t)$ is the **drift function** and $\mathbf{L}(\mathbf{x}, t)$ is the **dispersion matrix**.

White noise

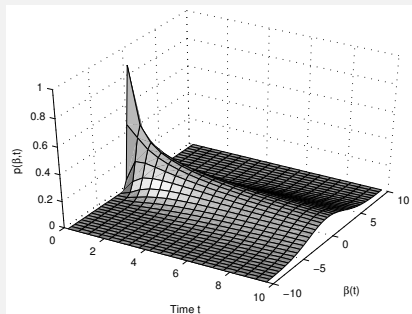
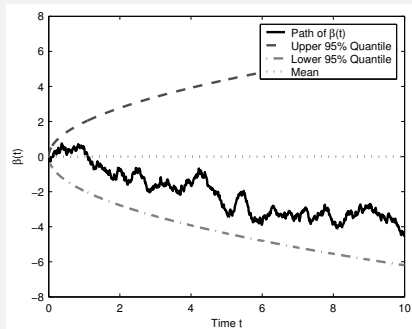
- 1 $\mathbf{w}(t_1)$ and $\mathbf{w}(t_2)$ are independent if $t_1 \neq t_2$.
- 2 $t \mapsto \mathbf{w}(t)$ is a Gaussian process with the mean and covariance:

$$\begin{aligned}E[\mathbf{w}(t)] &= \mathbf{0} \\E[\mathbf{w}(t) \mathbf{w}^T(s)] &= \delta(t - s) \mathbf{Q}.\end{aligned}$$



- \mathbf{Q} is the **spectral density** of the process.
- The sample path $t \mapsto \mathbf{w}(t)$ is **discontinuous almost everywhere**.
- White noise is **unbounded** and it takes arbitrarily large positive and negative values at any finite interval.

What does a solution of SDE look like?

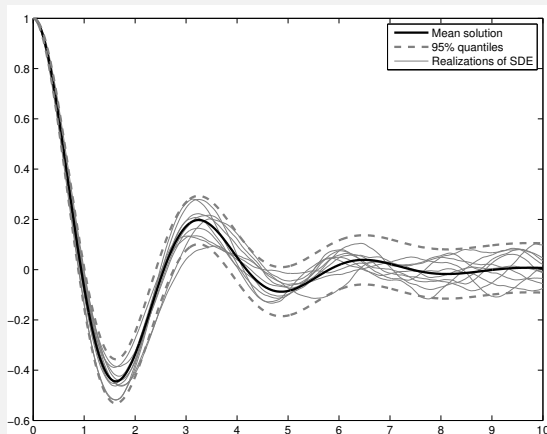


- **Left:** Path of a Brownian motion which is solution to stochastic differential equation

$$\frac{dx}{dt} = w(t)$$

- **Right:** Evolution of probability density of Brownian motion.

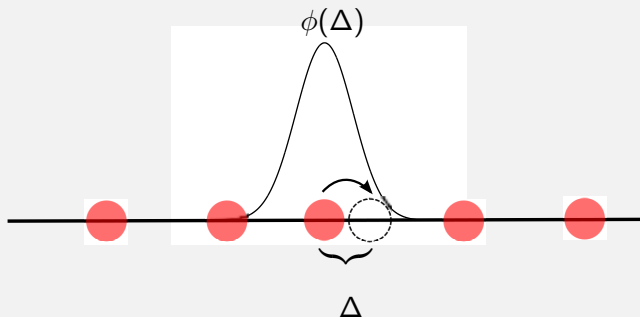
What does a solution of SDE look like? (cont.)



