Lecture 5: Unscented Kalman filter and General Gaussian Filtering

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Linearization Based Gaussian Approximation

- Problem: Determine the mean and covariance of y:

$$
\begin{aligned}
x &\sim \mathrm{N}(\mu, \sigma^2) \\
y &= \sin(x)
\end{aligned}
$$

- Linearization based approximation:

$$
y = \sin(\mu) + \frac{\partial \sin(\mu)}{\partial \mu} (x - \mu) + \ldots
$$

which gives

$$
\begin{aligned}
\mathrm{E}[y] &\approx \mathrm{E}[\sin(\mu) + \cos(\mu)(x - \mu)] = \sin(\mu) \\
\mathrm{Cov}[y] &\approx \mathrm{E}[(\sin(\mu) + \cos(\mu)(x - \mu) - \sin(\mu))^2] = \cos^2(\mu)\, \sigma^2.
\end{aligned}
$$

Principle of Unscented Transform [1/3]

- Form 3 sigma points as follows:

$$
\begin{aligned}
X_0 &= \mu \\
X_1 &= \mu + \sigma \\
X_2 &= \mu - \sigma.
\end{aligned}
$$

- We may now select some weights W_0, W_1, W_2 such that the original mean and (co)variance can be always recovered by

$$
\begin{aligned}
\mu &= \sum_i W_i \, x_i \\
\sigma^2 &= \sum_i W_i \, (X_i - \mu)^2.
\end{aligned}
$$

Principle of Unscented Transform [2/3]

- Use the same formula for approximating the distribution of $y = \sin(x)$ as follows:

$$\mu_y = \sum_i W_i \sin(X_i)$$

$$\sigma_y^2 = \sum_i W_i \, (\sin(X_i) - \mu_y)^2.$$

- For vectors $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ the generalization of standard deviation σ is the Cholesky factor $\mathbf{L} = \sqrt{\mathbf{P}}$:

$$\mathbf{P} = \mathbf{L} \, \mathbf{L}^T.$$

- The sigma points can be formed using columns of $\mathbf{L}$ (here c is a suitable positive constant):

$$\begin{aligned} \mathbf{X}_0 &= \mathbf{m} \\ \mathbf{X}_i &= \mathbf{m} + c \, \mathbf{L}_i \\ \mathbf{X}_{n+i} &= \mathbf{m} - c \, \mathbf{L}_i \end{aligned}$$

Principle of Unscented Transform [3/3]

- For transformation $\mathbf{y} = \mathbf{g}(\mathbf{x})$ the approximation is:

$$\mu_y = \sum_i W_i \, \mathbf{g}(\mathbf{X}_i)$$
$$\Sigma_y = \sum_i W_i \, (\mathbf{g}(\mathbf{X}_i) - \mu_y) \, (\mathbf{g}(\mathbf{X}_i) - \mu_y)^T.$$

- Joint distribution of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ is then given as

$$\mathrm{E}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{q} \right] \approx \sum_i W_i \begin{pmatrix} \mathbf{X}_i \\ \mathbf{g}(\mathbf{X}_i) \end{pmatrix} = \begin{pmatrix} \mathbf{m} \\ \mu_y \end{pmatrix}$$

$$\mathrm{Cov}\left[\begin{pmatrix} \mathbf{x} \\ \mathbf{g}(\mathbf{x}) + \mathbf{q} \end{pmatrix} \,\middle|\, \mathbf{q} \right]$$
$$\approx \sum_i W_i \begin{pmatrix} (\mathbf{X}_i - \mathbf{m}) \, (\mathbf{X}_i - \mathbf{m})^T & (\mathbf{X}_i - \mathbf{m}) \, (\mathbf{g}(\mathbf{X}_i) - \mu_y)^T \\ (\mathbf{g}(\mathbf{X}_i) - \mu_y) \, (\mathbf{X}_i - \mathbf{m})^T & (\mathbf{g}(\mathbf{X}_i) - \mu_y) \, (\mathbf{g}(\mathbf{X}_i) - \mu_y)^T \end{pmatrix}$$

