Lexicographic priorities in default logic

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Abstract

Resolving conflicts between default rules is a major subtask in performing default reasoning. A declarative way of controlling the resolution of conflicts is to assign priorities to default rules, and to prevent conflict resolution in ways that violate the priorities. This work extends Reiter's default logic with a priority mechanism that is based on lexicographic comparison. Given a default theory and a partial ordering on the defaults, the preferred extensions are the lexicographically best extensions. We discuss alternative ways of using lexicographic comparison, and investigate their properties and relations between them. The applicability of the priority mechanism to inheritance reasoning is investigated by presenting two translations from inheritance networks to prioritized default theories, and relating them to inheritance theories presented earlier by Gelfond and Przymusinska and by Brewka.

1 Introduction

Default logic [22] and other nonmonotonic logics [17,18] were devised for solving the frame problem in temporal reasoning and for expressing rules with exceptions. Most default theories of interest have default rules that conflict, that is, two applicable default rules have mutually contradictory conclusions. Only one of two conflicting default rules may be applied. In many cases there is a reason to give priority to one of them and ignore the other. Priorities in nonmonotonic logics have been investigated by several researchers [14,6,7,1,8]. The idea in these approaches is that there is an ordering on the default rules or propositions associated with the default rules.

In this paper, we introduce a priority mechanism to Reiter's default logic [22], and investigate its properties and applicability to a basic formalization of default reasoning, inheritance reasoning. Unlike earlier work on priorities and default logic by Baader and Hollunder, Brewka, and Marek and Truszczyński [1,7,16], this priority mechanism is based on lexicographic comparison that is the most common method of ordering sequences of letters or numbers. Lexicographic comparison is widely used in computer science and in research areas related to nonmonotonic reasoning. For example Przymusinski [21] defined the perfect model semantics for stratified logic programs in terms of lexicographic comparison. Lifschitz [14] proposed a way of incorporating priorities into circumscription by using lexicographic comparison, and it has also been used by Brewka [6], Ryan [26], and Geffner and Pearl [8].

The second topic of this paper is the representation of inheritance networks in nonmonotonic logics. Gelfond and Przymusinska present translations of inheritance networks with single and multiple inheritance to autoepistemic logic [10]. Brewka shows how inheritance in so-called class-property networks can be represented in predicate logic with circumscription [4]. We present translations of inheritance networks to prioritized default logic, and show that they are equivalent to the afore-mentioned formalizations for those classes of inheritance networks the former are defined. The difference between our translations is that one of them sanctions reasoning by contraposition. This is also a difference between the work by Gelfond and Przymusinska and by Brewka.

The outline of the paper is as follows. We start by giving the basic definitions related to Reiter's default logic. Section 3.1 introduces our priority mechanism on top of default logic. Sections 4.1 and 4.2 show that Brewka's preferred subtheories and Ryan's ordered theory presentations are special cases of prioritized default logic. In Section 5 we show how inheritance networks can be translated to prioritized default logic, and in Section 6 we show the relation of the inheritance theories of Gelfond and Przymusinska [10] and Brewka [4] to ours.

2 Default logic

Reiter's default logic [22] is one of the main formalizations of nonmonotonic reasoning, and its propositional version is the basis of this work. The language of the classical propositional logic is denoted by the symbol $\mathcal{L}$. The set of all default rules $\{\alpha{:}\beta_1, \ldots, \beta_n/\gamma \,|\, n \geq 0, \{\alpha, \beta_1, \ldots, \beta_n, \gamma\} \subseteq \mathcal{L}\}$ is denoted by $\mathcal{D}$.

Definition 1 ([22]) *Let $\Delta = \langle D, W \rangle$ be a default theory. For any set of formulae $S \subseteq \mathcal{L}$, let $\Gamma(S)$ be the smallest set such that $W \subseteq \Gamma(S)$, $\mathrm{Cn}(\Gamma(S)) =$*

$\Gamma(S)$, and if $\alpha{:}\beta_1, \ldots, \beta_n/\gamma \in D$ and $\alpha \in \Gamma(S)$ and $\{\neg\beta_1, \ldots, \neg\beta_n\} \cap S = \emptyset$, then $\gamma \in \Gamma(S)$. A set of formulae $E \subseteq \mathcal{L}$ is an extension for Δ if and only if $\Gamma(E) = E$.

A definition of extensions resembling the semiconstructive definition of extensions of Reiter [22] is presented by Gelfond, Lifschitz, Przymusinska, and Truszczyński[2]. This definition is equivalent to Definition 1.

Definition 2 ([9]) *Let $\langle D, W \rangle$ be a default theory. Let E be a set of formulae. Define*

$$D^E = \left\{ \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D \,\middle|\, \{\neg\beta_1, \ldots, \neg\beta_n\} \cap E = \emptyset \right\}.$$

Define $\mathrm{Cn}_W(D)$ as $\bigcup_{i \geq 0} \mathrm{Cn}(C_i)$ where for all $i \geq 0$, $C_0 = W, C_{i+1} = C_i \cup \{\gamma \,|\, \alpha{:}\beta_1, \ldots, \beta_n/\gamma \in D, C_i \models \alpha\}$. Then E is an extension of $\langle D, W \rangle$ if and only if $E = \mathrm{Cn}_W(D^E)$.

The notion of *generating defaults* of an extension is useful because by using it the extensions can be represented as a finite set of formulae whenever the set of defaults and the set of objective formulae are finite.

Definition 3 ([22]) *Suppose $\Delta = \langle D, W \rangle$ is a default theory and E is an extension for Δ. The set of* generating defaults *for E with respect to Δ is*

$$\mathrm{GD}(E, \Delta) = \left\{ \frac{\alpha : \beta_1, \ldots, \beta_n}{\gamma} \in D \,\middle|\, \alpha \in E \text{ and } \{\neg\beta_1, \ldots, \neg\beta_n\} \cap E = \emptyset \right\}.$$

Theorem 4 ([22]) *Suppose E is an extension of a default theory $\Delta = \langle D, W \rangle$. Then $E = \mathrm{Cn}(W \cup \{\gamma \,|\, \alpha{:}\beta_1, \ldots, \beta_n/\gamma \in \mathrm{GD}(E, \Delta)\})$.*

The standard consequence relation of default logic is *cautious reasoning* $\models_c$.

Definition 5 *Let $\Delta = \langle D, W \rangle$ be a default theory and $\phi \in \mathcal{L}$ a formula. Then $\Delta \models_c \phi$ if and only if $\phi \in E$ for all extensions E of Δ.*

The following terminology is used in referring to default rules of certain syntactic forms. Defaults of the form $\alpha{:}\beta/\beta$ and $\top{:}\beta_1, \ldots, \beta_n/\gamma$ are respectively *normal* and *prerequisite-free*. Prerequisite-free defaults are often written without prerequisites as $:\beta_1, \ldots, \beta_n/\gamma$. We shall denote sequences $\beta_1, \ldots, \beta_n$ of formulae by symbols $\sigma, \sigma', \sigma_1$ and so on.

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[2] Actually, unlike Gelfond et al., the definition we give here does not eliminate the justifications of defaults when constructing D^E and makes it explicit what "closed under inference rules" means.
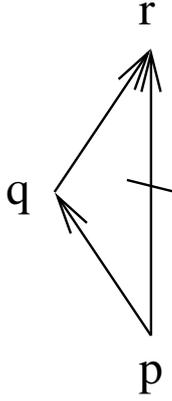
Fig. 1. The penguin triangle

3 Prioritized default logic

In this section we introduce our framework of default reasoning with priorities. Priorities express the plausibility of pieces of knowledge. For a given knowledge base expressed in default logic there may be several extensions each of which is a possible meaning of the knowledge base. However, an extension can be unacceptable because of the way a conflict between defaults is resolved. The simplest example of this is the penguin triangle.

Example 6 *Consider the inheritance network in Figure 1[3]. The letters p, q and r respectively express the statements that "x is a penguin", "x is a bird", and "x is able to fly". The arrows can be understood as stating that penguins are usually birds, birds are usually able to fly, and penguins are not usually able to fly. In default logic, these defaults can be formalized as $D = \{p{:}q/q, p{:}\neg r/\neg r, q{:}r/r\}$. It would be correct to conclude that a given penguin is not able to fly. For the default theory $\langle D, \{p\}\rangle$, however, default logic gives the extensions $\mathrm{Cn}(\{p, q, r\})$ and $\mathrm{Cn}(\{p, q, \neg r\})$ that respectively correspond to the application of the defaults $q{:}r/r$ and $p{:}\neg r/\neg r$. The conclusion that x does not fly is not supported by the extension $\mathrm{Cn}(\{p, q, r\})$.*

When sequences of letters are ordered lexicographically there is a unique lexicographically best sequence. However, in inheritance reasoning, there in general is no unique best extension. The standard example of this phenomenon is the Nixon diamond [24] in Figure 2. In this case the defaults should not be totally ordered: the default stating that quakers usually are pacifists does not have a higher priority than the default stating that republicans usually are not pacifists, or vice versa. Lexicographic comparison is usually understood only in the setting where the criteria according to which two entities are ordered

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[3] Default theories can be depicted as directed graphs with two kinds of arrows. Ordinary arrows going from p to q represent defaults $p{:}q/q$, and broken arrows represent defaults $p{:}\neg q/\neg q$.
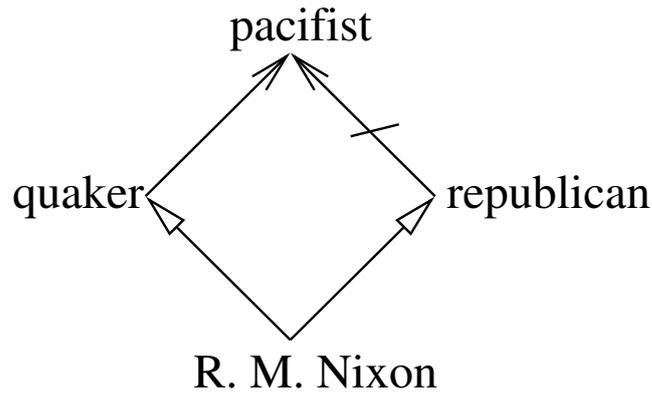
Fig. 2. The Nixon diamond

is a totally ordered set. In our setting the criteria are not totally ordered, and hence a generalization of lexicographic comparison to any strict partial order is needed.

A natural way to generalize lexicographic comparison to partially ordered criteria is to reduce it to lexicographic comparison with totally ordered criteria. It turns out that there are two ways of doing this. In the first case a candidate is better than another candidate if the former is lexicographically better than the other according to all total orders that extend the partial order on the criteria. Now the best candidates are those for which there are no better candidates. In the second case a candidate is best if there is a total order on the criteria that extends the partial order, and the candidate is better than any other candidate according to lexicographic comparison by using this total order. Hence, instead of requiring that a candidate is pairwise no worse than any other candidate by using lexicographic comparison with some total order, it is required that the candidate is better than any other candidate by using the same total order for all pairwise comparisons. Clearly, this second definition is stronger than the first definition in the sense that the set of best candidates is a subset of the best candidates in the first case.

These two definitions of lexicographically best candidates stem from different ways of utilizing the partiality in partial orders. A definition that we show equivalent to the first definition has been used for example by Geffner and Pearl [8] and Ryan [26]. The second definition has been used by Brewka [6]. Neither Geffner and Pearl nor Ryan explicitly do the reduction to lexicographic comparison with total orders. Brewka's definition, on the other hand, does not mention lexicographic comparison as it is given as a recursive procedure.

3.1 The definition

The significance of defaults is represented by strict partial orders, that is, transitive and asymmetric relations $\mathcal{P}$. If defaults δ and δ' are related by $\mathcal{P}$ then the acceptance of δ is more desirable than the acceptance of δ', and the default δ is more significant or has a higher priority. There are several different interpretations of what it means for a default to be accepted or not to be accepted. The first definition we use is based on the notion of generating defaults in Definition 3. We say that a generating default of an extension is applied in the extension. Hence we classify defaults to those that are applied and to those that are not.

Definition 7 (Application) *A default* $\alpha{:}\beta_1,\ldots,\beta_n/\gamma$ *is applied in* $E \subseteq \mathcal{L}$, *if* $E \models \alpha$ *and* $\{\neg\beta_1,\ldots,\neg\beta_n\}\cap\mathrm{Cn}(E) = \emptyset$. *This is denoted by* $\mathrm{appl}(\alpha{:}\beta_1,\ldots,\beta_n/\gamma, E)$.

Another way of classifying defaults is the following.

Definition 8 (Defeat) *A default* $\alpha{:}\beta_1,\ldots,\beta_n/\gamma$ *is defeated in* $E \subseteq \mathcal{L}$, *if* $E \models \alpha$ *and* $\{\neg\beta_1,\ldots,\neg\beta_n\}\cap\mathrm{Cn}(E) \neq \emptyset$. *This is denoted by* $\mathrm{def}(\alpha{:}\beta_1,\ldots,\beta_n/\gamma, E)$.

Notice that $\mathrm{appl}(\delta, E)$ implies not $\mathrm{def}(\delta, E)$, and $\mathrm{def}(\delta, E)$ implies not $\mathrm{appl}(\delta, E)$, but the converses do not in general hold. For convenience, we use the notation $\mathrm{appl}(\delta, E, E')$ for $\mathrm{appl}(\delta, E)$ and not $\mathrm{appl}(\delta, E')$. Similarly, $\mathrm{def}(\delta, E, E')$ means that $\mathrm{def}(\delta, E)$ and not $\mathrm{def}(\delta, E')$.

Definition 9 (Preferredness) *Let* $\Delta = \langle D, W \rangle$ *be a default theory and* $\mathcal{P}$ *a strict partial order on* D. *Let* E *be an extension of* Δ. *Then* E *is a* $\mathcal{P}$*-preferred extension of* Δ, *if there is a strict total order* $\mathcal{T}$ *on* D *such that* $\mathcal{P} \subseteq \mathcal{T}$ *and for all extensions* E' *of* Δ *and* $\delta \in D$,

$\mathrm{appl}(\delta, E', E)$ *implies that for some* $\epsilon \in D, \epsilon \mathcal{T} \delta$ *and* $\mathrm{appl}(\epsilon, E, E')$.

Such a strict total order is a $\Delta, \mathcal{P}$*-ordering for* E.

Definition 10 *The consequence relation* $\models\,\subseteq ((2^{\mathcal{D}} \times 2^{\mathcal{L}}) \times 2^{\mathcal{D} \times \mathcal{D}}) \times \mathcal{L}$ *is defined by* $\Delta \models_{\mathcal{P}} \phi$ *if and only if* ϕ *is in all* $\mathcal{P}$*-preferred extensions of* Δ.

We use the notation $\models_{\mathcal{P}}$ for the consequence relation of prioritized default logic to distinguish it from the logical consequence relation $\models$ of the classical propositional logic.

Example 11 *The default theory* $\langle D, W \rangle$ *with* $D = \{p{:}q/q, p{:}\neg r/\neg r, q{:}r/r\}$ *and* $W = \{p\}$ *is depicted in Figure 1. Let* $\mathcal{P} = \{\langle p{:}\neg r/\neg r, q{:}r/r\rangle\}$ *be a strict partial order on* D. *The default theory has two extensions,* $E_1 = \mathrm{Cn}(\{p, q, \neg r\})$ *where the defaults* $p{:}\neg r/\neg r$ *and* $p{:}q/q$ *are applied, and* $E_2 = \mathrm{Cn}(\{p, q, r\})$

where $p{:}q/q$ and $q{:}r/r$ are applied. These extensions and all strict total orders $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$ are depicted below. The most significant defaults are the lowest. The symbol $\bullet$ signifies that the default is applied and $\circ$ that it is not applied.

