Lecture 5: Unscented Kalman filter and General Gaussian Filtering

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Contents

1. Idea of Unscented Transform
2. Unscented Transform
3. Unscented Kalman Filter Algorithm
4. Unscented Kalman Filter Properties
5. Gaussian Moment Matching
6. Gaussian Assumed Density Filter
7. Summary and Demonstration
Problem: Determine the mean and covariance of $y$:

$$x \sim \mathcal{N}(\mu, \sigma^2)$$
$$y = \sin(x)$$

Linearization based approximation:

$$y = \sin(\mu) + \frac{\partial \sin(\mu)}{\partial \mu} (x - \mu) + \ldots$$

which gives

$$\mathbb{E}[y] \approx \mathbb{E}[\sin(\mu) + \cos(\mu)(x - \mu)] = \sin(\mu)$$
$$\text{Cov}[y] \approx \mathbb{E}[(\sin(\mu) + \cos(\mu)(x - \mu) - \sin(\mu))^2] = \cos^2(\mu) \sigma^2.$$
Form 3 sigma points as follows:

\[ X_0 = \mu \]
\[ X_1 = \mu + \sigma \]
\[ X_2 = \mu - \sigma. \]

We may now select some weights \( W_0, W_1, W_2 \) such that the original mean and (co)variance can be always recovered by

\[ \mu = \sum_i W_i x_i \]
\[ \sigma^2 = \sum_i W_i (X_i - \mu)^2. \]
Use the same formula for approximating the distribution of $y = \sin(x)$ as follows:

$$
\mu_y = \sum_i W_i \sin(X_i)
$$

$$
\sigma_y^2 = \sum_i W_i (\sin(X_i) - \mu_y)^2.
$$

For vectors $x \sim N(m, P)$ the generalization of standard deviation $\sigma$ is the **Cholesky factor** $L = \sqrt{P}$:

$$
P = LL^T.
$$

The **sigma points** can be formed using columns of $L$ (here $c$ is a suitable positive constant):

$$
X_0 = m
$$

$$
X_i = m + c L_i
$$

$$
X_{n+i} = m - c L_i
$$
For transformation $y = g(x)$ the approximation is:

$$
\mu_y = \sum_i W_i g(X_i)
$$

$$
\Sigma_y = \sum_i W_i (g(X_i) - \mu_y) (g(X_i) - \mu_y)^T.
$$

Joint distribution of $x$ and $y = g(x) + q$ is then given as

$$
E \left[ \begin{pmatrix} x \\ g(x) + q \end{pmatrix} \middle| q \right] \approx \sum_i W_i \begin{pmatrix} X_i \\ g(X_i) \end{pmatrix} = \begin{pmatrix} m \\ \mu_y \end{pmatrix}
$$

$$
\text{Cov} \left[ \begin{pmatrix} x \\ g(x) + q \end{pmatrix} \middle| q \right] \approx \sum_i W_i \begin{pmatrix} (X_i - m) (X_i - m)^T & (X_i - m) (g(X_i) - \mu_y)^T \\ (g(X_i) - \mu_y) (X_i - m)^T & (g(X_i) - \mu_y) (g(X_i) - \mu_y)^T \end{pmatrix}
$$
Unscented Transform Approximation of Non-Linear Transforms [1/3]

The unscented transform approximation to the joint distribution of \( x \) and \( y = g(x) + q \) where \( x \sim N(m, P) \) and \( q \sim N(0, Q) \) is

\[
\begin{pmatrix} x \\ y \end{pmatrix} \sim N \left( \begin{pmatrix} m \\ \mu_U \end{pmatrix}, \begin{pmatrix} P & C_U \\ C_U^T & S_U \end{pmatrix} \right),
\]

The sub-matrices are formed as follows:

1. Form the matrix of sigma points \( X \) as

\[
X = \begin{bmatrix} m & \cdots & m \end{bmatrix} + \sqrt{n + \lambda} \begin{bmatrix} 0 & \sqrt{P} & -\sqrt{P} \end{bmatrix},
\]

[continues in the next slide...]
Unscented Transform Approximation of Non-Linear Transforms [2/3]

Unscented transform (cont.)