Unscented Transform Approximation of Non-Linear Transforms [1/3]

Unscented transform

The unscented transform approximation to the joint distribution of $\mathbf{x}$ and $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_U \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_U \\ \mathbf{C}_U^T & \mathbf{S}_U \end{pmatrix} \right),$$

The sub-matrices are formed as follows:

1. Form the matrix of sigma points $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} \mathbf{m} & \cdots & \mathbf{m} \end{bmatrix} + \sqrt{n+\lambda} \begin{bmatrix} 0 & \sqrt{\mathbf{P}} & -\sqrt{\mathbf{P}} \end{bmatrix},$$

[continues in the next slide...]

Unscented Transform Approximation of Non-Linear Transforms [2/3]

Unscented transform (cont.)

2. Propagate the sigma points through $\mathbf{g}(\cdot)$:

$$\mathbf{Y}_i = \mathbf{g}(\mathbf{X}_i), \quad i = 1 \ldots 2n+1,$$

3. The sub-matrices are then given as:

$$\begin{aligned}
\boldsymbol{\mu}_U &= \sum_i W_{i-1}^{(m)} \, \mathbf{Y}_i \\
\mathbf{S}_U &= \sum_i W_{i-1}^{(c)} \, (\mathbf{Y}_i - \boldsymbol{\mu}_U) \, (\mathbf{Y}_i - \boldsymbol{\mu}_U)^T + \mathbf{Q} \\
\mathbf{C}_U &= \sum_i W_{i-1}^{(c)} \, (\mathbf{X}_i - \mathbf{m}) \, (\mathbf{Y}_i - \boldsymbol{\mu}_U)^T,
\end{aligned}$$

Unscented Transform Approximation of Non-Linear Transforms [3/3]

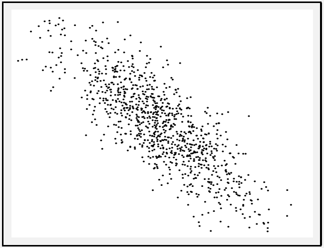
Unscented transform (cont.)

- λ is a scaling parameter defined as $\lambda = \alpha^2 (n + \kappa) - n$.
- α and κ determine the spread of the sigma points.
- Weights $W_i^{(m)}$ and $W_i^{(c)}$ are given as follows:

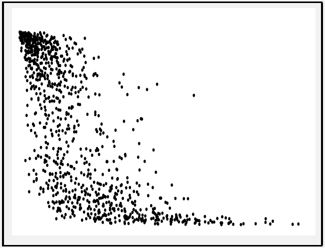
$$\begin{aligned}
W_0^{(m)} &= \lambda/(n+\lambda) \\
W_0^{(c)} &= \lambda/(n+\lambda) + (1-\alpha^2+\beta) \\
W_i^{(m)} &= 1/\{2(n+\lambda)\}, \quad i = 1,\ldots,2n \\
W_i^{(c)} &= 1/\{2(n+\lambda)\}, \quad i = 1,\ldots,2n,
\end{aligned}$$

- β can be used for incorporating prior information on the (non-Gaussian) distribution of $\mathbf{x}$.

Linearization/UT Example



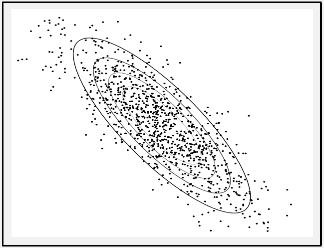
$$\Longrightarrow$$



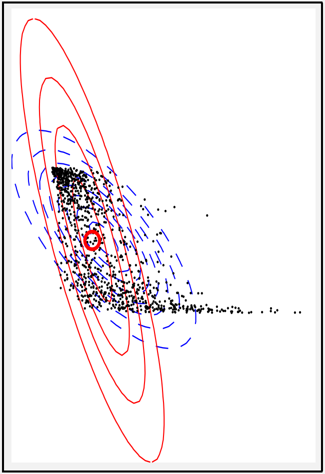
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} \right)$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = \exp(-y_1), \quad y_1(0) = x_1$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = -\frac{1}{2} y_2^3, \quad y_2(0) = x_2$$

