Exercise Round 2 (November 25, 2013).

Exercise 1. (Fokker–Planck–Kolmogorov (FKP) equation)

A) Write down the FKP for

$$\mathrm{d}x = \tanh(x)\ \mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{1}$$

where $\beta(t)$ is a standard Brownian motion, and check that the following solves it:

$$p(x,t) = \frac{1}{\sqrt{2\pi\,t}}\ \cosh(x)\ \exp\left(-\frac{1}{2}\,t\right)\ \exp\left(-\frac{1}{2t}\,x^2\right).$$

B) Plot the evolution of the probability density at times $t \in [0,5]$.

C) Simulate 1000 trajectories from the SDE using Euler-Maruyama method and check that the histogram matches the correct density at time $t = 5$.

Exercise 2. (Numerical solution of FPK)

Use finite-differences method to solve the FPK for the Equation (4). For simplicity, you can select a finite range $x \in [-L, L]$ and use the Diriclet boundary conditions $p(-L,t) = p(L,t) = 0$.

A) Divide the range to n grid points and let $h = 1/(n+1)$. On the grid, approximate the partial derivatives of $p(x,t)$ via

$$\begin{aligned}\frac{\partial p(x,t)}{\partial x} &\approx \frac{p(x+h,t) - p(x-h,t)}{2h} \\ \frac{\partial^2 p(x,t)}{\partial x^2} &\approx \frac{p(x+h,t) - 2\,p(x,t) + p(x-h,t)}{h^2}.\end{aligned} \tag{2}$$

B) Let $\mathbf{p}(t) = (p(h,t)\ p(2h,t)\ \cdots\ p(nh,t))^{\mathsf{T}}$ and from the above, form an equation of the form

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}\,\mathbf{p}. \tag{3}$$

C) Solve the above equation using (1) backward Euler (2) by numerical computation of $\exp(\mathbf{F}\,t)$ and by (3) forward Euler. Check that the results match the solution in the previous exercise.

Exercise 3. (1.5 order Itô–Taylor)

Simulate trajectories from the following SDE with 1.5 order strong Itô–Taylor series based method

$$\mathrm{d}x = \tanh(x)\ \mathrm{d}t + \mathrm{d}\beta, \qquad x(0) = 0, \tag{4}$$

where $\beta(t)$ is a standard Brownian motion, and compare the resulting histogram to the exact solution

$$p(x,t) = \frac{1}{\sqrt{2\pi\,t}}\ \cosh(x)\ \exp\left(-\frac{1}{2}\,t\right)\ \exp\left(-\frac{1}{2t}\,x^2\right).$$

Exercise 4. (Milstein's method)

Consider the following scalar SDE:

$$\begin{aligned} \mathrm{d}x &= -c\,x\ \mathrm{d}t + g\,x\ \mathrm{d}\beta \\ x(0) &= x_0 \end{aligned} \tag{5}$$

where a, g and x_0 are positive constants and $\beta(t)$ is a standard Brownian motion.

A) Check using the Itô formula that the solution to this equation is

$$x(t) = x_0\ \exp\left[(-c - g^2/2)\,t + g\,\beta(t)\right] \tag{6}$$

Hint: $\phi(\beta(t),t) = x_0\ \exp\left[(-c - g^2/2)\,t + g\,\beta(t)\right]$.

B) Simulate the equation using Milstein's method with parameters $x_0 = 1$, $c = 1/10$, $g = 1/10$, and check that the histogram at $t = 1$ looks the same as obtained by simulating the above exact solution.

Exercise 5. (Gaussian approximation of SDE)

A) Form a Gaussian assumed density approximation to the SDE in Equation (4) on times $t \in [0,5]$ and compare it to the exact solution. Compute the Gaussian integrals numerically on a uniform grid.

B) Form Gaussian assumed density approximation to the Equation (5) and numerically compare it to the histogram obtained in 4 B).