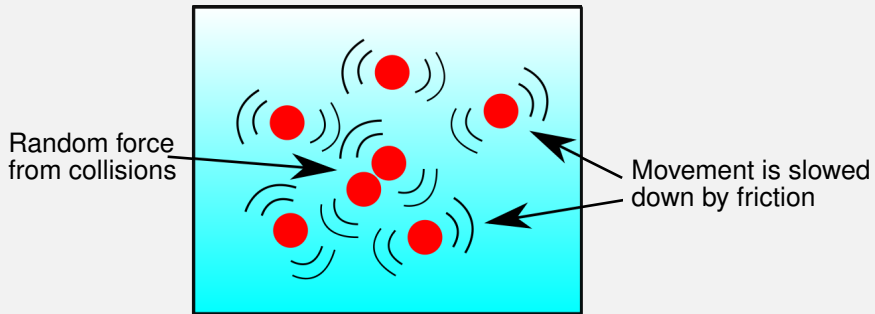
Paths of **stochastic spring model**

$$\frac{d^2x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + \nu^2 x(t) = w(t).$$

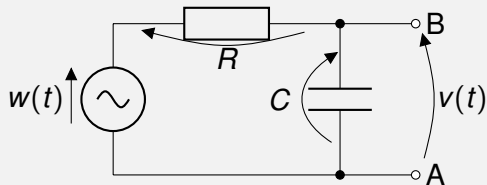
Einstein's construction of Brownian motion



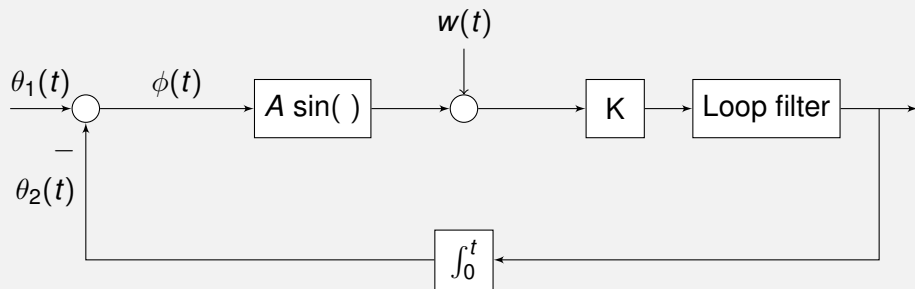
Langevin's construction of Brownian motion



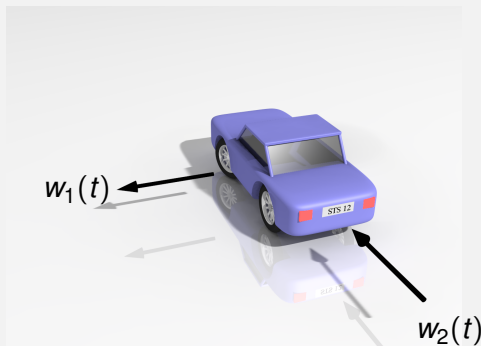
Noisy RC-circuit



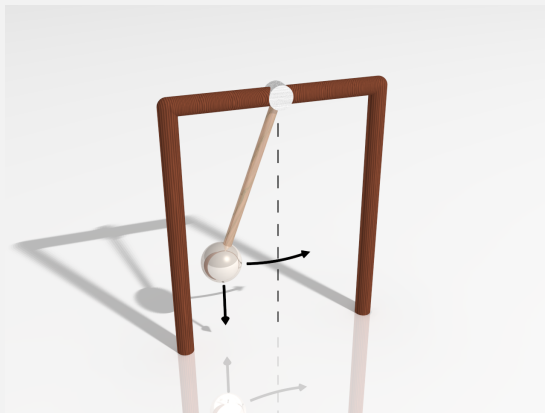
Noisy Phase Locked Loop (PLL)



Car model for navigation



Noisy pendulum model



Solutions of LTI SDEs

- Linear time-invariant stochastic differential equation (LTI SDE):

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F} \mathbf{x}(t) + \mathbf{L} \mathbf{w}(t), \quad \mathbf{x}(t_0) \sim \mathbf{N}(\mathbf{m}_0, \mathbf{P}_0).$$

- We can now take a “leap of faith” and solve this as if it was a deterministic ODE:

- Move $\mathbf{F} \mathbf{x}(t)$ to left and multiply by **integrating factor** $\exp(-\mathbf{F} t)$:

$$\exp(-\mathbf{F} t) \frac{d\mathbf{x}(t)}{dt} - \exp(-\mathbf{F} t) \mathbf{F} \mathbf{x}(t) = \exp(-\mathbf{F} t) \mathbf{L} \mathbf{w}(t).$$