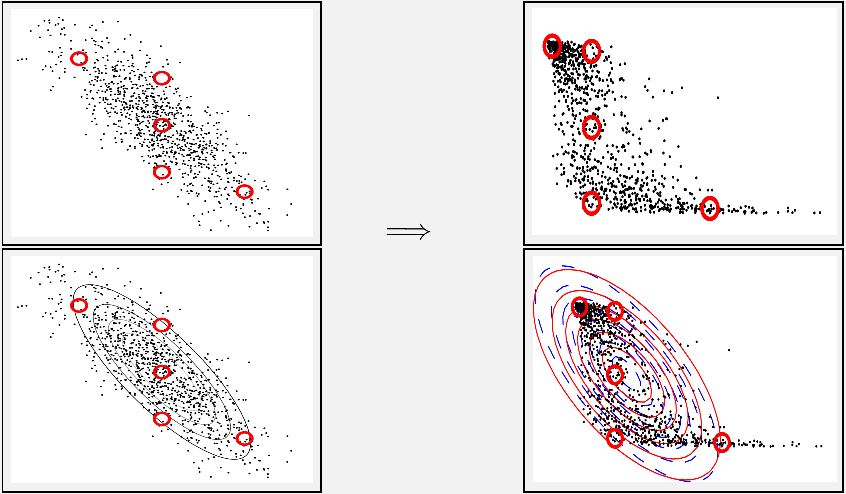
Linearization Approximation



$\Longrightarrow$



UT Approximation



Unscented Kalman Filter (UKF): Derivation [1/4]

- Assume that the filtering distribution of previous step is Gaussian

$$p(\mathbf{x}_{k-1} \,|\, \mathbf{y}_{1:k-1}) \approx \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1})$$

- The joint distribution of $\mathbf{x}_{k-1}$ and $\mathbf{x}_k = \mathbf{f}(\mathbf{x}_{k-1}) + \mathbf{q}_{k-1}$ can be approximated with UT as Gaussian

$$p(\mathbf{x}_{k-1}, \mathbf{x}_k, \,|\, \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_{k-1} \\ \mathbf{x}_k \end{bmatrix} \,\Big|\, \begin{pmatrix} \mathbf{m}'_1 \\ \mathbf{m}'_2 \end{pmatrix}, \begin{pmatrix} \mathbf{P}'_{11} & \mathbf{P}'_{12} \\ (\mathbf{P}'_{12})^T & \mathbf{P}'_{22} \end{pmatrix} \right),$$

- Form the sigma points $\mathbf{X}_i$ of $\mathbf{x}_{k-1} \sim \mathrm{N}(\mathbf{m}_{k-1}, \mathbf{P}_{k-1})$ and compute the transformed sigma points as $\hat{\mathbf{X}}_i = \mathbf{f}(\mathbf{X}_i)$.
- The expected values can now be expressed as:

$$\mathbf{m}'_1 = \mathbf{m}_{k-1}$$
$$\mathbf{m}'_2 = \sum_i W_{i-1}^{(m)} \, \hat{\mathbf{X}}_i$$

Unscented Kalman Filter (UKF): Derivation [2/4]

- The blocks of covariance can be expressed as:

$$
\begin{aligned}
\mathbf{P}'_{11} &= \mathbf{P}_{k-1} \\
\mathbf{P}'_{12} &= \sum_i W_{i-1}^{(c)} (\mathbf{X}_i - \mathbf{m}_{k-1}) \, (\hat{\mathbf{X}}_i - \mathbf{m}'_2)^T \\
\mathbf{P}'_{22} &= \sum_i W_{i-1}^{(c)} (\hat{\mathbf{X}}_i - \mathbf{m}'_2) \, (\hat{\mathbf{X}}_i - \mathbf{m}'_2)^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