2 Propagate the sigma points through $g(\cdot)$:

$$Y_i = g(X_i), \quad i = 1 \ldots 2n + 1,$$

3 The sub-matrices are then given as:

$$\mu_U = \sum_i W^{(m)}_{i-1} Y_i$$

$$S_U = \sum_i W^{(c)}_{i-1} (Y_i - \mu_U) (Y_i - \mu_U)^T + Q$$

$$C_U = \sum_i W^{(c)}_{i-1} (X_i - m) (Y_i - \mu_U)^T,$$
Unscented Transform Approximation of Non-Linear Transforms [3/3]

Unscented transform (cont.)

- \( \lambda \) is a scaling parameter defined as \( \lambda = \alpha^2 (n + \kappa) - n \).
- \( \alpha \) and \( \kappa \) determine the spread of the sigma points.
- Weights \( W_i^{(m)} \) and \( W_i^{(c)} \) are given as follows:

\[
W_0^{(m)} = \lambda/(n + \lambda) \\
W_0^{(c)} = \lambda/(n + \lambda) + (1 - \alpha^2 + \beta) \\
W_i^{(m)} = 1/\{2(n + \lambda)\}, \quad i = 1, \ldots, 2n \\
W_i^{(c)} = 1/\{2(n + \lambda)\}, \quad i = 1, \ldots, 2n,
\]

- \( \beta \) can be used for incorporating prior information on the (non-Gaussian) distribution of \( \mathbf{x} \).
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \right)
\]

\[
\begin{align*}
\frac{dy_1}{dt} &= \exp(-y_1), \quad y_1(0) = x_1 \\
\frac{dy_2}{dt} &= -\frac{1}{2}y_2^3, \quad y_2(0) = x_2
\end{align*}
\]
Linearization Approximation

\[ \text{Simo Särkkä} \] Lecture 5: UKF and GGF
UT Approximation

Simo Särkkä

Lecture 5: UKF and GGF
Unscented Kalman Filter (UKF): Derivation [1/4]

- Assume that the filtering distribution of previous step is Gaussian

\[ p(x_{k-1} \mid y_{1:k-1}) \approx N(x_{k-1} \mid m_{k-1}, P_{k-1}) \]

- The joint distribution of \( x_{k-1} \) and \( x_k = f(x_{k-1}) + q_{k-1} \) can be approximated with UT as Gaussian

\[ p(x_{k-1}, x_k, \mid y_{1:k-1}) \approx N \left( \begin{bmatrix} x_{k-1} \\ x_k \end{bmatrix} \mid \begin{bmatrix} m'_{1} \\ m'_{2} \end{bmatrix}, \begin{bmatrix} P'_{11} & P'_{12} \\ (P'_{12})^T & P'_{22} \end{bmatrix} \right) \]

- Form the sigma points \( X_i \) of \( x_{k-1} \sim N(m_{k-1}, P_{k-1}) \) and compute the transformed sigma points as \( \hat{X}_i = f(X_i) \).

- The expected values can now be expressed as:

\[
\begin{align*}
     m'_{1} &= m_{k-1} \\
     m'_{2} &= \sum_{i} W_{i-1}^{(m)} \hat{X}_i 
\end{align*}
\]
The blocks of covariance can be expressed as:

\[ P'_{11} = P_{k-1} \]
\[ P'_{12} = \sum_{i} W_{i}^{(c)} (X_i - m_{k-1}) (\hat{X}_i - m'_2)^T \]
\[ P'_{22} = \sum_{i} W_{i}^{(c)} (\hat{X}_i - m'_2) (\hat{X}_i - m'_2)^T + Q_{k-1} \]

The prediction mean and covariance of \( x_k \) are then \( m'_2 \) and \( P'_{22} \), and thus we get

\[ m_k^- = \sum_{i} W_{i}^{(m)} \hat{X}_i \]
\[ P_k^- = \sum_{i} W_{i}^{(c)} (\hat{X}_i - m_k^-) (\hat{X}_i - m_k^-)^T + Q_{k-1} \]
For the joint distribution of $x_k$ and $y_k = h(x_k) + r_k$ we similarly get

$$p(x_k, y_k, | y_{1:k-1}) \approx N \left( \begin{bmatrix} x_k \\ y_k \end{bmatrix} \mid \begin{bmatrix} m''_1 \\ m''_2 \end{bmatrix}, \begin{bmatrix} P''_{11} & P''_{12} \\ (P''_{12})^T & P''_{22} \end{bmatrix} \right),$$

