

Exercise Round 1 (November 18, 2013).**Exercise 1. (Mean and covariance equations)**

A) Complete the missing steps in the derivation of the covariance (2.37).

B) Derive the mean and covariance differential equations (2.38) by differentiating the equations (2.36) and (2.37).

Exercise 2. (Solution of Ornstein–Uhlenbeck process)

A) Find the complete solution $x(t)$ as well as the mean $m(t)$ and variance $P(t)$ of the following scalar stochastic differential equation:

$$\frac{dx(t)}{dt} = -\lambda x(t) + w(t), \quad x(0) = x_0, \quad (1)$$

where x_0 and $\lambda > 0$ are given constants and the white noise $w(t)$ has the spectral density q .

B) Compute the limit of the mean and variance when $t \rightarrow \infty$ (1) directly via $\lim_{t \rightarrow \infty} P(t)$ and (2) by solving the stationary state of the variance differential equation $dP/dt = 0$.

Exercise 3. (Euler–Maruyama solution of O–U process)

Simulate 1000 trajectories on the time interval $t \in [0, 1]$ from the Ornstein–Uhlenbeck process in the previous exercise using the Euler–Maruyama method with $\lambda = 1/2$, $q = 1$, $\Delta t = 1/100$, $x_0 = 1$ and check that the mean and covariance trajectories approximately agree with the theoretical values.

Exercise 4. (Usage of Itô formula)

A) Compute the Itô differential of

$$\phi(\beta) = t + \exp(\beta)$$

where $\beta(t)$ is a Brownian motion with diffusion constant q .

B) Compute the Itô differential of

$$\phi(x) = x^2,$$

where x solves the scalar SDE

$$dx = f(x) dt + \sigma d\beta,$$

σ is a constant, and $\beta(t)$ is a standard Brownian motion ($q = 1$).

C) Compute the Itô differential of

$$\phi(\mathbf{x}) = \mathbf{x}^\top \mathbf{x}$$

where

$$d\mathbf{x} = \mathbf{F} \mathbf{x} dt + d\boldsymbol{\beta}$$

where \mathbf{F} is a constant matrix and the joint diffusion matrix of $\boldsymbol{\beta}$ is \mathbf{Q} .

Exercise 5. (Stochastic Differential Equations)

A) Check that

$$x(t) = \exp(\beta(t))$$

solves the SDE

$$dx = \frac{1}{2} x dt + x d\beta,$$

where $\beta(t)$ is a standard Brownian motion ($q = 1$).

B) Solve the following SDE by changing the variable to $y = \ln x$:

$$dx = -cx d\beta$$

$c > 0$ is a constant, and $\beta(t)$ is a standard Brownian motion.

C) Convert the following Stratonovich SDE equation into the equivalent Itô SDE:

$$dx_1 = -x_2 \circ d\beta_1$$

$$dx_2 = x_1 \circ d\beta_2$$

where β_1 and β_2 are independent standard Brownian motions.

Exercise 6. (Mean and variance differential equations)

Derive the mean and covariance equations for the scalar SDE

$$dx = f(x) dt + \sigma(x) d\beta, \quad (2)$$

where β has the diffusion coefficient q , as follows:

A) Conclude from the definition of Itô integral that

$$\mathbb{E} \left[\int_u^v \sigma(x(t)) d\beta(t) \right] = 0$$

for any u and v .

B) Take expectations from both sides of the SDE (2) and formally divide by dt to get the differential equation for the mean $m(t)$.

C) Apply Itô formula to $\phi(x, t) = (x - m(t))^2$ and take expectation of the resulting equation to derive the differential equation for the variance.

D) Write down the mean and covariance differential equations for the scalar SDE

$$dx = -\lambda x dt + d\beta,$$

where $\lambda > 0$ and solve them with $x(0) = x_0$.