- The prediction mean and covariance of $\mathbf{x}_k$ are then $\mathbf{m}'_2$ and $\mathbf{P}'_{22}$, and thus we get

$$
\begin{aligned}
\mathbf{m}_k^- &= \sum_i W_{i-1}^{(m)} \, \hat{\mathbf{X}}_i \\
\mathbf{P}_k^- &= \sum_i W_{i-1}^{(c)} (\hat{\mathbf{X}}_i - \mathbf{m}_k^-) \, (\hat{\mathbf{X}}_i - \mathbf{m}_k^-)^T + \mathbf{Q}_{k-1}
\end{aligned}
$$

Unscented Kalman Filter (UKF): Derivation [3/4]

- For the joint distribution of $\mathbf{x}_k$ and $\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{r}_k$ we similarly get

$$p(\mathbf{x}_k, \mathbf{y}_k, \mid \mathbf{y}_{1:k-1}) \approx \mathrm{N}\left(\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} \Bigm| \begin{pmatrix} \mathbf{m}''_1 \\ \mathbf{m}''_2 \end{pmatrix}, \begin{pmatrix} \mathbf{P}''_{11} & \mathbf{P}''_{12} \\ (\mathbf{P}''_{12})^T & \mathbf{P}''_{22} \end{pmatrix} \right),$$

- If $\mathbf{X}_i^-$ are the sigma points of $\mathbf{x}_k \sim \mathrm{N}(\mathbf{m}_k^-, \mathbf{P}_k^-)$ and $\hat{\mathbf{Y}}_i = \mathbf{f}(\mathbf{X}_i^-)$, we get:

$$\begin{aligned}
\mathbf{m}''_1 &= \mathbf{m}_k^- \\
\mathbf{m}''_2 &= \sum_i W_{i-1}^{(m)} \hat{\mathbf{Y}}_i \\
\mathbf{P}''_{11} &= \mathbf{P}_k^- \\
\mathbf{P}''_{12} &= \sum_i W_{i-1}^{(c)} (\mathbf{X}_i^- - \mathbf{m}_k^-) (\hat{\mathbf{Y}}_i - \mathbf{m}''_2)^T \\
\mathbf{P}''_{22} &= \sum_i W_{i-1}^{(c)} (\hat{\mathbf{Y}}_i - \mathbf{m}''_2) (\hat{\mathbf{Y}}_i - \mathbf{m}''_2)^T + \mathbf{R}_k
\end{aligned}$$

- Recall that if

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}, \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix} \right),$$

then

$$\mathbf{x} \mid \mathbf{y} \sim \mathrm{N}(\mathbf{a} + \mathbf{C}\,\mathbf{B}^{-1}\,(\mathbf{y} - \mathbf{b}), \mathbf{A} - \mathbf{C}\,\mathbf{B}^{-1}\mathbf{C}^T).$$

- Thus we get the conditional mean and covariance:

$$\begin{aligned} \mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{P}_{12}''\,(\mathbf{P}_{22}'')^{-1}(\mathbf{y}_k - \mathbf{m}_2'') \\ \mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{P}_{12}''\,(\mathbf{P}_{22}'')^{-1}\,(\mathbf{P}_{12}'')^T. \end{aligned}$$

Unscented Kalman Filter (UKF): Algorithm [1/3]

Unscented Kalman filter: Prediction step

1. Form the matrix of sigma points:

$$\mathbf{X}_{k-1} = \left[\mathbf{m}_{k-1} \quad \cdots \quad \mathbf{m}_{k-1}\right] + \sqrt{n+\lambda}\left[0 \quad \sqrt{\mathbf{P}_{k-1}} \quad -\sqrt{\mathbf{P}_{k-1}}\right].$$

2. Propagate the sigma points through the dynamic model:

$$\hat{\mathbf{X}}_{k,i} = \mathbf{f}(\mathbf{X}_{k-1,i}), \quad i = 1 \ldots 2n+1.$$

3. Compute the predicted mean and covariance:

$$\mathbf{m}_k^- = \sum_i W_{i-1}^{(m)}\, \hat{\mathbf{X}}_{k,i}$$

$$\mathbf{P}_k^- = \sum_i W_{i-1}^{(c)}\, (\hat{\mathbf{X}}_{k,i} - \mathbf{m}_k^-)\, (\hat{\mathbf{X}}_{k,i} - \mathbf{m}_k^-)^T + \mathbf{Q}_{k-1}.$$