$\mathcal{T}_1$	E_1	E_2	$\mathcal{T}_2$	E_1	E_2	$\mathcal{T}_3$	E_1	E_2
$p{:}q/q$	$\bullet$	$\bullet$	$q{:}r/r$	$\circ$	$\bullet$	$q{:}r/r$	$\circ$	$\bullet$
$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$
$q{:}r/r$	$\circ$	$\bullet$	$p{:}q/q$	$\bullet$	$\bullet$	$p{:}\neg r/\neg r$	$\bullet$	$\circ$
$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$	$\mid$
$p{:}\neg r/\neg r$	$\bullet$	$\circ$	$p{:}\neg r/\neg r$	$\bullet$	$\circ$	$p{:}q/q$	$\bullet$	$\bullet$

The extension E_1 is a $\mathcal{P}$-preferred extension because the leftmost strict total order $\mathcal{T}_1$ is a $\Delta, \mathcal{P}$-ordering for E_1: $q{:}r/r$ is the only default δ such that $\mathrm{appl}(\delta, E_2, E_1)$, and $\mathrm{appl}(p{:}\neg r/\neg r, E_1, E_2)$ and $p{:}\neg r/\neg r \mathcal{T}_1 q{:}r/r$. The extension E_2 is not a $\mathcal{P}$-preferred extension because none of the three strict total orders $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ is a $\Delta, \mathcal{P}$-ordering for E_2: for all $i \in \{1, 2, 3\}$, there is the default $p{:}\neg r/\neg r$ such that $\mathrm{appl}(p{:}\neg r/\neg r, E_1, E_2)$ and there is no default δ such that $\delta \mathcal{T}_i p{:}\neg r/\neg r$ and $\mathrm{appl}(\delta, E_2, E_1)$.

The alternative way of using the partiality is given in the next definition.

Definition 12 (Preferredness$_2$) *Let $\Delta = \langle D, W \rangle$ be a default theory, and $\mathcal{P}$ a strict partial order on D. Let E be an extension of Δ. Then E is a $\mathcal{P}$-preferred$_2$ extension of Δ, if for all extensions E' of Δ there is a strict total order $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$ and for all $\delta \in D$,*

$\mathrm{appl}(\delta, E', E)$ *implies that for some* $\epsilon \in D, \epsilon \mathcal{T} \delta$ *and* $\mathrm{appl}(\epsilon, E, E')$.

Such a strict total order is a $\Delta, \mathcal{P}$-ordering against E' for E.
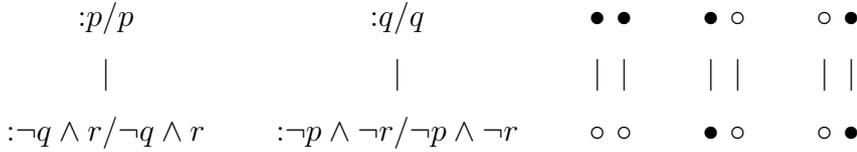
Definition 13 *The consequence relation $\models^2 \subseteq ((2^{\mathcal{D}} \times 2^{\mathcal{L}}) \times 2^{\mathcal{D} \times \mathcal{D}}) \times \mathcal{L}$ is defined by $\Delta \models^2_{\mathcal{P}} \phi$ if and only if ϕ is in all $\mathcal{P}$-preferred$_2$ extensions of Δ.*

By replacing $\mathrm{appl}(\delta, E', E)$ by $\mathrm{def}(\delta, E, E')$ and $\mathrm{appl}(\epsilon, E, E')$ by $\mathrm{def}(\epsilon, E', E)$ in the above definitions, we obtain the definitions of $\mathcal{P}$-preferred$_d$ and $\mathcal{P}$-preferred$_{2d}$ extensions. Similarly we obtain the consequence relations $\models^d_{\mathcal{P}}$ and $\models^{2d}_{\mathcal{P}}$. The subscript d is used similarly when referring to different kinds of $\Delta, \mathcal{P}$-orderings.

Lemma 14 (Inclusion) *Let $\Delta = \langle D, W \rangle$ be a default theory. Let $\mathcal{P}$ be a strict partial order on D. If E is a $\mathcal{P}$-preferred extension of Δ, then E is a $\mathcal{P}$-preferred$_2$ extension of Δ. The converse does not hold.*

PROOF. Let $\Delta = \langle D, W \rangle$ be a default theory and $\mathcal{P}$ a strict partial order on D. If an extension E of Δ is $\mathcal{P}$-preferred, then there is a $\Delta, \mathcal{P}$-ordering $\mathcal{T}$ for E. Now $\mathcal{T}$ fulfills the condition that for all extensions E' of Δ and all defaults δ, $\mathrm{appl}(\delta, E', E)$ implies that for some $\epsilon \in D$, $\epsilon \mathcal{T} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. Let E' be any extension of Δ. By definition of $\Delta, \mathcal{P}$-orderings against extensions, $\mathcal{T}$ is also a $\Delta, \mathcal{P}$-ordering against E' for E. Because this holds for any extension E', E is a $\mathcal{P}$-preferred$_2$ extension of Δ.

The following example shows that the converse does not hold. Let $\Delta = \langle D, \emptyset \rangle$ be a default theory with four defaults in D and the strict partial order $\mathcal{P} = \{\langle :\neg q \wedge r / \neg q \wedge r, :p/p \rangle, \langle :\neg p \wedge \neg r / \neg p \wedge \neg r, :q/q \rangle\}$ on D. The three extensions $E_1 = \mathrm{Cn}(\{p, q\})$, $E_2 = \mathrm{Cn}(\{\neg q \wedge r, p\})$, and $E_3 = \mathrm{Cn}(\{\neg p \wedge \neg r, q\})$ of Δ and $\mathcal{P}$ are depicted below.

$:p/p$	$:q/q$	$\bullet\ \bullet$	$\bullet\ \circ$	$\circ\ \bullet$
$\vert$	$\vert$	$\vert\ \ \vert$	$\vert\ \ \vert$	$\vert\ \ \vert$
$:\neg q \wedge r / \neg q \wedge r$	$:\neg p \wedge \neg r / \neg p \wedge \neg r$	$\circ\ \circ$	$\bullet\ \circ$	$\circ\ \bullet$

The extension E_1 is $\mathcal{P}$-preferred$_2$ because of the $\Delta, \mathcal{P}$-ordering $:\neg q \wedge r / \neg q \wedge r \mathcal{T}_2 :p/p \mathcal{T}_2 :\neg p \wedge \neg r / \neg p \wedge \neg r \mathcal{T}_2 :q/q$ against E_3, and the $\Delta, \mathcal{P}$-ordering $:\neg p \wedge \neg r / \neg p \wedge \neg r \mathcal{T}_3 :q/q \mathcal{T}_3 :\neg q \wedge r / \neg q \wedge r \mathcal{T}_3 :p/p$ against E_2. However, there are no $\Delta, \mathcal{P}$-orderings for E_1. In particular, $\mathcal{T}_2$ is a $\Delta, \mathcal{P}$-ordering for E_2, and $\mathcal{T}_3$ is a $\Delta, \mathcal{P}$-ordering for E_3. Brewka and Junker [12] give a similar example to show that the preference notions of Brewka [6] and Geffner and Pearl [8] differ. $\square$

Lemma 15 *Let $\Delta = \langle D, W \rangle$ be a default theory, ϕ a formula, and $\mathcal{P}$ a strict partial order on D. If $\Delta \models_c \phi$, then $\Delta \models_{\mathcal{P}}^2 \phi$. If $\Delta \models_{\mathcal{P}}^2 \phi$, then $\Delta \models_{\mathcal{P}} \phi$.*

Our definitions of preferred and preferred$_2$ extensions directly determine a subset of all the extensions of a default theory. This is in contrast with some earlier work in prioritized nonmonotonic reasoning where priorities impose an ordering on propositional models or extensions, and the preferred models or extensions are the minimal or maximal elements [8]. It seems that there are no natural definitions of preferred extensions that use orderings of extensions in that way. However, preferred$_2$ extensions can be obtained as the minimal elements of partial orders on extensions.

Definition 16 *Let $\Delta = \langle D, W \rangle$ be a default theory and let E and E' be extensions of Δ. Let $\mathcal{P}$ be a strict partial order on D. Then $E \sqsubset_{\mathcal{P}}^{\Delta} E'$ if and only if $E \neq E'$ and for all $\delta \in D$ and strict total orders $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$,*

$\mathrm{appl}(\delta, E', E)$ *implies that for some* $\epsilon \in D$, $\epsilon \mathcal{T} \delta$ *and* $\mathrm{appl}(\epsilon, E, E')$.

The assumption concerning $\mathcal{P}$ in the next theorem guarantees that all strict total orders $\mathcal{T}$ on D that extend $\mathcal{P}$ are well-orderings, that is, for any non-empty subset of D there is a $\mathcal{T}$-least element.

Theorem 17 *Let $\Delta = \langle D, W \rangle$ be a default theory. Let $\mathcal{P}$ be a strict partial order on D such that for each $\delta \in D$ the set $\{\delta' \in D | not \, \delta \mathcal{P} \delta'\}$ is finite. Then the relation $\sqsubset_{\mathcal{P}}^{\Delta}$ is transitive and asymmetric.*

PROOF. Let $\mathcal{T}$ be a strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$, and let δ be any member of D. By assumption the set $B = \{\delta' \in D | not \, \delta \mathcal{P} \delta'\}$ is finite. Because the set $B' = \{\delta' \in D | not \, \delta \mathcal{T} \delta'\}$ is a subset of B, it is finite. Hence there are no infinite descending chains from any default in D and all strict total orders extending $\mathcal{P}$ are well-orderings.

Transitivity: Suppose that $E_1 \sqsubset_{\mathcal{P}}^{\Delta} E_2$ and $E_2 \sqsubset_{\mathcal{P}}^{\Delta} E_3$. Let $\mathcal{T}$ be a strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$. The result that for all $\delta \in D$ such that appl(δ, E_3, E_1) there is $\epsilon \in D$ such that $\epsilon \mathcal{T} \delta$ and appl(ϵ, E_1, E_3), is easily obtained by analyzing the case where appl(δ, E_3, E_1) and not appl(δ, E_2), and the case where appl(δ, E_3, E_1) and appl(δ, E_2). The assumption that strict total orderings that extend $\mathcal{P}$ are well-orderings is used here. Hence $E_1 \sqsubset_{\mathcal{P}}^{\Delta} E_3$.

Asymmetry: From the assumption that $E_1 \sqsubset_{\mathcal{P}}^{\Delta} E_2$ and $E_2 \sqsubset_{\mathcal{P}}^{\Delta} E_1$ for some extensions E_1 and E_2 of Δ, a contradiction is derived with the inexistence of infinite descending sequences of defaults in D. Hence $\sqsubset_{\mathcal{P}}^{\Delta}$ is asymmetric. □

The following lemma and theorem establish the fact that the minimal elements of $\sqsubset_{\mathcal{P}}^{\Delta}$ are exactly the $\mathcal{P}$-preferred$_2$ extensions of Δ.

Lemma 18 ($\sqsubset_{\mathcal{P}}^{\Delta}$ monotonicity) *Let $\Delta = \langle D, W \rangle$ be a default theory, and $\mathcal{P}$ and $\mathcal{P}'$ strict partial orders on D such that $\mathcal{P}' \subseteq \mathcal{P}$. Then $\sqsubset_{\mathcal{P}'}^{\Delta} \subseteq \sqsubset_{\mathcal{P}}^{\Delta}$.*

PROOF. Let E and E' be extensions of Δ and $E \sqsubset_{\mathcal{P}'}^{\Delta} E'$. Let $\mathcal{T}$ be any strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$. Because $\mathcal{P}' \subseteq \mathcal{P}$, also $\mathcal{P}' \subseteq \mathcal{T}$. Because $E \sqsubset_{\mathcal{P}'}^{\Delta} E'$, for all $\delta \in D$, appl(δ, E', E) implies that for some $\epsilon \in D, \epsilon \mathcal{T} \delta$ and appl(ϵ, E, E'). As this holds for any strict total order $\mathcal{T}$ such that $\mathcal{P} \subseteq \mathcal{T}$, by definition $E \sqsubset_{\mathcal{P}}^{\Delta} E'$. Therefore $\sqsubset_{\mathcal{P}'}^{\Delta} \subseteq \sqsubset_{\mathcal{P}}^{\Delta}$. □

Theorem 19 *Let $\Delta = \langle D, W \rangle$ be a default theory, E an extension of Δ, and $\mathcal{P}$ a strict partial order on D such that for each $\delta \in D$ the set $\{\delta' \in D | not \, \delta \mathcal{P} \delta'\}$ is finite. Then E is a $\mathcal{P}$-preferred$_2$ extension of Δ if and only if there is no E' such that $E' \sqsubset_{\mathcal{P}}^{\Delta} E$.*

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PROOF. Assume that E is a $\mathcal{P}$-preferred$_2$ extension of Δ and E' is any extension of Δ such that $E \neq E'$. By definition of $\mathcal{P}$-preferredness$_2$, there is a $\Delta, \mathcal{P}$-ordering $\mathcal{T}$ against E' for E. By definitions of $\Delta, \mathcal{P}$-orderings and $\sqsubset_{\mathcal{T}}^{\Delta}$, $E \sqsubset_{\mathcal{T}}^{\Delta} E'$. Because by Theorem 17 $\sqsubset_{\mathcal{T}}^{\Delta}$ is asymmetric, not $E' \sqsubset_{\mathcal{T}}^{\Delta} E$. By Lemma 18 and the fact that $\mathcal{P} \subseteq \mathcal{T}$, not $E' \sqsubset_{\mathcal{P}}^{\Delta} E$.

Assume E is an extension of Δ that is not $\mathcal{P}$-preferred$_2$. Then for some extension E' there are no $\Delta, \mathcal{P}$-orderings against E' for E. Now for all strict total orders $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$ there is $\delta_{\mathcal{T}} \in D$ such that $\mathrm{appl}(\delta_{\mathcal{T}}, E', E)$ and there is no $\delta' \mathcal{T} \delta_{\mathcal{T}}$ such that $\mathrm{appl}(\delta', E, E')$. Take any strict total order $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$. Let δ be any default in D such that $\mathrm{appl}(\delta, E, E')$. Now $\delta_{\mathcal{T}} \mathcal{T} \delta$ and $\mathrm{appl}(\delta_{\mathcal{T}}, E', E)$. Therefore $E' \sqsubset_{\mathcal{P}}^{\Delta} E$. $\quad\square$

The relation $\sqsubset_{\mathcal{P}}^{\Delta}$ can be defined directly on the basis of $\mathcal{P}$ without reducing lexicographic comparison with partial orders to lexicographic comparison with total orders. This kind of definition is used by Geffner and Pearl [8] and Ryan [26].

Theorem 20 *Let E and E' be extensions of a default theory $\Delta = \langle D, W \rangle$. Let $\mathcal{P}$ be a strict partial order on D. Then $E \sqsubset_{\mathcal{P}}^{\Delta} E'$ if and only if $E \neq E'$ and for all $\delta \in D$,*

$\mathrm{appl}(\delta, E', E)$ *implies that for some $\epsilon \in D, \epsilon \mathcal{P} \delta$ and $\mathrm{appl}(\epsilon, E, E')$.*

PROOF. Assume that for all $\delta \in D$, $\mathrm{appl}(\delta, E', E)$ implies that for some $\epsilon \in D$, $\epsilon \mathcal{P} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. Let $\mathcal{T}$ be any strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$. Let δ be any member of D. If $\mathrm{appl}(\delta, E', E)$, then by the above assumption there is $\epsilon \in D$ such that $\epsilon \mathcal{P} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. Because $\mathcal{P} \subseteq \mathcal{T}$, also $\epsilon \mathcal{T} \delta$. Because this holds for all $\mathcal{T}$ such that $\mathcal{P} \subseteq \mathcal{T}$, by definition $E \sqsubset_{\mathcal{P}}^{\Delta} E'$.

Assume that it is not the case that for all $\delta \in D$, $\mathrm{appl}(\delta, E', E)$ implies that for some $\epsilon \in D, \epsilon \mathcal{P} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. That is, for some $\delta \in D$, $\mathrm{appl}(\delta, E', E)$ and there is no $\epsilon \in D$ such that $\epsilon \mathcal{P} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. Take any strict total order $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$ and $\delta \mathcal{T} \mu$ for all $\mu \in D$ such that not $\mu \mathcal{P} \delta$. Hence there is no $\epsilon \in D$ such that $\epsilon \mathcal{T} \delta$ and $\mathrm{appl}(\epsilon, E, E')$. By definition of $\sqsubset_{\mathcal{P}}^{\Delta}$ we conclude that not $E \sqsubset_{\mathcal{P}}^{\Delta} E'$. $\quad\square$

A subclass of strict partial orders that corresponds to the idea of priorities as integer-valued functions, is that of layered strict partial orders. With layered strict partial orders there cannot be two defaults such that one has a higher priority than the other and both have a priority no better nor worse than a third default. Brewka [6] and Benferhat et al. [3] discuss this class of priorities in the context of maximally consistent subsets of formulae. For this class of

strict partial orders there is no difference between preferred and preferred$_2$ extensions.

