Exercise Round 3 (December 2, 2013).

Exercise 1. (Wiener velocity model)

Derive the expressions for matrices $\mathbf{A}_k$ and $\mathbf{Q}_k$ in the exactly discretized version of the model

$$\frac{d^2 x(t)}{dt^2} = w(t), \tag{1}$$

where $w(t)$ is a white noise with spectral density q, and the sampling period is $t_k - t_{k-1} = \Delta t$.

Exercise 2. (Kalman filter and RTS smoother for O-U model)

Consider the model

$$\begin{aligned} dx &= -\lambda \, x \, dt + d\beta \\ y_k &= x(t_k) + e_k, \end{aligned} \tag{2}$$

with $\lambda = 1/2$, $q = 1$, $x(0) \sim N(0, P_\infty)$, $e_k \sim N(0, 1)$, where P_∞ is the stationary variance of the SDE.

A) Simulate data from the model by using Euler–Maruyama with step size $\Delta t = 1/100$ over the time period $[0, 10]$, and generate measurements at the time steps $t_k = k$ for $k = 1, \ldots, 10$.

B) Implement a Kalman filter to the model with Method A.

C) Implement an RTS smoother (method A) to the problem.

D) How would you compute the smoothing solution at an arbitrary t?

Exercise 3. (Continuous-time filtering)

A) Write down the Kushner-Stratonovich equation for the model

$$\begin{aligned} dx &= -\lambda \, x \, dt + d\beta \\ dy &= x \, dt + d\eta, \end{aligned} \tag{3}$$

where β and η are independent standard Brownian motions.

B) Write down the corresponding Zakai equation

C) Write down the Kalman-Bucy filter for the model

D) Show that the filters in A), B), and C) are equivalent

Exercise 4. (GP-regression with O-U covariance function)

A) Implement GP regression for the model

$$\begin{aligned} f(t) &\sim \mathcal{GP}(0, k(t, t')) \\ y_k &= f(t_k) + e_k, \end{aligned} \tag{4}$$

where the covariance function is

$$k(t, t') = \frac{q}{2\,\lambda} \exp(-\lambda\,|t - t'|) \tag{5}$$

with the same parameters as in Exercise 2. Apply the GP-regressor to the data simulated in Exercise 2.

B) Compute the spectrum, do the spectral factorization, and form the corresponding state-space representation.

C) Check that the RTS smoother in Exercise 2 gives the same solution as the GP regression above.

Exercise 5. (Approximation of sinusoidal with Matern $3/2$)

Recall that the Matern $3/2$ covariance function and its spectral density are given as

$$\begin{aligned} k(t, t') &= \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\sqrt{2\nu}\,\frac{|t - t'|}{l}\right)^{\nu} K_\nu\left(\sqrt{2\nu}\,\frac{|t - t'|}{l}\right) \\ S(\omega) &= \sigma^2\,\frac{2\pi^{1/2}\Gamma(\nu + 1/2)}{\Gamma(\nu)} \lambda^{2\nu} (\lambda^2 + \omega^2)^{-(\nu+1/2)}, \end{aligned} \tag{6}$$

where $\lambda = \sqrt{2\nu}/l$ and $\nu = 3/2$.

A) Generate data from the function $\sin(t)$ on grid $\Delta t = 1/100$ and measurements at times $1, 2, 3, \ldots, 10$ with standard deviation $1/10$.

B) Implement GP regression with the Matern $3/2$ covariance function with $l = 2$, $\sigma = 1/2$. and compute the GP-regression solution at ever point on the grid.

C) Derive the state-space representation of the $3/2$ Matern process.

D) Implement Kalman filter / RTS smoother solution to the state-space representation of the Matern $3/2$ and check that the result matches that of the GP solution in B) above.