Unscented Kalman filter: Update step

1. Form the matrix of sigma points:

$$\mathbf{X}_k^- = \left[\mathbf{m}_k^- \quad \cdots \quad \mathbf{m}_k^-\right] + \sqrt{n+\lambda}\left[\mathbf{0} \quad \sqrt{\mathbf{P}_k^-} \quad -\sqrt{\mathbf{P}_k^-}\right].$$

2. Propagate sigma points through the measurement model:

$$\hat{\mathbf{Y}}_{k,i} = \mathbf{h}(\mathbf{X}_{k,i}^-), \quad i = 1 \ldots 2n+1.$$

3. Compute the following terms:

$$\begin{aligned}
\boldsymbol{\mu}_k &= \sum_i W_{i-1}^{(m)} \, \hat{\mathbf{Y}}_{k,i} \\
\mathbf{S}_k &= \sum_i W_{i-1}^{(c)} \, (\hat{\mathbf{Y}}_{k,i} - \boldsymbol{\mu}_k) \, (\hat{\mathbf{Y}}_{k,i} - \mu_k)^T + \mathbf{R}_k \\
\mathbf{C}_k &= \sum_i W_{i-1}^{(c)} \, (\mathbf{X}_{k,i}^- - \mathbf{m}_k^-) \, (\hat{\mathbf{Y}}_{k,i} - \boldsymbol{\mu}_k)^T.
\end{aligned}$$

Unscented Kalman Filter (UKF): Algorithm [3/3]

Unscented Kalman filter: Update step (cont.)

4. Compute the filter gain $\mathbf{K}_k$ and the filtered state mean $\mathbf{m}_k$ and covariance $\mathbf{P}_k$, conditional to the measurement $\mathbf{y}_k$:

$$\begin{aligned}
\mathbf{K}_k &= \mathbf{C}_k \, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k \, [\mathbf{y}_k - \boldsymbol{\mu}_k] \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k \, \mathbf{S}_k \, \mathbf{K}_k^T.
\end{aligned}$$

Unscented Kalman Filter (UKF): Example

- Recall the discretized pendulum model

$$\begin{pmatrix} x_k^1 \\ x_k^2 \end{pmatrix} = \underbrace{\begin{pmatrix} x_{k-1}^1 + x_{k-1}^2 \, \Delta t \\ x_{k-1}^2 - g \, \sin(x_{k-1}^1) \, \Delta t \end{pmatrix}}_{\mathbf{f}(\mathbf{x_{k-1}})} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}$$

$$y_k = \underbrace{\sin(x_k^1)}_{\mathbf{h}(\mathbf{x_k})} + r_k,$$

- $\Rightarrow$ *Matlab demonstration*

Unscented Kalman Filter (UKF): Advantages

- No closed form derivatives or expectations needed.
- Not a local approximation, but based on values on a larger area.
- Functions $\mathbf{f}$ and $\mathbf{h}$ do not need to be differentiable.
- Theoretically, captures higher order moments of distribution than linearization.

Unscented Kalman Filter (UKF): Disadvantage

- Not a truly global approximation, based on a small set of trial points.
- Does not work well with nearly singular covariances, i.e., with nearly deterministic systems.
- Requires more computations than EKF or SLF, e.g., Cholesky factorizations on every step.
- Can only be applied to models driven by Gaussian noises.