If $X_i^-$ are the sigma points of $x_k \sim N(m_k^-, P_k^-)$ and $\hat{Y}_i = f(X_i^-)$, we get:

$$m''_1 = m_k^-$$

$$m''_2 = \sum_i W_i^{(m)} \hat{Y}_i$$

$$P''_{11} = P_k^-$$

$$P''_{12} = \sum_i W_i^{(c)} (X_i^- - m_k^-) (\hat{Y}_i - m''_2)^T$$

$$P''_{22} = \sum_i W_i^{(c)} (\hat{Y}_i - m''_2) (\hat{Y}_i - m''_2)^T + R_k$$
Recall that if
\[
\begin{pmatrix} x \\ y \end{pmatrix} \sim N \left( \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \right),
\]
then
\[
x | y \sim N \left( a + C B^{-1} (y - b), A - C B^{-1} C^T \right).
\]

Thus we get the conditional mean and covariance:
\[
\begin{align*}
m_k &= m_k^- + P''_{12} (P''_{22})^{-1} (y_k - m''_2) \\
P_k &= P_k^- - P''_{12} (P''_{22})^{-1} (P''_{12})^T.
\end{align*}
\]
Unscented Kalman filter: Prediction step

1. Form the matrix of sigma points:

\[
X_{k-1} = \begin{bmatrix} m_{k-1} & \cdots & m_{k-1} \end{bmatrix} + \sqrt{n + \lambda} \begin{bmatrix} 0 & \sqrt{P_{k-1}} & -\sqrt{P_{k-1}} \end{bmatrix}.
\]

2. Propagate the sigma points through the dynamic model:

\[
\hat{X}_{k,i} = f(X_{k-1,i}), \quad i = 1 \ldots 2n + 1.
\]

3. Compute the predicted mean and covariance:

\[
m_k^- = \sum_i W_{i-1}^{(m)} \hat{X}_{k,i}
\]

\[
P_k^- = \sum_i W_{i-1}^{(c)} (\hat{X}_{k,i} - m_k^-) (\hat{X}_{k,i} - m_k^-)^T + Q_{k-1}.
\]
Unscented Kalman Filter (UKF): Algorithm [2/3]

Unscented Kalman filter: Update step

1. Form the matrix of sigma points:

\[
X_k^- = \begin{bmatrix} m_k^- & \cdots & m_k^- \end{bmatrix} + \sqrt{n+\lambda} \begin{bmatrix} 0 & \sqrt{P_k^-} & -\sqrt{P_k^-} \end{bmatrix}.
\]

2. Propagate sigma points through the measurement model:

\[
\hat{Y}_{k,i} = h(X_{k,i}^-), \quad i = 1 \ldots 2n + 1.
\]

3. Compute the following terms:

\[
\mu_k = \sum_i W_{i-1}^{(m)} \hat{Y}_{k,i}
\]

\[
S_k = \sum_i W_{i-1}^{(c)} (\hat{Y}_{k,i} - \mu_k)(\hat{Y}_{k,i} - \mu_k)^T + R_k
\]

\[
C_k = \sum_i W_{i-1}^{(c)} (X_{k,i}^- - m_k^-)(\hat{Y}_{k,i} - \mu_k)^T.
\]
4. Compute the filter gain $K_k$ and the filtered state mean $m_k$ and covariance $P_k$, conditional to the measurement $y_k$:

$$K_k = C_k S_k^{-1}$$

$$m_k = m_k^- + K_k [y_k - \mu_k]$$

$$P_k = P_k^- - K_k S_k K_k^T.$$
Recall the discretized pendulum model

\[
\begin{pmatrix} x^1_k \\ x^2_k \\ x_k \end{pmatrix} = \underbrace{\begin{pmatrix} x^1_{k-1} + x^2_{k-1} \Delta t \\ x^2_{k-1} - g \sin(x^1_{k-1}) \Delta t \end{pmatrix}}_{f(x_{k-1})} + \begin{pmatrix} 0 \\ q_{k-1} \end{pmatrix}
\]