Theorem 21 (Layered priorities) *Let $\Delta = \langle D, W \rangle$ be a default theory and $\mathcal{P}$ a strict partial order on D such that $\mathcal{P} = \{\langle \delta, \delta' \rangle | \delta \in D_i, \delta' \in D_j, i < j\}$ for some sets $D_1, D_2, \ldots, D_n$. Then for all extensions E of Δ, E is a $\mathcal{P}$-preferred extension if and only if E is a $\mathcal{P}$-preferred$_2$ extension.*

PROOF. By Lemma 14 each $\mathcal{P}$-preferred extension of Δ is $\mathcal{P}$-preferred$_2$. We have to show that for layered $\mathcal{P}$, each $\mathcal{P}$-preferred$_2$ extension of Δ is $\mathcal{P}$-preferred. Let E be a $\mathcal{P}$-preferred$_2$ extension of Δ. Let $\mathcal{T}$ be a strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$ and $\delta \mathcal{T} \delta'$ whenever $\mathrm{appl}(\delta, E)$ and not $\mathrm{appl}(\delta', E)$ and neither $\delta \mathcal{P} \delta'$ nor $\delta' \mathcal{P} \delta$. Such an ordering can be constructed by taking defaults in D layer by layer starting from the most significant one, and taking the ones applied in E before the ones on the same layer that are not applied in E. Let E' be an extension of Δ such that $E \neq E'$. Because E is $\mathcal{P}$-preferred$_2$, there is a $\Delta, \mathcal{P}$-ordering $\mathcal{T}'$ against E' for E. Assume $\mathrm{appl}(\delta, E', E)$ for some $\delta \in D$. Now there is $\delta' \in D$ such that $\mathrm{appl}(\delta', E, E')$ and $\delta' \mathcal{T}' \delta$. If $\delta' \mathcal{P} \delta$, then clearly also $\delta' \mathcal{T} \delta$. Otherwise, as $\mathcal{P} \subseteq \mathcal{T}'$, not $\delta \mathcal{P} \delta'$, and hence the defaults are on the same layer. Because $\mathrm{appl}(\delta', E)$ and not $\mathrm{appl}(\delta, E)$, by definition $\delta' \mathcal{T} \delta$. Hence $\mathcal{T}$ is a $\Delta, \mathcal{P}$-ordering for E, and E is a $\mathcal{P}$-preferred extension of Δ. $\square$

Reiter's default logic is a special case of prioritized default logic. The embedding is the restriction to empty priorities.

Lemma 22 *Let Δ be a default theory and $\mathcal{P}$ the empty relation. Then all extensions of Δ are $\mathcal{P}$-preferred and $\mathcal{P}$-preferred$_2$.*

PROOF. Take any extension E of $\Delta = \langle D, W \rangle$. Take any strict total order $\mathcal{T}$ on D such that $\mathcal{P} \subseteq \mathcal{T}$ and for all $\delta \in D$ and $\delta' \in D$, $\delta \mathcal{T} \delta'$ whenever $\mathrm{appl}(\delta, E)$ and not $\mathrm{appl}(\delta', E)$. Assume that there is an extension E' such that for some $\delta' \in D$, $\mathrm{appl}(\delta', E', E)$ but there is no $\delta \in D$ such that $\delta \mathcal{T} \delta'$ and $\mathrm{appl}(\delta, E, E')$. By definition of $\mathcal{T}$, $\delta \mathcal{T} \delta'$ for all δ such that $\mathrm{appl}(\delta, E, E')$. This means that $\{\delta \in D | \mathrm{appl}(\delta, E')\} \subset \{\delta \in D | \mathrm{appl}(\delta, E)\}$. But this contradicts Theorems 2.4 and 2.5 in [22]. Hence $\mathcal{T}$ is a $\Delta, \mathcal{P}$-ordering for E and E is a $\mathcal{P}$-preferred extension of Δ. By Lemma 14 E is a $\mathcal{P}$-preferred$_2$ extension of Δ. $\square$

The purpose of priorities is to indicate which ways of resolving a conflict between defaults are acceptable. If priorities do not determine how a certain conflict should be resolved, a form of case analysis is performed by resolving

conflicts in all possible ways. If more priorities are given, the number of possible ways of resolving the conflicts decreases, and the number of conclusions increases. Our definition of priorities has this monotonicity property.

Lemma 23 (Monotonicity) *Let $\Delta = \langle D, W \rangle$ be a default theory. Let E be a $\mathcal{P}$-preferred ($\mathcal{P}$-preferred$_d$, $\mathcal{P}$-preferred$_2$, $\mathcal{P}$-preferred$_{2d}$) extension of Δ for some strict partial order $\mathcal{P}$ on D. Let $\mathcal{P}'$ be a strict partial order on D such that $\mathcal{P}' \subseteq \mathcal{P}$. Then E is a $\mathcal{P}'$-preferred ($\mathcal{P}'$-preferred$_d$, $\mathcal{P}'$-preferred$_2$, $\mathcal{P}'$-preferred$_{2d}$) extension of Δ.*

PROOF. If E is a $\mathcal{P}$-preferred extension of Δ, there is a $\Delta, \mathcal{P}$-ordering $\mathcal{T}$ for E. Because $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{T}$ is also a $\Delta, \mathcal{P}'$-ordering for E, and hence E is a $\mathcal{P}'$-preferred extension. Similarly if E is a $\mathcal{P}$-preferred$_2$ extension of Δ, for all extensions E' of Δ there is a $\Delta, \mathcal{P}$-ordering $\mathcal{T}$ against E' for E. Because $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{T}$ is also a $\Delta, \mathcal{P}'$-ordering against E' for E, and hence E is a $\mathcal{P}'$-preferred$_2$ extension. Proofs for preferred$_d$ and preferred$_{2d}$ extensions are similar. $\square$

Results characterizing preferred$_2$ extensions as minimal elements of strict partial orders on extensions indirectly demonstrate that for default theories with a finite non-empty set of extensions the existence of preferred$_2$ extensions is guaranteed for any strict partial order on the defaults. The next results answer the question of the existence of preferred extensions under the same assumptions. For an infinite number of defaults the existence of preferred extensions is not guaranteed, and therefore we restrict to finite default theories.

Lemma 24 *Let $\Delta = \langle D, W \rangle$ be a default theory where D is finite, and let $\mathcal{P}$ be a strict total order on D. Let Δ have at least one extension. Then there is exactly one $\mathcal{P}$-preferred extension of Δ that is also the unique $\mathcal{P}$-preferred$_d$ extension of Δ.*

PROOF. We show that there is an extension E of Δ such that $\mathcal{P}$ is the $\Delta, \mathcal{P}$-ordering for E. Let $\delta_1, \ldots, \delta_n$ be the ordering $\mathcal{P}$ of D. Define $D_i = \{\delta_1, \ldots, \delta_i\}$ for all $i \in \{0, \ldots, n\}$. Define for all $i \in \{1, \ldots, n-1\}$,

$$X_0 = \{E \subseteq \mathcal{L} | E \text{ is an extension of } \Delta\}, \text{ and}$$

$$X_{i+1} = \begin{cases} \{E \in X_i | \text{appl}(\delta_{i+1}, E)\} & \text{if appl}(\delta_{i+1}, E) \text{ for some } E \in X_i, \\ X_i & \text{otherwise.} \end{cases}$$

Induction hypothesis: For $j \in \{0, \ldots, i\}$, 1. the set X_j is non-empty, 2. for all $E \in X_j$ and $E' \in X_j$ and $\delta \in D_j$, appl(δ, E) if and only if appl(δ, E'), and 3.

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for all $E \in X_j$ and $E' \in X_0 \backslash X_j$ there is $\delta \in D_j$ such that $\text{appl}(\delta, E, E')$ and there is no $\delta' \in D$ such that $\delta' \mathcal{P} \delta$ and $\text{appl}(\delta', E', E)$.

The proofs of both the base case and the inductive case are straightforward.

The claim of the lemma is obtained from the facts established in the induction proof as follows. By 1 the set X_n is non-empty. By 2 and Theorems 2.4 and 2.5 in [22] $|X_n| \leq 1$. Hence X_n is a singleton $\{E\}$. Let E' be any extension of Δ. Assume that there is $\delta' \in D$ such that $\text{appl}(\delta, E', E)$. Hence $E' \neq E$ and $E' \in X_0 \backslash X_n$. Now by 3 there is $\delta \in D_n = D$ such that $\text{appl}(\delta, E, E')$ and there is no δ'' such that $\delta'' \mathcal{P} \delta$ and $\text{appl}(\delta'', E', E)$. Therefore not $\delta' \mathcal{P} \delta$, and because $\mathcal{P}$ is a strict total order, it is the case that $\delta \mathcal{P} \delta'$. Because this holds for all $\delta' \in D$ and all extension E' of Δ, $\mathcal{P}$ is a $\Delta, \mathcal{P}$-ordering for E. Therefore E is a $\mathcal{P}$-preferred extension of Δ. Let E' be any extension such that $E \neq E'$. Now $E \in X_n$ and $E' \notin X_n$, and therefore by 3 there is $\delta \in D$ such that $\text{appl}(\delta, E, E')$ and there is no $\delta' \in D$ such that $\delta' \mathcal{P} \delta$ and $\text{appl}(\delta', E', E)$. Hence E is the only $\mathcal{P}$-preferred extension of Δ.

The proof works for preferred_d extensions with the obvious modifications. $\square$

Lemma 25 *Let $\Delta = \langle D, W \rangle$ be a default theory where D is finite, and let $\mathcal{P}$ be any strict total order on D. Let Δ have at least one extension. Then there is exactly one $\mathcal{P}$-preferred$_2$ extension of Δ.*

PROOF. By Lemma 24 there is exactly one $\mathcal{P}$-preferred extension. Because $\mathcal{P}$ is layered, by Theorem 21 $\mathcal{P}$-preferredness coincides with $\mathcal{P}$-preferredness$_2$. $\square$

Existence easily follows for strict partial orders in general.

Theorem 26 *Let $\Delta = \langle D, W \rangle$ be a default theory where D is finite. If Δ has extensions, it has both $\mathcal{P}$-preferred and $\mathcal{P}$-preferred$_2$ extensions for any strict partial order $\mathcal{P}$ on D.*

PROOF. Let $\mathcal{P}$ be any strict partial order on D. Let $\mathcal{T}$ be a strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$. By Lemmata 24 and 25 there are $\mathcal{T}$-preferred and $\mathcal{T}$-preferred$_2$ extensions of Δ. By Lemma 23 they are also $\mathcal{P}$-preferred and $\mathcal{P}$-preferred$_2$. $\square$

4 Related work on priorities

In this section we show the connection of the lexicographic prioritized default logic to the work by Brewka [6] and Ryan [26]. Other related work include Geffner and Pearl [8] who use lexicographic comparison in the same way as Ryan does. The prioritization in prioritized default logics of Brewka [7], Baader and Hollunder [1], and Marek and Truszczyński [16] is lexicographic when the defaults are prerequisite-free normal.

4.1 Priorities and maximal consistent subsets

Poole [20] investigates default reasoning in a framework that is based on maximal consistent subsets of sets of formulae. Brewka [6] extends this framework with priorities. He calls this generalization *preferred subtheories*. Brewka gives two definitions, one of which is more general than the other and in which formulae are ordered by strict partial orders.

Definition 27 ([6]) *Let $\mathcal{B}$ be a strict partial order on a finite set of formulae T. Let O be a set of formulae. The set S is* a preferred subtheory *of $\langle T, O, \mathcal{B} \rangle$ if and only if there is a strict total order $t_1, t_2, \ldots, t_n$ on T that extends $\mathcal{B}$ and $S = S_n$ where $S_0 = O$, and*

$$
S_{i+1} = \begin{cases} S_i \cup \{t_{i+1}\} & \text{if } S_i \cup \{t_{i+1}\} \text{ is consistent} \\ S_i & \text{otherwise} \end{cases} \quad \text{for all } i \in \{0, \ldots n\}.
$$

We have taken the liberty to extend Brewka's definition slightly. Brewka does not have the second component O. Whenever O is consistent, O could be taken to be the formulae with the highest priority. The reason for our modification is that it makes the presentation of some of the results in later sections more convenient.

The next lemmata and theorems show the connection between preferred subtheories and prioritized default logic. Brewka has earlier defined a prioritized version of default logic that restricts to prerequisite-free normal defaults [5].

Definition 28 *Let $\Theta = \langle T, O, \mathcal{B} \rangle$ where T and O are sets of formulae and $\mathcal{B}$ is a strict partial order on T. Then define*

$$
\text{tr}_{\text{ps}}(\Theta) = \left\langle \left\{ \frac{:\phi}{\phi} \,\middle|\, \phi \in T \right\}, O \right\rangle, \text{ and}
$$

$$p_{\mathrm{ps}}(\Theta) = \left\{ \left\langle \frac{:\phi}{\phi}, \frac{:\psi}{\psi} \right\rangle \middle| \phi \mathcal{B} \psi \right\}.$$

Lemma 29 *Let O be a consistent set of formulae and T a set of formulae. The sets S_i for $i \in \{0, \ldots, n\}$ and $\langle T, O, \mathcal{B} \rangle$ in Definition 27 are maximal consistent subsets of $\{t_1, \ldots, t_i\} \cup O$.*

Lemma 30 *Let T be a set of formulae and O a consistent set of formulae. Let S be a maximal consistent subset of $T \cup O$ that contains O. Then $\mathrm{Cn}(S)$ is an extension of the default theory $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$.*

Lemma 31 *Let T be a set of formulae, O a consistent set of formulae, and E an extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$. Then $S = E \cap (T \cup O)$ is a maximal consistent subset of $T \cup O$ and $E = \mathrm{Cn}(S)$.*

Lemma 32 *Let T be a set of formulae and O a consistent set of formulae. Then for all $\phi \in T$ and all extensions E of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$, $\neg\phi \notin E$ if and only if $\phi \in E$.*

Theorem 33 *Let S be a preferred subtheory of $\langle T, O, \mathcal{B} \rangle$. Then $\mathrm{Cn}(S)$ is a $p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-preferred extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$.*

PROOF. Assume O is inconsistent. Then $S = O$ is the only preferred subtheory of $\langle T, O, \mathcal{B} \rangle$ and $\mathrm{Cn}(S) = \mathcal{L}$ is the only extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$. By Theorem 26 $\mathrm{Cn}(S)$ is a $p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-preferred extension. Assume O is consistent. Because S is a preferred subtheory of $\langle T, O, \mathcal{B} \rangle$, there is a total ordering $t_1, \ldots, t_n$ of T as given in Definition 27. Let $\mathcal{T}$ be the total ordering $:t_1/t_1, \ldots, :t_n/t_n$. Clearly $p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle) \subseteq \mathcal{T}$. We show that $\mathcal{T}$ is a $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle), p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-ordering for $\mathrm{Cn}(S)$. Assume that for an extension E of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$ there is $t_k \in T$ such that $\mathrm{appl}(:t_k/t_k, E, \mathrm{Cn}(S))$. We derive a contradiction from the assumption that there is no $l \in \{1, \ldots, k-1\}$ such that $\mathrm{appl}(:t_l/t_l, \mathrm{Cn}(S), E)$. Now $\mathrm{Cn}(S) \cap \{t_1, \ldots, t_{k-1}\} \subseteq E \cap \{t_1, \ldots, t_{k-1}\}$. Because $\neg t_k \in \mathrm{Cn}(S)$, $(S \cap (\{t_1, \ldots, t_{k-1}\} \cup O)) \cup \{t_k\}$ is inconsistent. By Lemma 31 E is consistent and hence also its subset $(E \cap (\{t_1, \ldots, t_{k-1}\} \cup O)) \cup \{t_k\}$ is consistent. Because of this contradiction, there must be $t \in \{t_1, \ldots, t_{k-1}\}$ such that $\mathrm{appl}(:t/t, \mathrm{Cn}(S), E)$. Therefore $\mathcal{T}$ is a $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle), p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-ordering for $\mathrm{Cn}(S)$, and $\mathrm{Cn}(S)$ is a $p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-preferred extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$. $\quad\square$

Theorem 34 *Let E be a $p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-preferred extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$. Then $E \cap (T \cup O)$ is a preferred subtheory of $\langle T, O, \mathcal{B} \rangle$ and $E = \mathrm{Cn}(E \cap (T \cup O))$.*

PROOF. Assume E is inconsistent. Then O is inconsistent by Lemma 80, and O is the only preferred subtheory of $\langle T, O, \mathcal{B} \rangle$. So assume that E is consistent. Then there is a $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle), p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-ordering $\mathcal{T}$ for E. Let $\mathcal{T}$

be $:t_1/t_1, \ldots, :t_n/t_n$. The proof that $E \cap (T \cup O)$ is a preferred subtheory is by induction. We construct a preferred subtheory S_n according to Definition 27 so that $E = \mathrm{Cn}(S_n)$.