- Rewrite this as

$$\frac{d}{dt} [\exp(-\mathbf{F} t) \mathbf{x}(t)] = \exp(-\mathbf{F} t) \mathbf{L} \mathbf{w}(t).$$

- Integrate from t_0 to t :

$$\exp(-\mathbf{F} t) \mathbf{x}(t) - \exp(-\mathbf{F} t_0) \mathbf{x}(t_0) = \int_{t_0}^t \exp(-\mathbf{F} \tau) \mathbf{L} \mathbf{w}(\tau) d\tau.$$

- Rearranging then gives the **solution**:

$$\mathbf{x}(t) = \exp(\mathbf{F}(t - t_0)) \mathbf{x}(t_0) + \int_{t_0}^t \exp(\mathbf{F}(t - \tau)) \mathbf{L} \mathbf{w}(\tau) d\tau.$$

- We have assumed that $\mathbf{w}(t)$ is an **ordinary function**, which it is **not**.
- Here we are lucky, because for **linear SDEs** we get the **right solution**, but **generally not**.
- The source of the problem is the **integral of a non-integrable function** on the right hand side.

Mean and covariance of LTI SDEs

- The mean can be computed by **taking expectations**:

$$E[\mathbf{x}(t)] = E[\exp(\mathbf{F}(t - t_0)) \mathbf{x}(t_0)] + E\left[\int_{t_0}^t \exp(\mathbf{F}(t - \tau)) \mathbf{L} \mathbf{w}(\tau) d\tau\right]$$

- Recalling that $E[\mathbf{x}(t_0)] = \mathbf{m}_0$ and $E[\mathbf{w}(t)] = 0$ then gives **the mean**

$$\mathbf{m}(t) = \exp(\mathbf{F}(t - t_0)) \mathbf{m}_0.$$

- We also get the following **covariance** (see the exercises...):

$$\begin{aligned} \mathbf{P}(t) &= E\left[(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T\right] \\ &= \exp(\mathbf{F}t) \mathbf{P}_0 \exp(\mathbf{F}t)^T \\ &\quad + \int_0^t \exp(\mathbf{F}(t - \tau)) \mathbf{L} \mathbf{Q} \mathbf{L}^T \exp(\mathbf{F}(t - \tau))^T d\tau. \end{aligned}$$

Mean and covariance of LTI SDEs (cont.)

- By differentiating the mean and covariance expression we can derive the following **differential equations for the mean and covariance**:

$$\begin{aligned}\frac{d\mathbf{m}(t)}{dt} &= \mathbf{F} \mathbf{m}(t) \\ \frac{d\mathbf{P}(t)}{dt} &= \mathbf{F} \mathbf{P}(t) + \mathbf{P}(t) \mathbf{F}^T + \mathbf{L} \mathbf{Q} \mathbf{L}^T.\end{aligned}$$

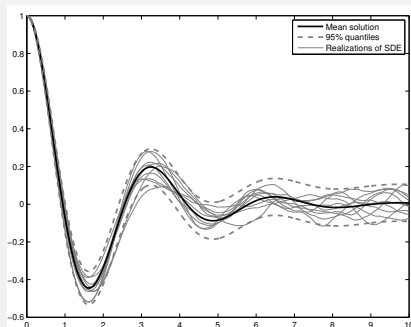
- For example, let's consider the **spring model**:

$$\underbrace{\begin{pmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{pmatrix}}_{d\mathbf{x}(t)/dt} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix}}_{\mathbf{F}} \underbrace{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\mathbf{L}} w(t).$$

Mean and covariance of LTI SDEs (cont.)