\[y_k = \underbrace{\sin(x^1_k)}_{h(x_k)} + r_k,\]

\[\Rightarrow \text{Matlab demonstration}\]
- No closed form derivatives or expectations needed.
- Not a local approximation, but based on values on a larger area.
- Functions $\mathbf{f}$ and $\mathbf{h}$ do not need to be differentiable.
- Theoretically, captures higher order moments of distribution than linearization.
Unscented Kalman Filter (UKF): Disadvantage

- Not a truly global approximation, based on a small set of trial points.
- Does not work well with nearly singular covariances, i.e., with nearly deterministic systems.
- Requires more computations than EKF or SLF, e.g., Cholesky factorizations on every step.
- Can only be applied to models driven by Gaussian noises.
Consider the transformation of $x$ into $y$:

$$x \sim N(m, P)$$

$$y = g(x).$$

Form Gaussian approximation to $(x, y)$ by directly approximating the integrals:

$$\mu_M = \int g(x) \ N(x \mid m, P) \ dx$$

$$S_M = \int (g(x) - \mu_M)(g(x) - \mu_M)^T \ N(x \mid m, P) \ dx$$

$$C_M = \int (x - m)(g(x) - \mu_M)^T \ N(x \mid m, P) \ dx.$$
Gaussian moment matching

The moment matching based Gaussian approximation to the joint distribution of $x$ and the transformed random variable $y = g(x) + q$ where $x \sim N(m, P)$ and $q \sim N(0, Q)$ is given as

$$
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
\sim N
\left( 
\begin{pmatrix}
  m \\
  \mu_M
\end{pmatrix},
\begin{pmatrix}
  P & C_M \\
  C_M^T & S_M
\end{pmatrix}
\right), \quad (1)
$$

where

$$
\mu_M = \int g(x) \ N(x \mid m, P) \ dx
$$

$$
S_M = \int (g(x) - \mu_M)(g(x) - \mu_M)^T \ N(x \mid m, P) \ dx + Q \quad (2)
$$

$$
C_M = \int (x - m)(g(x) - \mu_M)^T \ N(x \mid m, P) \ dx.
$$
Gaussian Assumed Density Filter [1/3]

Gaussian assumed density filter prediction

\[
\begin{align*}
\mathbf{m}_k^- &= \int \mathbf{f}(\mathbf{x}_{k-1}) \, \mathcal{N}(\mathbf{x}_{k-1} \mid \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} \\
\mathbf{P}_k^- &= \int (\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)(\mathbf{f}(\mathbf{x}_{k-1}) - \mathbf{m}_k^-)^T \\
&\quad \times \mathcal{N}(\mathbf{x}_{k-1} \mid \mathbf{m}_{k-1}, \mathbf{P}_{k-1}) \, d\mathbf{x}_{k-1} + \mathbf{Q}_{k-1}.
\end{align*}
\]
Special case of assumed density filtering (ADF).

Multidimensional Gauss-Hermite quadrature ⇒ Gauss Hermite Kalman filter (GHKF).

Cubature integration ⇒ Cubature Kalman filter (CKF).

Monte Carlo integration ⇒ Monte Carlo Kalman filter (MCKF).

Gaussian process / Bayes-Hermite Kalman filter: Form Gaussian process regression model from set of sample points and integrate the approximation.

Linearization, unscented transform, central differences, divided differences can be considered as special cases.
Unscented transform (UT) approximates transformations of Gaussian variables by propagating sigma points through the non-linearity.

In UT the mean and covariance are approximated as linear combination of the sigma points.

The unscented Kalman filter uses unscented transform for computing the approximate means and covariance in non-linear filtering problems.

A non-linear transformation can also be approximated with Gaussian moment matching.

Gaussian assumed density filter is based on matching the moments with numerical integration ⇒ many kinds of Kalman filters.
[Tracking of pendulum with EKF, SLF and UKF]