Induction hypothesis: $S_i = E \cap (\{t_1, \ldots, t_i\} \cup O)$.

Base case $i = 0$: $S_0 = O = E \cap (\emptyset \cup O)$ by definition of S_0 and because $O \subseteq E$.

Inductive case $i \geq 1$: Assume $S_{i-1} \cup \{t_i\}$ is consistent. Now $S_i = S_{i-1} \cup \{t_i\}$. There is a maximal consistent subset S of $T \cup O$ such that $S_i \subseteq S$. Hence $\mathrm{Cn}(S)$ is an extension of $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$ by Lemma 30. Clearly not $\mathrm{appl}(\delta, E, \mathrm{Cn}(S))$ for all $\delta \mathcal{T}:t_i/t_i$. Because $\mathcal{T}$ is a $\mathrm{tr}_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle), p_{\mathrm{ps}}(\langle T, O, \mathcal{B} \rangle)$-ordering for E, not $\mathrm{appl}(:t_i/t_i, \mathrm{Cn}(S), E)$. Hence $\mathrm{appl}(:t_i/t_i, E)$ and $t_i \in E$. Assume $S_{i-1} \cup \{t_i\}$ is inconsistent. Hence $S_i = S_{i-1}$ and $t_i \notin S_i$. Because $S_i \subseteq E$ and E is consistent by Lemma 80, $t_i \notin E$. Hence $S_i = E \cap (\{t_1, \ldots, t_i\} \cup O)$. $\square$

4.2 Ordered theory presentations

Ryan [26] defines ordered theory presentations (OTPs) as partial orders on sets of formulae. The ordering expresses the plausibility of the formulae and the maximal elements are the most plausible. An OTP determines a partial order on the models of the language $\mathcal{L}$. The consequences of an OTP are the formulae that are true in all the maximal models. An interesting novelty in OTPs is the use of natural consequences of formulae. Natural consequences are logical consequences that as truth-functions fulfill certain monotonicity conditions. In this section we show the connection of OTPs to prioritized default logic.

Definition 35 ([26]) *A formula ϕ is* monotonic *in p (written $p \in \phi^+$), if for all models M and N such that $N \models p$ and for all propositional variables $q \neq p$ $M \models q$ if and only if $N \models q$, $M \models \phi$ implies $N \models \phi$.*

A formula ϕ is anti-monotonic *in p (written $p \in \phi^-$), if for all models M and N such that $N \not\models p$ and for all propositional variables $q \neq p$ $M \models q$ if and only if $N \models q$, $M \models \phi$ implies $N \models \phi$.*

A formula ϕ is a natural consequence *of ϕ, written $\phi \models^{\smile} \psi$, if $\phi \models \psi$ and $\phi^+ \subseteq \psi^+$ and $\phi^- \subseteq \psi^-$.*

The motivation for introducing natural consequences is that even when a formula cannot be satisfied simultaneously with some other formulae, some part of it can. For example, some of the natural consequences of $p \wedge q$ are p, q, and $p \vee q$. The set of natural consequences of a formula is infinite. If ϕ is a natural consequence of a formula, then so are $\neg\neg\phi, \neg\neg\neg\neg\phi$ and so on.

In the propositional case, instead of the set of all natural consequences of a formulae, equivalently a finite set of natural consequences can be used. This is because the syntactic properties of the natural consequences are not used in the definitions, and therefore it suffices to consider for each formula ϕ only one natural consequence from each equivalence class of logically equivalent natural consequences, of which there are only a finite number.

The following is the definition of OTPs and ordering relations induced by OTPs on propositional models.

Definition 36 ([26]) *Let X be a set, $\leq$ a partial order (an antisymmetric, reflexive and transitive relation) on X and $F : X \to \mathcal{L}$ a function that maps elements of X to formulae. Then $\Gamma = \langle X, \leq, F \rangle$ is an OTP.*

Let M and N belong to $\mathcal{M}$ (the collection of all models.) Then $M \sqsubseteq_\phi N$, if for all $\psi \in \mathcal{L}$, $\phi \models^{\smile} \psi$ implies ($M \models \psi$ implies $N \models \psi$).

Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP, and let $M, N \in \mathcal{M}$. Then $M \sqsubseteq^\Gamma N$ if for each $x \in X$, if $M \not\sqsubseteq_{F(x)} N$ then there is $y \leq x$ such that $M \sqsubset_{F(y)} N$ (that is, $M \sqsubseteq_{F(y)} N$ and $N \not\sqsubseteq_{F(y)} M$.)

Let Γ be an OTP, and M an element of $\mathcal{M}$. Then $M \models \Gamma$ if M is $\sqsubseteq^\Gamma$-maximal.

Let Γ be an OTP. Then $\Gamma \models \phi$, if for each $M \in \mathcal{M}, M \models \Gamma$ implies $M \models \phi$.

The following translation of OTPs to prioritized default logic resembles the translation of preferred subtheories in Section 4.1. By $<$ we mean the strict (asymmetric) partial order $\{\langle x, y \rangle | x \leq y, x \neq y\}$.

Definition 37 *Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP. The translation of Γ to prioritized default logic is defined by the following functions.*

$$\mathrm{tr}_o(\Gamma) = \left\langle \left\{ \left. \frac{:\phi}{\phi} \right| x \in X, F(x) \models^{\smile} \phi \right\}, \emptyset \right\rangle$$

$$p_o(\Gamma) = \left\{ \left. \left\langle \frac{:\phi}{\phi}, \frac{:\psi}{\psi} \right\rangle \right| x \in X, y \in X, x < y, F(x) \models^{\smile} \phi, F(y) \models^{\smile} \psi \right\}$$

Because Ryan allows multiple occurrences of the same formula in an OTP and different formulae may share natural consequences, there is a problem with our definition of priorities where there may be only one occurrence of each default in the partial order that expresses the priorities. It can be solved by renaming each propositional variable p occurring in a natural consequence of a formula $F(x)$ to p_x and extending the translation of an OTP with equivalences $p \leftrightarrow p_x$. Our discussion of OTPs from now on assumes that formulae do not

share natural consequences.

The main result about OTPs is Theorem 41 the proof of which uses the following three lemmata.

Lemma 38 *Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP where $\leq$ is a well-ordering. Let E and E' be extensions of $\mathrm{tr}_o(\Gamma)$ and let M and M' be models of E and E', respectively. Then $E \sqsubset_{p_o(\Gamma)}^{\mathrm{tr}_o(\Gamma)} E'$ if and only if $M' \sqsubset^{\Gamma} M$.*

PROOF. In this proof we use the alternative definition of $\sqsubset_{\mathcal{P}}$ that is given in Theorem 20. Let $\mathcal{P} = p_o(\Gamma)$ and $\Delta = \langle D, \emptyset \rangle = \mathrm{tr}_o(\Gamma)$.

($\Rightarrow$) Assume $E \sqsubset_{\mathcal{P}}^{\Delta} E'$. Assume $M' \not\sqsubseteq_{F(x)} M$ for some $x \in X$. That is, for some ϕ, $F(x) \models^{\smile} \phi$ and $M' \models \phi$ and $M \not\models \phi$. By Lemma 32 $\phi \in E' \backslash E$ and appl(:$\phi/\phi, E', E$). Because $E \sqsubset_{\mathcal{P}}^{\Delta} E'$ there is :$\psi/\psi \in D$ such that :$\psi/\psi \mathcal{P}$:ϕ/ϕ and appl(:$\psi/\psi, E, E'$). Take :ψ/ψ to be a $\mathcal{P}$-minimal element of D such that :$\psi/\psi \mathcal{P}$:ϕ/ϕ and appl(:$\psi/\psi, E, E'$). Such an element exists because $\mathcal{P}$ is a well-ordering. Because $\psi \in E \backslash E'$, by Lemma 32 $M \models \psi$ and $M' \not\models \psi$, and because :$\psi/\psi \mathcal{P}$:ϕ/ϕ there is $y \in X$ such that $F(y) \models^{\smile} \psi$ and $y < x$. Hence $M \not\sqsubseteq_{F(y)} M'$. It remains to show that $M' \sqsubseteq_{F(y)} M$. Assume the opposite, that is, for some ω, $F(y) \models^{\smile} \omega$ and $M' \models \omega$ and $M \not\models \omega$. This implies by Lemma 32 that $\omega \in E' \backslash E$. Now because $E \sqsubset_{\mathcal{P}}^{\Delta} E'$ there should be :$\theta/\theta \in D$ such that :$\theta/\theta \mathcal{P}$:$\omega/\omega$ and appl(:$\theta/\theta, E, E'$), and consequently there is $z \in X$ such that $F(z) \models^{\smile} \theta$. Because $F(y) \models^{\smile} \omega$ and :$\theta/\theta \mathcal{P}$:$\omega/\omega$, $z < y$. Hence :$\theta/\theta \mathcal{P}$:$\psi/\psi \mathcal{P}$:$\phi/\phi$. This contradicts the assumption that :ψ/ψ is a $\mathcal{P}$-minimal element such that :$\psi/\psi \mathcal{P}$:ϕ/ϕ and appl(:$\psi/\psi, E, E'$). Therefore for all ω such that $F(y) \models^{\smile} \omega$, $M' \models \omega$ implies $M \models \omega$. Therefore $M' \sqsubseteq_{F(y)} M$. Because $M \not\sqsubseteq_{F(y)} M'$ and $M' \sqsubseteq_{F(y)} M$, $M' \sqsubset_{F(y)} M$. This completes the proof of $M' \sqsubseteq^{\Gamma} M$.

To prove that $M \not\sqsubseteq^{\Gamma} M'$, we show that there is $x \in X$ such that $M \not\sqsubseteq_{F(x)} M'$ and there is no $y \in X$ such that $y \leq x$ and $M \sqsubset_{F(y)} M'$. Take :ϕ/ϕ to be such that appl(:$\phi/\phi, E, E'$) and there is no δ such that $\delta \mathcal{P}$:ϕ/ϕ and appl(δ, E, E'). Such a :ϕ/ϕ exists because $\mathcal{P}$ is a well-ordering, and as $E \sqsubset_{\mathcal{P}}^{\Delta} E'$, $E \neq E'$ and by Theorems 2.4 and 2.5 in [22] there must be some :ϕ/ϕ such that appl(:$\phi/\phi, E, E'$). Now $M \models \phi$ and $M' \not\models \phi$. By definition of Δ there is $x \in X$ such that $F(x) \models^{\smile} \phi$. Now $M \not\sqsubseteq_{F(x)} M'$. Assume there is y such that $y \leq x$ and $M \sqsubset_{F(y)} M'$. Now there is ψ such that $F(y) \models^{\smile} \psi$ and $M \not\models \psi$ and $M' \models \psi$, and therefore $\psi \in E' \backslash E$ by Lemma 32 and :$\psi/\psi \mathcal{P}$:ϕ/ϕ by the definition of $\mathcal{P}$. Because $E \sqsubset_{\mathcal{P}}^{\Delta} E'$ there is :$\chi/\chi \mathcal{P}$:ψ/ψ such that appl(:$\chi/\chi, E, E'$). Now :$\chi/\chi \mathcal{P}$:$\psi/\psi \mathcal{P}$:ϕ/ϕ. This, however, contradicts our assumption that :ϕ/ϕ is a $\mathcal{P}$-minimal default such that appl(:$\phi/\phi, E, E'$). Hence there is no $y \in X$ such that $y \leq x$ and $M \sqsubseteq_{F(y)} M'$. Therefore $M' \sqsubset^{\Gamma} M$.

($\Leftarrow$) Assume $M' \sqsubset^{\Gamma} M$, that is, $M' \sqsubseteq^{\Gamma} M$ and $M \not\sqsubseteq^{\Gamma} M'$. We show that $E \sqsubset_{\mathcal{P}}^{\Delta} E'$, that is, for all $\delta \in D$, if appl(δ, E', E) then there is $\delta' \mathcal{P} \delta$ such

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that $\text{appl}(\delta', E, E')$. Assume $\text{appl}(:\phi/\phi, E', E)$. Now $M \not\models \phi$ and $M' \models \phi$ and $F(x) \models^{\smile} \phi$ for some $x \in X$. Hence $M' \not\sqsubseteq_{F(x)} M$. Because $M' \sqsubseteq^{\Gamma} M$, $M' \not\sqsubseteq_{F(x)} M$ implies the existence of $y \in X$ such that $y \leq x$ and $M' \sqsubset_{F(y)} M$. Now it must be the case that $y \neq x$ and hence $y < x$. Further, for some ω, $F(y) \models^{\smile} \omega$ and $M' \not\models \omega$ and $M \models \omega$. By definition of $\mathcal{P}$, $:\omega/\omega\mathcal{P}:\phi/\phi$. By Lemma 32, $\omega \in E \backslash E'$, and thus $\text{appl}(:\omega/\omega, E, E')$. Hence there is $:\omega/\omega \in D$ such that $\text{appl}(:\omega/\omega, E, E')$ and $:\omega/\omega\mathcal{P}:\phi/\phi$, and finally $E \sqsubset^{\Delta}_{\mathcal{P}} E'$. $\square$

Lemma 39 *Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP, and $\phi \in \mathcal{L}$ a formula. If $\phi \notin E$ for some $p_o(\Gamma)$-preferred$_2$ extension E, then there is a $\sqsubseteq^{\Gamma}$-maximal model M such that $M \not\models \phi$.*

PROOF. Let E be a $p_o(\Gamma)$-preferred$_2$ extension of $\text{tr}_o(\Gamma)$ such that $\phi \notin E$. Hence there is a model M such that $M \models E \cup \{\neg\phi\}$. Assume there is N such that $M \sqsubset^{\Gamma} N$. The set $\Omega = \{\phi | N \models \phi, F(x) \models^{\smile} \phi, x \in X\}$ is not necessarily a maximal consistent subset of $T = \{\phi | F(x) \models^{\smile} \phi, x \in X\}$. Let Ω' be a maximal consistent set of T such that $\Omega \subseteq \Omega' \subseteq T$. Because Ω' is consistent, it has a model L. Now $N \sqsubseteq_{F(x)} L$ for all $x \in X$, because for all ϕ and $x \in X$ such that $F(x) \models^{\smile} \phi$, $N \models \phi$ implies $L \models \phi$. Therefore $N \sqsubseteq^{\Gamma} L$ and by the transitivity of $\sqsubseteq^{\Gamma}$, $M \sqsubset^{\Gamma} L$. Because Ω' is a maximal consistent subset of T, there is by Lemma 30 an extension $E' = \text{Cn}(\Omega')$ of $\text{tr}_o(\Gamma)$. By Lemma 38 $E' \sqsubset_{p_o(\Gamma)} E$. This contradicts the $\mathcal{P}$-preferredness$_2$ of E, and hence M is $\sqsubseteq^{\Gamma}$-maximal. $\square$

Lemma 40 *Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP. Let $\phi \in \mathcal{L}$ be a formula. If $M \not\models \phi$ for some $\sqsubseteq^{\Gamma}$-maximal model M, then there is a $p_o(\Gamma)$-preferred$_2$ extension E of $\text{tr}_o(\Gamma)$ such that $\phi \notin E$.*

PROOF. Let $T = \{\phi | F(x) \models^{\smile} \phi, x \in X\}$. Let M be a $\sqsubseteq^{\Gamma}$-maximal model. Clearly, $\Pi = \{\phi \in T | M \models \phi\}$ is a consistent subset of T. Assume Π is not maximal, that is, there is a consistent set $\Omega \subseteq T$ such that $\Pi \subset \Omega$. Now $M' \models \Omega$ for some model M'. Clearly, for no $\chi \in T$, $M \models \chi$ and $M' \not\models \chi$. Hence $M \sqsubseteq^{\Gamma} M'$. Take any $\chi \in \Omega \backslash \Pi$. Now $M' \models \chi$ and $M \not\models \chi$. Hence $M' \not\sqsubseteq^{\Gamma} M$, and finally $M \sqsubset^{\Gamma} M'$. This contradicts the $\sqsubseteq^{\Gamma}$-maximality of M, so Π must be a maximal consistent subset of T. Hence by Lemma 30 $\text{Cn}(\Pi)$ is an extension of $\text{tr}_o(\Gamma)$. Because $M \models \Pi$ and $M \not\models \phi$, $\phi \notin \text{Cn}(\Pi)$. Assume $E = \text{Cn}(\Pi)$ is not $p_o(\Gamma)$-preferred$_2$. By Lemma 38, there would be a model N for which $M \sqsubset^{\Gamma} N$. This would contradict the $\sqsubseteq^{\Gamma}$-maximality of M. Hence E is $p_o(\Gamma)$-preferred$_2$ and $\phi \notin E$. $\square$

Theorem 41 *Let $\Gamma = \langle X, \leq, F \rangle$ be an OTP where $\leq$ is a well-ordering. Then $\Gamma \models \phi$ if and only if ϕ belongs to all $p_o(\Gamma)$-preferred$_2$ extensions of $\text{tr}_o(\Gamma)$.*

PROOF.