The mean and covariance equations:

$$\begin{aligned} \begin{pmatrix} \frac{dm_1}{dt} \\ \frac{dm_2}{dt} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \\ \begin{pmatrix} \frac{dP_{11}}{dt} & \frac{dP_{12}}{dt} \\ \frac{dP_{21}}{dt} & \frac{dP_{22}}{dt} \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \\ &+ \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\nu^2 & -\gamma \end{pmatrix}^T \\ &+ \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \end{aligned}$$



Alternative derivation of mean and covariance

- We can also attempt to derive **mean and covariance equations** directly from

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F} \mathbf{x}(t) + \mathbf{L} \mathbf{w}(t), \quad \mathbf{x}(t_0) \sim N(\mathbf{m}_0, \mathbf{P}_0).$$

- By taking expectations from both sides gives

$$\mathbb{E} \left[\frac{d\mathbf{x}(t)}{dt} \right] = \frac{d\mathbb{E}[\mathbf{x}(t)]}{dt} = \mathbb{E} [\mathbf{F} \mathbf{x}(t) + \mathbf{L} \mathbf{w}(t)] = \mathbf{F} \mathbb{E}[\mathbf{x}(t)].$$

- This thus gives the correct mean differential equation

$$\frac{d\mathbf{m}(t)}{dt} = \mathbf{F} \mathbf{m}(t)$$

Alternative derivation of mean and covariance (cont.)

- For the covariance we use

$$\begin{aligned} \frac{d}{dt} [(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T] &= \left(\frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right) (\mathbf{x} - \mathbf{m})^T \\ &\quad + (\mathbf{x} - \mathbf{m}) \left(\frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right)^T \end{aligned}$$

- Substitute $d\mathbf{x}(t)/dt = \mathbf{F}\mathbf{x}(t) + \mathbf{L}\mathbf{w}(t)$ and take expectation:

$$\begin{aligned} \frac{d}{dt} \mathbb{E} [(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^T] &= \mathbf{F} \mathbb{E} [(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T] \\ &\quad + \mathbb{E} [(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^T] \mathbf{F}^T \end{aligned}$$

- This implies the covariance differential equation

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{F}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T.$$

- But this solution is wrong!**

Alternative derivation of mean and covariance (cont.)

- Our mistake was to assume

$$\begin{aligned} \frac{d}{dt} [(\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^\top] &= \left(\frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right) (\mathbf{x} - \mathbf{m})^\top \\ &\quad + (\mathbf{x} - \mathbf{m}) \left(\frac{d\mathbf{x}}{dt} - \frac{d\mathbf{m}}{dt} \right)^\top \end{aligned}$$

- However, this result from basic calculus is **not valid** when $\mathbf{x}(t)$ is stochastic.
- The mean equation was ok, because its derivation did not involve the usage of **chain rule** (or product rule) above.
- But **which results** are **right** and which **wrong**?
- We need to develop a **whole new calculus** to deal with this. . .

Fourier domain solution of SDE

- Consider the scalar SDE (**Ornstein–Uhlenbeck process**):

$$\frac{dx(t)}{dt} = -\lambda x(t) + w(t)$$

- Let's take a **formal Fourier transform** (*Warning: $w(t)$ is not a square-integrable function!*):

$$(i\omega) X(i\omega) = -\lambda X(i\omega) + W(i\omega)$$

- Solving for $X(i\omega)$ gives

$$X(i\omega) = \frac{W(i\omega)}{(i\omega) + \lambda}$$

- This can be seen to have **the transfer function form**

$$X(i\omega) = H(i\omega) W(i\omega)$$

where the **transfer function** is

$$H(i\omega) = \frac{1}{(i\omega) + \lambda}$$

Fourier domain solution of SDE (cont.)

- By direct calculation we get

$$h(t) = \mathcal{F}^{-1}[H(i\omega)] = \exp(-\lambda t) u(t),$$

where $u(t)$ is the **Heaviside step function**.

- The solution can be expressed as **convolution**, which thus gives

$$\begin{aligned}x(t) &= h(t) * w(t) \\&= \int_{-\infty}^{\infty} \exp(-\lambda(t - \tau)) u(t - \tau) w(\tau) d\tau \\&= \int_0^t \exp(-\lambda(t - \tau)) w(\tau) d\tau\end{aligned}$$

provided that $w(t)$ is assumed to be zero for $t < 0$.