Gaussian Moment Matching [1/2]

- Consider the transformation of $\mathbf{x}$ into $\mathbf{y}$:

$$\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}).$$

- Form Gaussian approximation to $(\mathbf{x}, \mathbf{y})$ by directly approximating the integrals:

$$\mu_M = \int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{S}_M = \int (\mathbf{g}(\mathbf{x}) - \mu_M) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}$$
$$\mathbf{C}_M = \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \mu_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}.$$

Gaussian Moment Matching [2/2]

Gaussian moment matching

The moment matching based Gaussian approximation to the joint distribution of $\mathbf{x}$ and the transformed random variable $\mathbf{y} = \mathbf{g}(\mathbf{x}) + \mathbf{q}$ where $\mathbf{x} \sim \mathrm{N}(\mathbf{m}, \mathbf{P})$ and $\mathbf{q} \sim \mathrm{N}(\mathbf{0}, \mathbf{Q})$ is given as

$$\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu}_M \end{pmatrix}, \begin{pmatrix} \mathbf{P} & \mathbf{C}_M \\ \mathbf{C}_M^T & \mathbf{S}_M \end{pmatrix} \right), \tag{1}$$

where

$$\begin{aligned} \boldsymbol{\mu}_M &= \int \mathbf{g}(\mathbf{x}) \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} \\ \mathbf{S}_M &= \int (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x} + \mathbf{Q} \\ \mathbf{C}_M &= \int (\mathbf{x} - \mathbf{m}) \, (\mathbf{g}(\mathbf{x}) - \boldsymbol{\mu}_M)^T \, \mathrm{N}(\mathbf{x} \,|\, \mathbf{m}, \mathbf{P}) \, d\mathbf{x}. \end{aligned} \tag{2}$$

Gaussian assumed density filter prediction

$$
\begin{aligned}
\mathbf{m}_k^- &= \int \mathbf{f}(\mathbf{x}_{k-1}) \, \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} \\
\mathbf{P}_k^- &= \int (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-) \, (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)^T \\
&\qquad \times \mathrm{N}(\mathbf{x}_{k-1} \,|\, \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} + \mathbf{Q}_{k-1}.
\end{aligned}
$$

Gaussian Assumed Density Filter [2/3]

Gaussian assumed density filter update

$$
\begin{aligned}
\boldsymbol{\mu}_k &= \int \mathbf{h}(\mathbf{x}_k)\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k \\
\mathbf{S}_k &= \int (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)\, (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^T\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k + \mathbf{R}_k \\
\mathbf{C}_k &= \int (\mathbf{x}_k - \mathbf{m}^-)\, (\mathbf{h}(\mathbf{x}_k) - \boldsymbol{\mu}_k)^T\, \mathrm{N}(\mathbf{x}_k \,|\, \mathbf{m}_k^-, \mathbf{P}_k^-)\, d\mathbf{x}_k \\
\mathbf{K}_k &= \mathbf{C}_k\, \mathbf{S}_k^{-1} \\
\mathbf{m}_k &= \mathbf{m}_k^- + \mathbf{K}_k\, (\mathbf{y}_k - \boldsymbol{\mu}_k) \\
\mathbf{P}_k &= \mathbf{P}_k^- - \mathbf{K}_k\, \mathbf{S}_k\, \mathbf{K}_k^T.
\end{aligned}
$$

Gaussian Assumed Density Filter [3/3]

- Special case of assumed density filtering (ADF).
- Multidimensional Gauss-Hermite quadrature $\Rightarrow$ Gauss Hermite Kalman filter (GHKF).
- Cubature integration $\Rightarrow$ Cubature Kalman filter (CKF).
- Monte Carlo integration $\Rightarrow$ Monte Carlo Kalman filter (MCKF).
- Gaussian process / Bayes-Hermite Kalman filter: Form Gaussian process regression model from set of sample points and integrate the approximation.
- Linearization, unscented transform, central differences, divided differences can be considered as special cases.

Summary

- Unscented transform (UT) approximates transformations of Gaussian variables by propagating sigma points through the non-linearity.
- In UT the mean and covariance are approximated as linear combination of the sigma points.
- The unscented Kalman filter uses unscented transform for computing the approximate means and covariance in non-linear filtering problems.
- A non-linear transformation can also be approximated with Gaussian moment matching.
- Gaussian assumed density filter is based on matching the moments with numerical integration $\Rightarrow$ many kinds of Kalman filters.

Matlab Demo

[Tracking of pendulum with EKF, SLF and UKF]