$\Gamma \models \phi$ iff $M \models \phi$ for all $\sqsubseteq^\Gamma$-maximal models M

          iff there is no $\sqsubseteq^\Gamma$-maximal model M such that $M \not\models \phi$

          iff there is no $p_o(\Gamma)$-preferred$_2$ extension E such that $\phi \notin E$

          iff $\phi \in E$ for all $p_o(\Gamma)$-preferred$_2$ extensions E.

The third equivalence is by Lemmata 39 and 40: $\phi \notin E$ for some $p_o(\Gamma)$-preferred$_2$ extension E of $\text{tr}_o(\Gamma)$ if and only if $M \not\models \phi$ for some $\sqsubseteq^\Gamma$-maximal model M. $\square$

The use of natural consequences in OTPs is the first difference between pre-ferred subtheories of Brewka and Ryan's ordered theory presentations. How-ever, this difference is independent of the kind of model minimization used, and natural consequences of formulae can be used in conjunction of preferred subtheories as well. The second difference is that Ryan's preference notion corresponds to preferred$_2$ extensions in prioritized default logic, and Brewka's preferred subtheories correspond to preferred extensions. A preference notion equivalent to Ryan's has been used by Geffner and Pearl in their definition of conditional entailment [8].

5   Representation of inheritance networks in prioritized default logic

In this section we give two translations of inheritance networks to default logic. We prove that the second translation is equivalent to the first translation in the sense that it sanctions exactly the same positive literals as consequences. In Section 6 we show the connection between our translations and the trans-lational theories of Gelfond and Przymusinska [10] and Brewka [4]. A detailed overview of inheritance reasoning is given in [11].

5.1   Definition of inheritance networks

We define inheritance networks as quadruples $G = \langle V, E, P, N \rangle$. The set V consists of *nodes* that correspond to classes and properties, and $E \subseteq V \times V \times \{0, 1, 2, 3\}$ is a set of *links*. The *polarity* of a link $\langle v_1, v_2, p \rangle$ is p. If the polarity is 0 or 1, the link is *defeasible*. If the polarity is 2 or 3, the link is *strict*. If the polarity is 0 or 2, the link is *positive*. If the polarity is 1 or 3, the link is *negative*. We consider inheritance reasoning with one individual only. The sets

$P \subseteq V$ and $N \subseteq V$ state that the individual respectively belongs and does not belong to a class.

A sequence $\langle v_1, w_1, p_1 \rangle, \ldots, \langle v_n, w_n, p_n \rangle$ of links in E is *a path* from v_1 to w_n, if $w_i = v_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. The path is *positive*, if p_i is 0 or 2 for all $i \in \{1, \ldots, n\}$. The *empty path* from any node to itself is also a path. An inheritance network is *acyclic* if there is a function f from V to the integers such that $f(v_1) < f(v_2)$ whenever there is a link $\langle v_1, v_2, p \rangle \in E$. This is equivalent to the inexistence of a non-empty path from a node to itself. The *rank* $r(p)$ of a node $p \in V$ is defined as the length of a longest path in G that ends in p. This notion was used in [10].

5.2 The translation

A translation of inheritance networks to default theories is as follows.

Definition 42 *Assume* $G = \langle V, E, P, N \rangle$ *is an inheritance network. The translation of* G *into default logic is* $\Delta_G = \langle D, W \rangle$, *where*

$$D = \left\{ \frac{p : q}{q} \,\middle|\, \langle p, q, 0 \rangle \in E \right\} \cup \left\{ \frac{p : \neg q}{\neg q} \,\middle|\, \langle p, q, 1 \rangle \in E \right\}$$

$$\cup \left\{ \frac{p :}{q} \,\middle|\, \langle p, q, 2 \rangle \in E \right\} \cup \left\{ \frac{p :}{\neg q} \,\middle|\, \langle p, q, 3 \rangle \in E \right\}, \ and$$

$$W = P \cup \{\neg p | p \in N\}.$$

In addition to this translation, in Section 5.3 we present a closely related translation in which defaults are prerequisite-free normal.

We adopt the terminology concerning paths and ranks of nodes in inheritance networks also in the setting of default theories. If Δ is the translation of an inheritance network G, then there is a path from p to q in Δ if there is a path from p to q in G. Also, there is a path from $\neg p$ or p to $\neg q$ or q in Δ, if there is a path from p to q in G. The rank $r(l)$ of a literal $l = p$ or $l = \neg p$ in a default theory Δ is the rank $r(p)$ of the node p in G. For a default δ with the conclusion l we define $r(\delta) = r(l)$.

The above translation uses non-normal defaults, and not all non-normal default theories have extensions. We restrict to a subclass of inheritance networks such that the corresponding default theories have at least one extension. Like in most of the research on inheritance reasoning, our inheritance networks are acyclic and finite.

Definition 43 (Good) *Let $G = \langle V, E, P, N \rangle$ be an inheritance network. Then G is good if 1. G is acyclic, 2. V is finite, 3. $P \cap N = \emptyset$, 4. there is no node $p \in V$ such that $\langle q, p, 2 \rangle \in E$ and $\langle q', p, 3 \rangle \in E$ for some $q \in V$ and $q' \in V$, 5. there is no node $p \in V$ such that $\langle q, p, 2 \rangle \in E$ and $p \in N$ for some $q \in V$, 6. there is no node $p \in V$ such that $\langle q, p, 3 \rangle \in E$ and $p \in P$ for some $q \in V$, 7. $s = s'$ whenever $\langle p, q, s \rangle \in E$ and $\langle p, q, s' \rangle \in E$, and 8. there is no defeasible link $\langle p, q, n \rangle \in E$ and strict link $\langle p', q', n' \rangle \in E$ such that $q = p'$.*

Conditions 3, 4, 5 and 6 ensure that there are no unresolvable conflicts, that is, there is always a defeasible link involved in a conflict, and hence all conflicts can be resolved by rejecting a defeasible link.

Lemma 44 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Then the set $W \cup \{\beta | p:/\beta \in D\}$ is consistent.*

Lemma 45 *Let G be a good inheritance network. Then the default theory Δ_G given in Definition 42 has at least one extension, and all extensions of Δ_G are consistent.*

Definition 46 (Priorities) *Let G be an inheritance network. Let $\Delta_G = \langle D, W \rangle$ be the translation of G as given in Definition 42. Let $\mathcal{P}_G$ be the smallest transitive relation such that the following holds. For defaults $p{:}\pi/\pi, r{:}\overline{\pi}/\overline{\pi}$ and $s{:}\sigma/\phi$ in D, $p{:}\pi/\pi \mathcal{P}_G r{:}\overline{\pi}/\overline{\pi}$ whenever there is a non-empty positive path from p to r, and $p{:}\pi/\pi \mathcal{P}_G s{:}\sigma/\phi$ whenever there is a path from π to s. Here p, r and s are propositional variables, π and ϕ are literals, and σ is the empty sequence or consists of one literal.*

Lemma 47 *Let G be an acyclic inheritance network. Then $\mathcal{P}_G$, as given in Definition 46, is a strict partial order.*

In Section 6.1 we show that the translation given in this section coincides with a translational theory of inheritance presented by Gelfond and Przymusinska [10].

5.3 Inheritance reasoning with prerequisite-free defaults

Inheritance networks can also be represented as prerequisite-free normal default theories. In this case the links have the contraposition property as they are represented as defaults $:p \to \beta/p \to \beta$. The translation of normal defaults with prerequisites to prerequisite-free normal defaults uses the function $I(D) = \{:p \to \beta/p \to \beta | p{:}\beta/\beta \in D\}$. Also define $I^{-1}(D) = \{p{:}\beta/\beta | :p \to \beta/p \to \beta \in D\}$. For translation of strict links define $I_s(D) = \{p \to \beta | p{:}/\beta \in D\}$. The following lemmata show that for each extension of $\langle D, W \rangle$, there is an

extension of $\langle I(D), W \cup I_s(D) \rangle$ that defeats the corresponding defaults, and vice versa.

Lemma 48 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let $\Delta' = \langle I(D), W \cup I_s(D) \rangle$. Let E be an extension of Δ_G and $F = \{\delta \in D | \text{not } \mathrm{def}(\delta, E)\}$. Then there is an extension E' of Δ' such that $I(F) = \{\delta \in I(D) | \text{not } \mathrm{def}(\delta, E')\}$ and for all propositional variables p, $p \in E$ if and only if $p \in E'$.*

PROOF. Let $S = \{\phi | {:} \phi/\phi \in I(F)\} \cup W \cup I_s(D)$.

Claim A For all literals π, $\pi \in E$ implies $S \models \pi$. *Proof:* By Definition 2 and Lemma 84 $E = \mathrm{Cn}_W(F)$. Let $C_0, C_1, \ldots$ be the sets for $\mathrm{Cn}_W(F)$ in Definition 2. *Induction hypothesis*: For all literals π, $C_i \models \pi$ implies $S \models \pi$. *Base case $i = 0$:* Because $C_0 = W$ and $W \subseteq S$, $C_0 \models \pi$ implies $S \models \pi$ for all literals π. *Inductive case $i \geq 1$:* Assume $C_i \models \pi$. If $\pi \in \mathrm{Cn}(C_{i-1}) \subseteq \mathrm{Cn}(C_i)$, then by monotonicity of $\models$ and by induction hypothesis $S \models \pi$. If $\pi \in \mathrm{Cn}(C_i) \backslash \mathrm{Cn}(C_{i-1})$, then because E is consistent by Lemma 45, there is $q{:}\sigma/\pi \in F$ such that $C_{i-1} \models q$. By induction hypothesis $S \models q$. Because $q \rightarrow \pi \in \{\phi | {:} \phi/\phi \in I(F)\} \cup I_s(D) \subseteq S$, finally $S \models \pi$. Q.E.D.

Let M be a propositional model such that for all propositional variables p, $M \models p$ iff $p \in E$.

Claim B $M \models S$. *Proof:* Because G is a good inheritance network, E is consistent by Lemma 45. If $\neg p \in E$, then $p \notin E$ because E is consistent, and hence $M \models \neg p$. Therefore $M \models \pi$ for all literals $\pi \in E$. Specifically $M \models W$ because $W \subseteq E$ and W is a set of literals. Assume $p \rightarrow \beta \in \{\phi | {:} \phi/\phi \in I(F)\} \cup I_s(D) = S \backslash W$ and $M \models p$. Now either by definition of F not $\mathrm{def}(p{:}\beta/\beta, E)$, or because defaults without justifications cannot be defeated, not $\mathrm{def}(p{:}/\beta, E)$. Because $M \models p$, $p \in E$. Because not $\mathrm{def}(p{:}\sigma/\beta, E)$, $\beta \in E$. Hence $M \models \beta$ and $M \models p \rightarrow \beta$. If $M \not\models p$, then immediately $M \models p \rightarrow \beta$. Therefore $M \models S$. Q.E.D.

Claim C For all propositional variables p, $S \models p$ if and only if $M \models p$. *Proof:* Assume $S \models p$. Because $M \models S$ by Claim **B**, $M \models p$. Assume $S \not\models p$. By Claim **A** $p \notin E$. By definition of M, $M \not\models p$. Q.E.D.

Claim D $S \cup \{p \rightarrow \beta\}$ is inconsistent for all ${:}p \rightarrow \beta/p \rightarrow \beta \in I(D) \backslash I(F)$. *Proof:* Assume ${:}p \rightarrow \beta/p \rightarrow \beta \in I(D) \backslash I(F)$. Hence $\mathrm{def}(p{:}\beta/\beta, E)$, that is, $p \in E$ and $\neg \beta \in E$. By Claim **A** $S \models p$ and $s \models \neg \beta$. Hence $S \cup \{p \rightarrow \beta\}$ is inconsistent. Q.E.D.

By Claims **B** and **D** S is a maximal consistent subset of $\{\phi | {:} \phi/\phi \in I(D)\} \cup W \cup I_s(D)$, and by Lemma 30 $E' = \mathrm{Cn}(S)$ is an extension of $\langle I(D), W \cup I_s(D) \rangle$.

Clearly, no member of $I(F)$ is defeated in $\mathrm{Cn}(S)$, and members of $I(D)\backslash I(F)$ are defeated in $\mathrm{Cn}(S)$. By Claim **C** $p \in \mathrm{Cn}(S)$ if and only if $M \models p$, and $M \models p$ if and only if $p \in E$ by definition of M. $\quad\square$

The proof in the opposite direction requires priorities, because when links are represented as prerequisite-free defaults, a conflict between two links can lead to the defeat of a link lower in the inheritance network. This cannot happen in the first translation, and priorities are needed to prevent it in the second.

Lemma 49 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let $\Delta' = \langle I(D), W \cup I_s(D) \rangle$. Let $\mathcal{P}$ be a strict partial order on $I(D)$ such that $:p \to \beta/p \to \beta \mathcal{P}:p' \to \beta'/p' \to \beta'$ whenever $\beta = p'$. Let E' be a $\mathcal{P}$-preferred$_d$ extension of Δ' and $F' = \{\delta \in I(D)|\,not\,\mathrm{def}(\delta, E')\}$. Then there is an extension $E \subseteq E'$ of Δ_G such that $I^{-1}(F') \cup \{p:/\beta \in D|\} = \{\delta \in D|\,not\,\mathrm{def}(\delta, E)\}$ and for all propositional variables p, $p \in E$ if and only if $p \in E'$.*

PROOF. Define $\mathrm{Tr}(D, W, E) = \mathrm{Cn}(W \cup \{p \wedge \beta \in E\,|p:\sigma/\beta \in D\})$ for sets of defaults D and sets of formulae W and E. Let $E = \mathrm{Tr}(D, W, E')$. We first show that E is an extension of Δ_G. Because E' is a $\mathcal{P}$-preferred$_d$ extension of Δ' and defaults in Δ' are prerequisite-free normal, by Theorem 34 $E' = \mathrm{Cn}(S)$ for some preferred subtheory S of $\langle \{\phi|:\phi/\phi \in I(D)\}, W \cup I_s(D), \mathcal{P}' \rangle$ where $\mathcal{P}' = \{\langle \phi, \phi' \rangle|:\phi/\phi\mathcal{P}:\phi'/\phi'\}$. Let $n = |I(D)|$. Let $\mathcal{T}$ be the strict total order and $S_0, \ldots, S_n$ the sets in Definition 27 with the properties $\mathcal{P}' \subseteq \mathcal{T}$ and $S = S_n$. Let $p_1 \to \beta_1, \ldots, p_n \to \beta_n$ be the ordering $\mathcal{T}$. Define $\delta_i = p_i:\beta_i/\beta_i$ for all $i \in \{1, \ldots, n\}$. Define $D_i = \{\delta_1, \ldots, \delta_i\} \cup \{p:/\beta \in D|\}$ for all $i \in \{0, \ldots, n\}$.

Induction hypothesis: $E_i = \mathrm{Tr}(D_i, W, \mathrm{Cn}(S_i))$ is an extension of $\langle D_i, W \rangle$.