- Analogous derivation works for **multidimensional LTI SDEs**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F} \mathbf{x}(t) + \mathbf{L} w(t)$$

Spectral densities and covariance functions

- A useful quantity is the **spectral density** which is defined as

$$S_x(\omega) = E[|X(i\omega)|^2] = E[X(i\omega) X(-i\omega)].$$

- What makes it useful is that the stationary-state **covariance function** is its inverse Fourier transform:

$$C_x(\tau) = E[x(t) x(t + \tau)] = \mathcal{F}^{-1}[S_x(\omega)]$$

- For the **Ornstein–Uhlenbeck** process we get

$$S_x(\omega) = \frac{E[|W(i\omega)|^2]}{|(i\omega) + \lambda|^2} = \frac{q}{\omega^2 + \lambda^2},$$

and

$$C(\tau) = \frac{q}{2\lambda} \exp(-\lambda |\tau|).$$

Spectral densities and covariance functions (cont.)

- In **multidimensional** case we have (joint) spectral density matrix:

$$\mathbf{S}_x(\omega) = \mathbb{E}[\mathbf{X}(i\omega) \mathbf{X}^T(-i\omega)],$$

- The joint covariance matrix is its **inverse Fourier transform**

$$\mathbf{C}_x(\tau) = \mathcal{F}^{-1}[\mathbf{S}_x(\omega)].$$

- For **general LTI SDEs**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F} \mathbf{x}(t) + \mathbf{L} \mathbf{w}(t),$$

we get

$$\mathbf{S}_x(\omega) = (\mathbf{F} - (i\omega) \mathbf{I})^{-1} \mathbf{L} \mathbf{Q} \mathbf{L}^T (\mathbf{F} + (i\omega) \mathbf{I})^{-T}$$

$$\mathbf{C}_x(\tau) = \mathcal{F}^{-1}[(\mathbf{F} - (i\omega) \mathbf{I})^{-1} \mathbf{L} \mathbf{Q} \mathbf{L}^T (\mathbf{F} + (i\omega) \mathbf{I})^{-T}].$$

Problem with general solutions

- We could now attempt to analyze **non-linear SDEs** of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t) \mathbf{w}(t)$$

- We cannot solve the deterministic case—no possibility for a “**leap of faith**”.
- We don't know how to derive the **mean and covariance equations**.
- What we can do is to simulate by using **Euler–Maruyama**:

$$\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_k), t_k) \Delta t + \mathbf{L}(\hat{\mathbf{x}}(t_k), t_k) \Delta \beta_k,$$

where $\Delta \beta_k$ is a Gaussian random variable with distribution $\mathbf{N}(\mathbf{0}, \mathbf{Q} \Delta t)$.

- Note that the **variance** is proportional to Δt , not the standard deviation.

Problem with general solutions (cont.)

- **Picard–Lindelöf** theorem can be useful for analyzing existence and uniqueness of ODE solutions. Let's try that for

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t) \mathbf{w}(t)$$

- The basic assumption in the theorem for the right hand side of the differential equation were:
 - **Continuity** in both arguments.
 - **Lipschitz continuity** in the first argument.
- But white noise is **discontinuous everywhere!**
- We need a new **existence theory** for SDE solutions as well. . .

Summary

- **Stochastic differential equation (SDE)** is an ordinary differential equation (ODE) with a stochastic driving force.
- SDEs arise in various **physics and engineering** problems.
- Solutions for **linear SDEs** can be (heuristically) derived in the similar way as for deterministic ODEs.
- We can also compute the **mean and covariance** of the solutions of a linear SDE.
- **Fourier transform** solutions to linear **time-invariant (LTI) SDEs** lead to the useful concepts of spectral density and covariance function.
- The **heuristic treatment** only works for some analysis of **linear** SDEs, and for e.g. **non-linear equations** we need a new theory.
- One way to approximate solution of SDE is to simulate trajectories from it using the **Euler–Maruyama method**.

$$\frac{dx(t)}{dt} = -\lambda x(t) + w(t), \quad x(0) = x_0,$$