Base case $i = 0$: Let $p_1':/\beta_1', \ldots, p_m':/\beta_m'$ be a strict total order on D_0 so that $j < k$ whenever $\beta_j' = p_k'$. There is such an ordering because G is good and therefore acyclic. Define $D_j' = \{p_1':/\beta_1', \ldots, p_j':/\beta_j'\}$ for all $j \in \{0, \ldots, m\}$. Define $S_j' = W \cup \{p \to \beta|p:/\beta \in D_j'\}$. By Lemma 44 S_j' is consistent for all $j \in \{0, \ldots, m\}$. Clearly, S_j' is the unique extension of $\langle \emptyset, W \cup I_s(D_j') \rangle$ for all $j \in \{0, \ldots, m\}$. *Induction hypothesis*: $E_j' = \mathrm{Tr}(D_j', W, \mathrm{Cn}(S_j'))$ is an extension of $\langle D_j', W \rangle$. *Base case $j = 0$*: The result is immediate as $\mathrm{Tr}(D_0', W, \mathrm{Cn}(S_0')) = \mathrm{Cn}(W)$ and $\mathrm{Cn}(W)$ is an extension of $\langle \emptyset, W \rangle$. *Inductive case $j \geq 1$*: By definition $S_j' = S_{j-1}' \cup \{p_j' \to \beta_j'\}$. Assume $S_{j-1}' \models p_j'$. Hence $S_j' \models p_j' \wedge \beta_j'$ and $E_j' = \mathrm{Cn}(E_{j-1}' \cup \{p_j' \wedge \beta_j'\})$. Because $S_{j-1}' \models p_j'$, by Lemma 89 either $p_j' \in W$ or $S_{j-1}' \models p$ for some $p \to p_j' \in S_{j-1}'$. In both cases $p_j' \in E_{j-1}'$. Therefore by Lemma 88 E_j' is an extension of $\langle D_j', W \rangle$. Assume $S_{j-1}' \not\models p_j'$. By Lemma 90 the set of propositional variables in $\mathrm{Cn}(S_j')\backslash \mathrm{Cn}(S_{j-1}')$ is a subset of $\{\beta_j'\}$ and as G is acyclic, $p_j' \neq \beta_j'$. Hence $p_j' \notin \mathrm{Cn}(S_j')$, $p_j' \wedge \beta_j' \notin \mathrm{Cn}(S_j')$ and $p_j' \wedge \beta_j' \notin E_j'$.

Therefore $E'_j = E'_{j-1}$. By induction hypothesis and Lemma 87 E'_j is an extension of $\langle D'_j, W \rangle$.

Inductive case $i \geq 1$: Assume $S_{i-1} \cup \{p_i \to \beta_i\}$ is inconsistent. Therefore $S_i = S_{i-1}$. Now $p_i \wedge \overline{\beta_i} \in \text{Cn}(S_{i-1})$ and because S_{i-1} is consistent, $p_i \wedge \beta_i \notin \text{Cn}(S_{i-1})$, and hence $E_i = E_{i-1}$. Because $S_{i-1} \models \overline{\beta_i}$ and by assumption there are no formulae $\beta_i \to \beta' \in S_{i-1}$ for any β', by Lemma 89 either $\overline{\beta_i} \in W \subseteq S_{i-1}$ or $\beta' \to \overline{\beta_i} \in S_{i-1} \subseteq \{\phi | : \phi/\phi \in I(D)\} \cup I_s(D)$ and $S_{i-1} \models \beta'$ for some β'. In both cases $\overline{\beta_i} \in E_{i-1}$. Hence by Lemma 86 E_i is an extension of $\langle D_i, W \rangle$.

Assume $S_{i-1} \cup \{p_i \to \beta_i\}$ is consistent. Then $S_i = S_{i-1} \cup \{p_i \to \beta_i\}$. By Lemma 90 the set of propositional variables in $\text{Cn}(S_i) \backslash \text{Cn}(S_{i-1})$ is a subset of $\{\beta_i\}$. Because G is acyclic, $p_i \neq \beta_i$, and hence $p_i \in S_i$ if and only if $p_i \in S_{i-1}$. Assume $S_{i-1} \models p_i$. Hence $S_i \models p_i \wedge \beta_i$ and $E_i = \text{Cn}(E_{i-1} \cup \{p_i \wedge \beta_i\})$. Because $S_{i-1} \models p_i$, by Lemma 89 either $p_i \in W$ or $S_{i-1} \models p$ for some $p \to p_i \in S_{i-1}$. Hence $p_i \in E_{i-1}$. Therefore by Lemma 88 E_i is an extension of $\langle D_i, W \rangle$. Assume $S_{i-1} \not\models p_i$. Hence $p_i \wedge \beta_i \notin \text{Cn}(S_i)$ and $p_i \wedge \beta_i \notin E_i$. Therefore $E_i = E_{i-1}$. By induction hypothesis and Lemma 87 E_i is an extension of $\langle D_i, W \rangle$. This finishes the induction proof, and the case $i = n$ shows that $E = \text{Tr}(D_n, W, \text{Cn}(S_n))$ is an extension of $\langle D_n, W \rangle = \langle D, W \rangle$.

Because $E \subseteq E'$, for all propositional variables p, $p \in E$ implies $p \in E'$. Assume $p \in E' = \text{Cn}(S_n)$. By Lemma 89 $p \in W$ or $q \to p \in S_n$ and $S_n \models q$ for some q. Hence either $p \in W \subseteq E$ or by definition $q \wedge p \in \text{Tr}(D, W, \text{Cn}(S_n)) = E$. Hence for all propositional variables p, $p \in E$ if and only if $p \in E'$.

Next we show that $I^{-1}(F') \cup \{p{:}/\beta \in D|\} \subseteq \{\delta \in D | \text{not def}(\delta, E)\}$. For $p{:}/\beta \in D$ not $\text{def}(p{:}/\beta, E)$ is immediate because defaults without justifications are defeated in no extension. Assume $p{:}\beta/\beta \in I^{-1}(F')$. Hence $p \wedge \neg\beta \notin E'$. Because $E \subseteq E'$, $p \wedge \neg\beta \notin E$. Hence not $\text{def}(p{:}\beta/\beta, E)$. Hence no member of $I^{-1}(F') \cup \{p{:}/\beta \in D|\}$ is defeated in E.

To prove the inclusion in the opposite direction, we use Lemma 49 and the results obtained in the proof so far. Assume E' is an extension of Δ'. By the above results there is an extension E of Δ_G such that $I^{-1}(F') \cup \{p{:}/\beta \in D|\} \subseteq F$, where $F' = \{\delta \in I(D) | \text{not def}(\delta, E')\}$ and $F = \{\delta \in D | \text{not def}(\delta, E)\}$. Hence $F' \subseteq I(F)$. By Lemma 48 there is an extension E'_2 of Δ' such that $I(F) = \{\delta \in I(D) | \text{not def}(\delta, E'_2)\} = F'_2$. Now $F' \subseteq I(F) = F'_2$. By Theorem 2.4 in [22], Lemmata 84 and 82 and Definition 2 it cannot be the case that $F' \subset F'_2$. Hence $F' = I(F)$. Therefore $I^{-1}(F') \cup \{p{:}/\beta \in D|\} = \{\delta \in D | \text{not def}(\delta, E)\}$. $\square$

Lemma 50 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let $\mathcal{P}$ be a strict partial order on D such that $p{:}\beta/\beta \mathcal{P} p'{:}\beta'/\beta'$ whenever $\beta = p'$. Let $\Delta' = \langle I(D), W \cup I_s(D) \rangle$. Let*

$\mathcal{P}' = \{\langle:p \to \beta/p \to \beta, :p' \to \beta'/p' \to \beta'\rangle|p{:}\beta/\beta\mathcal{P}p'{:}\beta'/\beta'\}$. *1. If E' is a $\mathcal{P}'$-preferred$_d$ extension of Δ', then there is a $\mathcal{P}$-preferred$_d$ extension of Δ_G such that for all propositional variables p, $p \in E$ if and only if $p \in E'$. 2. If E is a $\mathcal{P}$-preferred$_d$ extension of Δ_G, then there is a $\mathcal{P}'$-preferred$_d$ extension E' of Δ' such that for all propositional variables p, $p \in E$ if and only if $p \in E'$.*

PROOF. 1. Let E' be a $\mathcal{P}'$-preferred$_d$ extension of Δ'. Let $F' = \{\delta \in I(D)|\text{not def}(\delta, E')\}$. Let $\mathcal{T}'$ be a $\Delta', \mathcal{P}'$-ordering$_d$ for E'. By Lemma 49 there is an extension E of Δ_G such that $I^{-1}(F') \cup \{p{:}/\beta \in D|\} = F = \{\delta \in D|\text{not def}(\delta, E)\}$. Let $\mathcal{T}$ be a strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$ and $\mathcal{T}' = \{\langle:p \to \beta/p \to \beta, :p' \to \beta'/p' \to \beta'\rangle|\langle p{:}\beta/\beta, p'{:}\beta'/\beta'\rangle \in \mathcal{T}\}$. We claim that $\mathcal{T}$ is a $\Delta_G, \mathcal{P}$-ordering$_d$ for E. Assume that this is not the case, that is, there is an extension E_2 and a default $\delta \in D$ such that $\text{def}(\delta, E, E_2)$ and there is no $\delta'\mathcal{T}\delta$ such that $\text{def}(\delta', E_2, E)$. Let $F_2 = \{\delta \in D|\text{not def}(\delta, E_2)\}$. By Lemma 48 there is E'' such that $I(F_2) = \{\delta \in I(D)|\text{not def}(\delta, E'')\} = F''$. Let $\{\delta_p\} = I(\{\delta\})$. Now $\text{def}(\delta_p, E', E'')$ and there is no $\delta'\mathcal{T}'\delta_p$ such that $\text{def}(\delta', E'', E')$. This contradicts the fact that $\mathcal{T}'$ is a $\Delta', \mathcal{P}'$-ordering$_d$ for E'. Hence $\mathcal{T}$ is a $\Delta_G, \mathcal{P}$-ordering$_d$ for E, and E is a $\mathcal{P}$-preferred$_d$ extension of Δ_G. By Lemma 49, for all propositional variables p, $p \in E$ if and only if $p \in E'$.

2. Proof is analogous to the first case. □

Finally, we establish the main result of this section.

Theorem 51 *Let G be a good inheritance network and Δ_G the translation of G as given in Definition 42. Let $\mathcal{P}_G$ be the strict total order for Δ_G given in Definition 46. Let $\mathcal{P} = \{\langle:p \to \beta/p \to \beta, :p' \to \beta'/p' \to \beta'\rangle|p{:}\beta/\beta\mathcal{P}_Gp'{:}\beta'/\beta'\}$. Then for all propositional variables p, $\langle D, W\rangle \models^d_{\mathcal{P}_G} p$ if and only if $\langle I(D), W \cup I_s(D)\rangle \models^d_{\mathcal{P}} p$.*

PROOF. The strict partial order $\mathcal{P}$ fulfills the property that $:p \to \beta/p \to \beta\mathcal{P}:p' \to \beta'/p' \to \beta'$ whenever $\beta = p'$. For propositional variables p, the existence of a $\mathcal{P}_G$-preferred$_d$ extension E of $\langle D, W\rangle$ such that $p \notin E$ is by Lemma 50 equivalent to the existence of a $\mathcal{P}$-preferred$_d$ extension E' of $\langle I(D), W \cup I_s(D)\rangle$ such that $p \notin E$. Hence for all propositional variables p, p is in all $\mathcal{P}_G$-preferred$_d$ extensions of $\langle D, W\rangle$ if and only if p is in all $\mathcal{P}$-preferred$_d$ extensions of $\langle I(D), W \cup I_s(D)\rangle$. □

In this section we discuss the inheritance theories presented in Sections 5.2 and 5.3 in terms of their behavior on sample inheritance networks. The first example is the simplest inheritance network with a resolvable conflict between two defaults.

Example 52 *For the penguin triangle (Figure 1) the translation in Section 5.2 yields the defaults $D = \{p{:}q/q, p{:}\neg r/\neg r, q{:}r/r\}$ and the priority relation $\mathcal{P}_G = \{\langle p{:}\neg r/\neg r, q{:}r/r\rangle, \langle p{:}q/q, q{:}r/r\rangle\}$. For $W = \{p\}$ the default theory $\langle D, W\rangle$ has exactly one $\mathcal{P}_G$-preferred$_d$ extension $E = \mathrm{Cn}(\{p, q, \neg r\})$.*

In the above example, and in inheritance reasoning in general, a conflict between two defaults is solved in favor of the default that is more specific, that is, a default that states something about a smaller set of individuals than the conflicting default. The meaning of "smaller" as a subset has not satisfied researchers, and less strict definitions of "smaller" have been considered, yielding different definitions of *preemption* between paths. The inheritance network in Figure 3 illustrates the difference between on-path and off-path preemption [11]. With on-path preemption, the path Bob→marine→man→beer-drinker could be preempted only by a negative link that starts from a node that is on the path Bob→marine→man. Hence with on-path preemption the conflict between the links ending in the node beer-drinker is unresolved. Off-path preemption allows also the interference of links that are on some path between *Bob* and *man*, not necessarily on the path Bob→marine→man. With off-path preemption the path Bob→marine→man→beer-drinker is preempted by the link from chaplain to beer-drinker, and hence the conclusion *not beer-drinker* is obtained. Both on-path and off-path preemption have their proponents [27]. The inheritance theories in the framework of prioritized default logic do not define a preemption relation between paths, and therefore they cannot directly be classified as on-path or off-path. Because the theories in Sections 5.2 and 5.3 assign $a \to p$ a priority higher than $b \to \neg p$ whenever a is lower than b in the inheritance network irrespective of whether b can be derived by using a sequence of links involving a, they behave more like path-based theories with off-path preemption.

Example 53 *The inheritance network in Figure 3 translates to the default theory $\Delta_G = \langle D, W\rangle$ where*

$$W = \{\text{chaplain}, \text{marine}\}, \textit{and}$$
$$D = \left\{ \frac{\text{chaplain} : \text{man}}{\text{man}}, \frac{\text{marine} : \text{man}}{\text{man}}, \right.$$
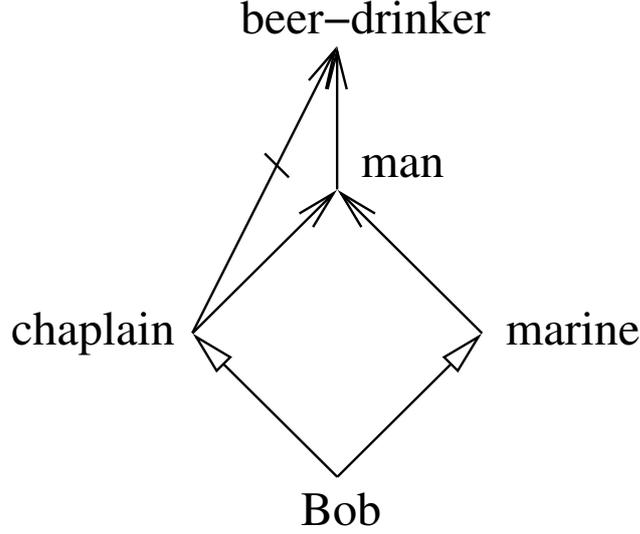
Fig. 3. The marine chaplain

$$\left. \frac{\text{chaplain} : \neg\text{beerdrinker}}{\neg\text{beerdrinker}}, \frac{\text{man} : \neg\text{beerdrinker}}{\neg\text{beerdrinker}} \right\}.$$

The priorities for Δ_G are

$$\mathcal{P}_G = \left\{ \left\langle \frac{\text{chaplain} : \neg\text{beerdrinker}}{\neg\text{beerdrinker}}, \frac{\text{man} : \text{beerdrinker}}{\text{beerdrinker}} \right\rangle, \right.$$
$$\left\langle \frac{\text{chaplain} : \text{man}}{\text{man}}, \frac{\text{man} : \text{beerdrinker}}{\text{beerdrinker}} \right\rangle,$$
$$\left. \left\langle \frac{\text{marine} : \text{man}}{\text{man}}, \frac{\text{man} : \text{beerdrinker}}{\text{beerdrinker}} \right\rangle \right\}.$$

The extension $E = \text{Cn}(\{\text{chaplain, marine, man}, \neg\text{beerdrinker}\})$ is the only $\mathcal{P}_G$-preferred$_d$ extension of Δ_G.

A link in an inheritance network is redundant if it is obtained by combining a sequence of consecutive links the last of which is positive or negative and the others are positive. Different notions of preemption treat redundant links differently, and there is a controversy between the proponents of on-path and off-path preemption on how redundant links should affect inheritance [11]. With our translations, redundant links can alter the behavior of an inheritance network.

Example 54 *Consider the inheritance network in Figure 4. The translation of the inheritance network is the set of defaults $D = \{a{:}b/b, b{:}c/c, c{:}d/d, b{:}\neg d/\neg d, a{:}d/d\}$ and the strict partial order $\mathcal{P}_G = \{\langle b{:}\neg d/\neg d, c{:}d/d\rangle, \langle a{:}d/d, b{:}\neg d/\neg d\rangle, \langle b{:}c/c, c{:}d/d\rangle, \langle a{:}b/b, b{:}c/c\rangle\}$. There are two extensions of $\langle D, \{a\}\rangle$, $E_1 = \text{Cn}(\{a, b, c, \neg d\})$ and $E_2 = \text{Cn}(\{a, b, c, d\})$. Only the extension E_2 is $\mathcal{P}_G$-preferred$_d$, and hence $\langle D, \{a\}\rangle \models^d_{\mathcal{P}_G} d$. The default*
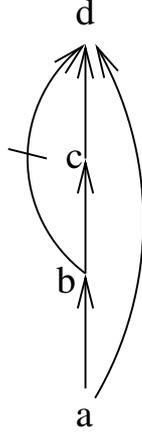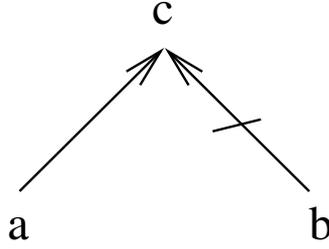
Fig. 4. A redundant link

Fig. 5. A simple inheritance network

theory $\langle D', \{a\} \rangle$ without the redundant link $a{:}d/d$, where $D' = \{a{:}b/b, b{:}c/c, c{:}d/d, b{:}\neg d/\neg d\}$, has the same extensions E_1 and E_2, but now only E_1 is $\mathcal{P}'_G$-preferred$_d$, where $\mathcal{P}'_G = \{\langle b{:}\neg d/\neg d, c{:}d/d \rangle, \langle b{:}c/c, c{:}d/d \rangle, \langle a{:}b/b, b{:}c/c \rangle\}$.

The inheritance theories in Sections 5.2 and 5.3 use defaults respectively with and without prerequisites. In the theory with prerequisite-free defaults links are represented as $:p \to \beta/p \to \beta$, and therefore reasoning by contraposition is possible. Hence, the negation of a prerequisite of a link may be derived when the conclusion is contradicted, and therefore these theories differ with respect to the negative literals they entail.

Example 55 *Figure 5 depicts an inheritance network with two links that correspond to the defaults $D = \{a{:}c/c, b{:}\neg c/\neg c\}$ (for the theory in Section 5.2) and $D' = \{:a \to c/a \to c, :b \to \neg c/b \to \neg c\}$ (for the theory in Section 5.3.) The priority relations are empty. The default theory $\langle D, \{b\} \rangle$ has the extension $E = \mathrm{Cn}(\{b, \neg c\})$, and the default theory $\langle D', \{b\} \rangle$ has the extension $E' = \mathrm{Cn}(\{b, b \to \neg c, a \to c\})$. The difference concerning negative literals is that $\neg a \in E' \backslash E$.*

6 Related work on inheritance reasoning

In this section we show that the inheritance theories by Gelfond and Przymusinska [10] and Brewka [4] are equivalent to the ones in Section 5.

6.1 An inheritance theory by Gelfond and Przymusinska

Gelfond and Przymusinska [10] present translations of inheritance networks to autoepistemic logic [18]. In Section 4b of their paper two formalizations of inheritance reasoning are given and proved equivalent. The first of these formalizations, defined for the class of good inheritance networks in Definition 43, uses prioritization that induces an ordering on the stable expansions of autoepistemic theories. In this section we show that Gelfond and Przymusinska's inheritance theory is equivalent to the one presented in Section 5.2.

Definition 56 (Autoepistemic logic) *The language $\mathcal{L}_{ae}$ of autoepistemic logic includes the formulae in $\mathcal{L}$ and is closed under the usual logical operators and the unary operator L. The consequence relation $\models_{ae}$ for autoepistemic formulae extends the logical consequence relation of classical propositional logic and treats formulae $L\phi$ like propositional variables. Sets $E = \{\phi \in \mathcal{L}_{ae} | \Sigma \cup \{L\phi \in \mathcal{L}_{ae} | \phi \in E\} \cup \{\neg L\phi \in \mathcal{L}_{ae} | \phi \notin E\} \models_{ae} \phi\}$ are the* stable expansions *of $\Sigma \subseteq \mathcal{L}_{ae}$.*

Definition 57 (Translation) *Let $G = \langle V, E, P, N \rangle$ be an acyclic inheritance network where V is finite. The translation $\mathrm{Th}(G)$ of G to autoepistemic logic consists of the following formulae.*

$$\{\mathrm{H}(x,c) | c \in P\} \cup \{\neg \mathrm{H}(x,c) | c \in N\} \cup$$
$$\{\mathrm{PD}(c,p) | \langle c,p,0 \rangle \in E\} \cup \{\mathrm{ND}(c,p) | \langle c,p,1 \rangle \in E\} \cup$$
$$\{\mathrm{PS}(c,p) | \langle c,p,2 \rangle \in E\} \cup \{\mathrm{NS}(c,p) | \langle c,p,3 \rangle \in E\} \cup$$
$$\{L\mathrm{PD}(c,p) \wedge L\mathrm{H}(x,c) \wedge \neg L\mathrm{ab}(x,c,p) \rightarrow \mathrm{H}(x,p) | c \in V, p \in V\} \cup$$
$$\{L\mathrm{ND}(c,p) \wedge L\mathrm{H}(x,c) \wedge \neg L\mathrm{ab}(x,c,p) \rightarrow \neg \mathrm{H}(x,p) | c \in V, p \in V\} \cup$$
$$\{L\mathrm{PS}(c,p) \wedge L\mathrm{H}(x,c) \rightarrow \mathrm{H}(x,p) | c \in V, p \in V\} \cup$$
$$\{L\mathrm{NS}(c,p) \wedge L\mathrm{H}(x,c) \rightarrow \neg \mathrm{H}(x,p) | c \in V, p \in V\}$$

Definition 58 (Explanation) *Let $\mathrm{Th}(G)$ be the translation of a finite and acyclic inheritance network G. A set X is an* explanation, *if X consists of formulae of the form $\mathrm{ab}(x,c_1,c_2)$ and $\mathrm{Th}(G) \cup X$ has a consistent stable expansion.*

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The purpose of an abnormality assumption $ab(x, c_1, c_2)$ is to prevent inheritance of properties for x from c_2 to c_1. For an explanation X, the set of abnormality assumptions $ab(x, c_1, c_2)$ with $c_2 = a$ is denoted by $X(a)$. The notion of explanation is motivated by the way inheritance networks G are translated to sets of formulae $\text{Th}(G)$: conflicts between defeasible links are not resolved by formulae in $\text{Th}(G)$, and in general $\text{Th}(G)$ does not have stable expansions. If X is an explanation, then $\text{Th}(G) \cup X$ has exactly one stable expansion.

Gelfond and Przymusinska define the relation *better than* on sets $X(a)$ and the relation *preferable* on explanations [4].

Definition 59 (Better than) *Let X and X' be explanations of an inheritance network G and a a node in G. Then $X(a)$ is* better than *$X'(a)$, if $X(a) \subset X'(a)$, or $X'(a)$ is non-empty [5] and for every $ab(x, c, a) \in X(a)$ there is $ab(x, c', a) \in X'(a)$ and a non-empty positive path from c' to c.*

Lemma 60 *The relation* better than *is a strict partial order.*

The preferability relation on explanations is defined in terms of the *better than* relation as follows.

Definition 61 (Preferable) *Let X and X' be explanations for an inheritance network G. Then X is* preferable *to X', if for some $m \geq 0$,*

(1) $X(a) = X'(a)$ for all nodes a in G such that $r(a) < m$,
(2) for all a such that $r(a) = m$, $X(a) = X'(a)$ or $X(a)$ is better than $X'(a)$, and
(3) for at least one a such that $r(a) = m$, $X(a)$ is better than $X'(a)$.

An explanation X is a best *explanation, if no explanation is preferable to X.*

The proof of the following lemma depends on the asymmetricity of *better than.*

Lemma 62 ([10]) *The preferability relation is a strict partial order.*

We rephrase Gelfond and Przymusinska's definitions in default logic. When E is an extension of a default theory $\langle D, W \rangle$, we denote by $E(a)$ the set of

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[4] In Proposition 1 in [10], the relation *preferable* is claimed to be a preorder. In the proof it is stated that by preorder an asymmetric and transitive relation is meant. However, the proposition is in error: neither the relation *better than* on which *preferable* is based nor *preferable*, as defined on Page 412 in their paper, are asymmetric. Our results that follow use a corrected definition of *better than.*
[5] This non-emptyness requirement is missing in the Gelfond and Przymusinska definition. It is needed because if both $X(a)$ and $X'(a)$ were empty, then each would be better than the other.

defaults $\delta \in D$ such that $\mathrm{def}(\delta, E)$ and the conclusion of δ is a or $\neg a$.

Definition 63 *Let $G = \langle V, E_d, P, N \rangle$ be an inheritance network and Δ_G the translation of G as given in Definition 42. Let E and E' be extensions of Δ_G, and $a \in V$. Then $E(a)$ is better than $E'(a)$, if $E(a) \subset E'(a)$, or for each $c{:}\alpha/\alpha \in E(a)$ there is $c'{:}\alpha'/\alpha' \in E'(a)$ such that there is a non-empty positive path from c' to c in G and $E'(a)$ is non-empty.*

Definition 64 *Let $G = \langle V, E_d, P, N \rangle$ be an inheritance network and Δ_G the translation of G as given in Definition 42. Let E and E' be extensions of Δ_G. Then E is* preferable *to E', if for some $m \geq 0$,*

(1) $E(a) = E'(a)$ for all $a \in V$ such that $r(a) < m$,
(2) for all $a \in V$ such that $r(a) = m$, $E(a) = E'(a)$ or $E(a)$ is better than $E'(a)$, and
(3) for at least one $a \in V$ such that $r(a) = m$, $E(a)$ is better than $E'(a)$.

An extension is a best *extension, if no extension is preferable to it.*

The main result in this section is Theorem 73 that shows that the best extensions of a default theory are the preferred$_d$ extensions. The proof of this theorem consists of two parts. The first part shows that every preferred$_{2d}$ extension (and therefore also every preferred$_d$ extension) is a best extension. This proof is based on assuming that there is a better extension, and then constructing with Lemma 70 an extension that violates the preferredness$_{2d}$ assumption. The proof of Lemma 70 uses Lemma 67 which in turn uses Lemma 66. In the second part of the proof of Theorem 73, for each best extension a $\Delta_G, \mathcal{P}_G$-ordering$_d$ is constructed, which directly implies that best extensions are $\mathcal{P}_G$-preferred$_d$ and $\mathcal{P}_G$-preferred$_{2d}$. The proof of Theorem 73 and its lemmata operate strictly within default logic. The correspondence to the autoepistemic theories defined by Gelfond and Przymusinska is due to the next lemma.

Lemma 65 *Let $G = \langle V, E_d, P, N \rangle$ be an inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let $\mathrm{Th}(G)$ be as given in Definition 57.*

If E is a best extension of Δ_G, then there is a best explanation X of $\mathrm{Th}(G)$ such that for all links from c_1 to c_2 in G and defaults $\delta \in \{c_1{:}c_2/c_2, c_1{:}\neg c_2/\neg c_2\}$, $\mathrm{def}(\delta, E)$ if and only if $\mathrm{ab}(x, c_1, c_2) \in X$. Furthermore, for all $q \in V$, $q \in E$ if and only if $\mathrm{H}(x, q) \in E'$, where E' is a stable expansion of $\mathrm{Th}(G) \cup X$, and for all $q \in V$, $\neg q \in E$ if and only if $\neg\mathrm{H}(x, q) \in E'$.

If X is a best explanation of $\mathrm{Th}(G)$, then there is a best extension E of Δ_G such that for all links from c_1 to c_2 in G and defaults $\delta = c_1{:}c_2/c_2$ or $\delta = c_1{:}\neg c_2/\neg c_2$, $\mathrm{def}(\delta, E)$ if and only if $\mathrm{ab}(x, c_1, c_2) \in X$. Furthermore, for all $q \in V$, $q \in E$ if and only if $\mathrm{H}(x, q) \in E'$, where E' is a stable expansion of $\mathrm{Th}(G) \cup X$, and

for all $q \in V$, $\neg q \in E$ if and only if $\neg\mathrm{H}(x, q) \in E'$.

PROOF. The proof relies on the close correspondence between autoepistemic logic and default logic [13]. The correspondence between best extensions and best explanations is due to the similarity of definitions of best. $\square$

Lemmata 66 and 67 are used in Lemma 70 that is used in the first part of the proof of Theorem 73. These lemmata construct an extension respectively by adding and removing a set of normal defaults that are the upper part of an inheritance hierarchy.

Lemma 66 *Let G be an acyclic inheritance network and $\Delta_G = \langle D \cup D', W\rangle$ the translation of G as given in Definition 42, and the propositional variables in conclusions of D' do not occur in any prerequisite of a default in D, and defaults in D' are normal. Let E be an extension of $\langle D, W\rangle$. Then there is an extension E' of Δ_G such that $E \subseteq E'$ and $\mathrm{def}(\delta, E')$ implies $\mathrm{def}(\delta, E)$ for all $\delta \in D$.*

PROOF. Let $\delta_1, \ldots, \delta_n$ be a total ordering of D' such that if $r(\delta_i) < r(\delta_j)$ then $i < j$. Such an ordering exists because G is acyclic. Let $p_i{:}\beta_i/\beta_i = \delta_i$ for all $i \in \{1, \ldots, n\}$. Define $D_i = \{\delta_1, \ldots, \delta_i\} \cup D$ for all $i \in \{0, \ldots, n\}$. We construct a sequence of sets $B_0, \ldots, B_n$ and prove by induction that $\mathrm{Cn}(B_i)$ is an extension of $\langle D_i, W\rangle$ for all $i \in \{0, \ldots, n\}$. As the case $i = n$ we obtain the fact that $\mathrm{Cn}(B_n)$ is an extension of $\langle D \cup D', W\rangle$. Define for all $i \in \{0, \ldots, n\}$,

$$B_0 = W \cup \left\{\gamma \left| \frac{\alpha : \sigma}{\gamma} \in \mathrm{GD}(E, \Delta)\right.\right\}$$

$$B_i = \begin{cases} B_{i-1} \cup \{\beta_i\}, & \text{if } B_{i-1} \models p_i \text{ and } B_{i-1} \not\models \neg\beta_i \\ B_{i-1}, & \text{otherwise.} \end{cases}$$

Induction hypothesis: $\mathrm{Cn}(B_i)$ is an extension of $\langle D_i, W\rangle$, and for all $\delta \in D$, $\mathrm{def}(\delta, \mathrm{Cn}(B_i))$ implies $\mathrm{def}(\delta, \mathrm{Cn}(B_0))$.

Base case $i = 0$: By assumption E is an extension of $\langle D, W\rangle$, and by Theorem 4 $\mathrm{Cn}(B_0) = E$, and by definition $D_0 = D$.

Inductive case $i \geq 1$: By induction hypothesis $\mathrm{Cn}(B_{i-1})$ is an extension of $\langle D_{i-1}, W\rangle$. Assume $B_{i-1} \models \neg\beta_i$. In this case $B_i = B_{i-1}$. By Lemma 86 $\mathrm{Cn}(B_i)$ is an extension of $\langle D_i, W\rangle = \langle D_{i-1} \cup \{\delta_i\}, W\rangle$. Assume $B_{i-1} \not\models p_i$. In this case $B_i = B_{i-1}$. By Lemma 87 $\mathrm{Cn}(B_i)$ is an extension of $\langle D_i, W\rangle = \langle D_{i-1} \cup$

$\{\delta_i\}, W\rangle$. Assume $B_{i-1} \models p_i$ and $B_{i-1} \not\models \neg\beta_i$. In this case $B_i = B_{i-1} \cup \{\beta_i\}$. By Lemma 88 $\mathrm{Cn}(B_i)$ is an extension of $\langle D_i, W\rangle$ because by assumption the conclusion of δ_i is not the prerequisite of any default in D, and by definition of rank and D_i the conclusion of δ_i is not the prerequisite of any default in D_i.

Assume $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_i))$ and $p{:}\beta/\beta \in D$. If $B_i = B_{i-1}$, then clearly $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_{i-1}))$ and by induction hypothesis $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_0))$. So assume $B_i = B_{i-1} \cup \{\beta_i\}$. Because B_i is a consistent set of literals, β_i is the only literal in $\mathrm{Cn}(B_i)\backslash\mathrm{Cn}(B_{i-1})$. Now $p \neq \beta_i$ because by assumption conclusions in D' do not occur in prerequisites in D. Hence $B_{i-1} \models p$. Because $B_i = B_{i-1} \cup \{\beta_i\}$, $\beta \neq \overline{\beta_i}$. Because $\beta_i \neq \overline{\beta}$, also $B_{i-1} \models \overline{\beta}$. Hence $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_{i-1}))$. Therefore by the induction hypothesis $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_i))$ implies $\mathrm{def}(p{:}\beta/\beta, \mathrm{Cn}(B_0))$. This finishes the induction proof. Hence $E' = \mathrm{Cn}(B_n)$ is an extension of $\langle D \cup D', W\rangle$ and $\mathrm{def}(\delta, E')$ implies $\mathrm{def}(\delta, E)$ for all $\delta \in D$. □

Lemma 67 *Let G be a good inheritance network and $\Delta_G = \langle D \cup D', W\rangle$ the translation of G as given in Definition 42. Assume that the propositional variables in conclusions of D' do not occur in any prerequisite of a default in D and the defaults in D' are normal. Let E' be an extension of $\langle D \cup D', W\rangle$. Then there is an extension E of $\langle D, W\rangle$ such that for all $\delta \in D$, if $\mathrm{def}(\delta, E)$ then $\mathrm{def}(\delta, E')$.*

PROOF. $E' = \mathrm{Cn}_W(\{\delta \in D \cup D' | \text{not } \mathrm{def}(\delta, E')\})$ by Definition 2 and Lemma 84. Let $D_{\mathrm{nd}} = \{\delta \in D | \text{not } \mathrm{def}(\delta, E')\}$. By Lemma 82 $E_0 = \mathrm{Cn}_W(D_{\mathrm{nd}}) \subseteq E'$. Let $D_{\mathrm{a}} = \{p{:}\sigma/\gamma \in D_{\mathrm{nd}} | p \in E_0\}$. By Lemma 85, E_0 is an extension of $\langle D_{\mathrm{a}}, W\rangle$ and $D_{\mathrm{a}} = \mathrm{GD}(E_0, \langle D_{\mathrm{a}}, W\rangle)$. By repeated application of Lemma 87 E_0 is an extension of $\langle D_{\mathrm{nd}}, W\rangle$. Because $E_0 \subseteq E'$, for all $\delta \in D$, if $\mathrm{def}(\delta, E_0)$, then $\mathrm{def}(\delta, E')$. By repeated application of Lemma 86 E_0 is an extension of $\langle D_{\mathrm{nd}} \cup D_{\mathrm{d}}, W\rangle$ where $D_{\mathrm{d}} = \{\delta \in D | \mathrm{def}(\delta, E_0)\}$.

Claim A *For all $p{:}\sigma/\gamma \in D_{\mathrm{nd}} \cup D_{\mathrm{d}}$, $p \in E'$ implies $p \in E_0$.* *Proof:* The proof is by induction. Let $C'_0, C'_1, \ldots$ be the sets in the definition of $E' = \mathrm{Cn}_W(\{\delta \in D \cup D' | \text{not } \mathrm{def}(\delta, E')\})$ and $C_0, C_1, \ldots$ the sets for $E_0 = \mathrm{Cn}_W(\{\delta \in D_{\mathrm{nd}} \cup D_{\mathrm{d}} | \text{not } \mathrm{def}(\delta, E_0)\})$. Notice that for all $\delta \in D_{\mathrm{nd}} \cup D_{\mathrm{d}}$, $\mathrm{def}(\delta, E_0)$ iff $\mathrm{def}(\delta, E')$. *Induction hypothesis*: If $p \in C'_i$ for $p{:}\sigma/\gamma \in D_{\mathrm{nd}} \cup D_{\mathrm{d}}$, then $p \in C_i$. *Base case $i = 0$*: By definition $C_0 = W = C'_0$. *Inductive case $i \geq 1$*: Assume $p \in C'_i$ for some $p{:}\sigma/\gamma \in D_{\mathrm{nd}} \cup D_{\mathrm{d}}$. If $p \in C'_{i-1}$ then by induction hypothesis $p \in C_{i-1} \subseteq C_i$. If $p \in C'_i\backslash C'_{i-1}$, then $C'_{i-1} \models p'$ for some $p'{:}\sigma'/p \in \{\delta \in D \cup D' | \text{not } \mathrm{def}(\delta, E')\}$. Because by assumption propositional variables in conclusions of defaults in D' do not occur in prerequisites of defaults in D, $p'{:}\sigma'/p \in D$. Because not $\mathrm{def}(p'{:}\sigma'/p, E')$, $p'{:}\sigma'/p \in D_{\mathrm{nd}}$. Because G is good, by Lemma 45 E' is consistent and its subset C'_{i-1} is consistent. Because C'_{i-1} is a consistent set of literals and $C'_{i-1} \models p'$, $p' \in C'_{i-1}$. Hence by induction

hypothesis $p' \in C_{i-1}$. Because $p':\sigma'/p \in D_{\mathrm{nd}}$, $p \in C_i$. Q.E.D.

Claim B $p \in E_0$ for all $p:\sigma/\gamma \in D \backslash D_{\mathrm{nd}}$. *Proof:* Let $p:\sigma/\gamma$ be any member of $D \backslash D_{\mathrm{nd}}$. By definition $\mathrm{def}(p:\sigma/\gamma, E')$ which implies $p \in E'$. Hence $p \in W$ or there is $p':\sigma'/p \in \mathrm{GD}(E', \Delta_G)$. In the first case $p \in E_0$ because $W \subseteq E_0$. Otherwise $p':\sigma'/p \in D$, because of the disjointness of the propositional variables in the conclusions of D and the prerequisites in D' and because $p:\sigma/\gamma \in D$. Hence $p':\sigma'/p \in D_{nd}$, and by Claim **A** $p \in E_0$. Q.E.D.

Claim C Defaults in $D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$ are normal. *Proof:* Assume $\delta \in D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$. Because $D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}}) \subseteq D \backslash D_{\mathrm{nd}} = D \backslash \{\delta \in D | \text{not def}(\delta, E')\} = \{\delta \in D | \mathrm{def}(\delta, E')\}$, $\mathrm{def}(\delta, E')$. Therefore δ has at least one justification and is normal. Q.E.D.

Claim D The propositional variables in conclusions of defaults in $D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$ do not occur in any prerequisite of a default in $D_{\mathrm{nd}} \cup D_{\mathrm{d}}$. *Proof:* First we show that for all $a:\pi/\pi \in D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$ there is $b:\overline{\pi}/\overline{\pi} \in D' \cap \mathrm{GD}(E', \Delta_G)$. Assume $a:\pi/\pi \in D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$. Because $a:\pi/\pi \in D \backslash D_{\mathrm{nd}}$, $\mathrm{def}(a:\pi/\pi, E')$. Hence $a \wedge \overline{\pi} \in E'$. Hence $\overline{\pi} \in W$ or there is $b:\overline{\pi}/\overline{\pi} \in \mathrm{GD}(E', \Delta_G)$. By Claim **B** $a \in E_0$. Because $a:\pi/\pi \in D \backslash D_{\mathrm{d}}$, not $\mathrm{def}(a:\pi/\pi, E_0)$, and hence $\overline{\pi} \notin E_0$. Because $W \subseteq E_0$, $\overline{\pi} \notin W$. Hence there is $b:\overline{\pi}/\overline{\pi} \in \mathrm{GD}(E', \Delta_G)$. Assume $b:\overline{\pi}/\overline{\pi} \in D$. Because not $\mathrm{def}(b:\overline{\pi}/\overline{\pi}, E')$, $b:\overline{\pi}/\overline{\pi} \in D_{\mathrm{nd}}$. Hence by Claim **A** $b \in E_0$. Because not $\mathrm{def}(b:\overline{\pi}/\overline{\pi}, E_0)$, $\overline{\pi} \in E_0$. Because $a \in E_0$, $\mathrm{def}(a:\pi/\pi, E_0)$, which contradicts the assumption that $a:\pi/\pi \in D \backslash D_n$ and hence not $\mathrm{def}(a:\pi/\pi, E_0)$. Hence $b:\overline{\pi}/\overline{\pi} \in D'$. Hence if $a:\pi/\pi \in D \backslash (D_{\mathrm{nd}} \cup D_{\mathrm{d}})$, then there is $b:\overline{\pi}/\overline{\pi} \in D'$. By the assumptions of the lemma the propositional variable in π does not occur in the prerequisite of any default in D. Q.E.D.

By Claims **C** and **D** and Lemma 66 we get from the extension E_0 of $\langle D_{\mathrm{nd}} \cup D_{\mathrm{d}}, W \rangle$ the extension E of $\langle D, W \rangle$ such that for all $\delta \in D$, if $\mathrm{def}(\delta, E)$ then $\mathrm{def}(\delta, E_0)$. As $\mathrm{def}(\delta, E_0)$ implies $\mathrm{def}(\delta, E')$ for all $\delta \in D$, $\mathrm{def}(\delta, E)$ implies $\mathrm{def}(\delta, E')$. □

Whenever all defaults with a rank less than some number are defeated in both of two extensions or in neither of them, then the conclusions of those defaults are in both of the extensions or in neither of them.

Lemma 68 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let E and E' be extensions of G such that $E(p) = E'(p)$ for all p such that $r(p) \leq n$. Then for all p such that $r(p) \leq n$, $p \in E$ if and only if $p \in E'$, and $\neg p \in E$ if and only if $\neg p \in E'$.*

PROOF. By induction on $r(\delta)$ for $\delta \in D$. □

Lemma 69 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let E and E' be extensions of G such that $E(p) \neq E'(p)$ and $E(q) = E'(q)$ for all q such that $r(q) < r(p)$. If $\pi \in E$, where $\pi \in \{p, \neg p\}$, then $\overline{\pi} \in E'$.*

PROOF. Assume $\pi \in E$. Because $E(p) \neq E'(p)$ there is $\delta = q{:}\beta/\beta \in E(p) \triangle E'(p)$. Hence $q \wedge \overline{\beta} \in E$ or $q \wedge \overline{\beta} \in E'$. By Lemma 68 $q \in E \cap E'$. If $\operatorname{def}(\delta, E, E')$, $\beta = \overline{\pi}$. Because not $\operatorname{def}(\delta, E')$, $\pi \notin E'$ and by definition of extensions $\beta = \overline{\pi} \in E'$. If $\operatorname{def}(\delta, E', E)$, $\overline{\beta} \in E'$. Because $q \in E$ and not $\operatorname{def}(\delta, E)$, $\beta = \pi$. Hence $\overline{\pi} \in E'$. $\square$

Whenever an extension is preferable to another extension, the latter can be improved at those lowest rank nodes where the former is better than the latter without making unrelated nodes worse.

Lemma 70 *Let $G = \langle V, E_d, P, N \rangle$ be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let E and E' be extensions of Δ_G. Let $n \geq 0$ be such that $E(c) = E'(c)$ for all nodes $c \in V$ such that $r(c) < n$. Let $B = \{b \in V | E'(b)$ is better than $E(b), r(b) = n\}$. Let $\mathcal{P}_G$ be the strict partial order on D given in Definition 46. Then there is an extension E'' of Δ_G such that $E''(b) = E'(b)$ for all $b \in B$, and $\operatorname{def}(\delta, E'')$ implies $\operatorname{def}(\delta, E)$ for all $\delta \in \{\delta \in D \backslash D_B | \delta' \mathcal{P}_G \delta$ for no $\delta' \in D_B\}$ where $D_B = \{p{:}\sigma/\gamma \in D | b \in B, \gamma \in \{b, \neg b\}\}$.*

PROOF. Define $D_L = \{\delta \in D \backslash D_B | \delta' \mathcal{P}_G \delta$ for no $\delta' \in D_B\}$, $D_H = \{\delta \in D \backslash D_B | \delta' \mathcal{P}_G \delta$ for some $\delta' \in D_B\}$, $D'_B = \{p{:}\gamma/\gamma \in D_B|\}$, and $D''_B = \{p{:}/\gamma \in D_B|\}$. Clearly $D = D_L \cup D_H \cup D_B$ and $D_B = D'_B \cup D''_B$.

Claim A If $q{:}/\beta \in D$ and $\beta \in \{a, \neg a\}$ for some $a \in V$ such that $r(a) = n$ and $E(a) \neq E'(a)$, then $q \notin E \cup E'$. *Proof:* Because $E(a) \neq E'(a)$, there is $p{:}\pi/\pi \in D$ such that $\pi \in \{a, \neg a\}$ and $\operatorname{def}(p{:}\pi/\pi, E, E')$ (proof for the case $\operatorname{def}(p{:}\pi/\pi, E', E)$ is similar.) Therefore $\overline{\pi} \in E$. By Lemma 69 $\pi \in E'$. Because G is good, by Lemma 45 $\pi \notin E$ and $\overline{\pi} \notin E'$. Because $q{:}/\beta \in D$ and $\beta \in \{a, \neg a\}$, by definition of extensions either $q \notin E$ or $q \notin E'$. By Lemma 68 $q \notin E \cup E'$. Q.E.D.

Claim B For all $b \in B$, either $b \in E$ or $\neg b \in E$. *Proof:* Let b be any member of B. Hence $E(b) \neq E'(b)$. Hence there is $\delta = p{:}\pi/\pi \in D$ with $\pi \in \{b, \neg b\}$ such that $\operatorname{def}(\delta, E, E')$ or $\operatorname{def}(\delta, E', E)$. Hence either $\{p, \overline{\pi}\} \subseteq E$ or $\{p, \overline{\pi}\} \subseteq E'$. If $\{p, \overline{\pi}\} \in E$, then we are done. Otherwise $\{p, \overline{\pi}\} \subseteq E'$. By Lemma 68 $p \in E$. Because E is an extension of $\langle D, W \rangle$ and $p \in E$ and not $\operatorname{def}(p{:}\pi/\pi, E)$, $\overline{\pi} \notin E$. By the definition of extensions $\pi \in E$. Q.E.D.

Claim C Defaults in D_H are normal. *Proof:* Let δ be a member of D_H. By definition there is $\delta' \in D_B$ such that $\delta' \mathcal{P}_G \delta$, that is, there is a path from the propositional variable in the conclusion of δ' to the prerequisite of δ. Because $\delta' \in D_B$ there is $\delta'' \in D_B$ such that δ' and δ'' have the same propositional variable in their conclusions and $\delta'' \in E(p) \triangle E'(p)$. Now δ'' is normal because it is defeated in an extension. Because there is a path from the propositional variable in the conclusion of δ'' to the prerequisite of δ and G is good, δ is normal. Q.E.D.

By repeated applications of Lemma 87 (justified by Claim **A**) E is an extension of $\langle D_L \cup D_H \cup D'_B, W \rangle$. Because defaults in D'_B are normal by definition and defaults in D_H are normal by Claim **C**, and the propositional variables in conclusions of defaults in $D_H \cup D'_B$ do not occur in the prerequisites of D_L (that would contradict the definitions of $\mathcal{P}_G$, D_H and D_L), by Lemma 67 there is an extension E_1 of $\langle D_L, W \rangle$ such that for all $\delta \in D_L$, $\text{def}(\delta, E_1)$ implies $\text{def}(\delta, E)$. Let $E_2 = \text{Cn}(E_1 \cup I)$ where $I = E' \cap (B \cup \{\neg p | p \in B\})$. Because members of B do not occur in D_L, $\text{def}(\delta, E_2)$ implies $\text{def}(\delta, E_1)$ for all $\delta \in D_L$. That E_2 is an extension of $\langle D_L \cup D'_B, W \rangle$ is by an induction proof that is based on the fact that for every formula in I there is a generating default with the conclusion in I and the precondition in E_1. By repeated applications of Lemma 87 (justified by Claim **A**) E_2 is an extension of $\langle D_L \cup D'_B \cup D''_B, W \rangle$. Because defaults in D_H are normal by Claim **C**, by Lemma 66 there is an extension E'' of $\langle D, W \rangle$ such that $E_2 \subseteq E''$ and $\text{def}(\delta, E'')$ implies $\text{def}(\delta, E_2)$ for all $\delta \in D_L$. Because $\text{def}(\delta, E'')$ implies $\text{def}(\delta, E_2)$ implies $\text{def}(\delta, E_1)$ implies $\text{def}(\delta, E)$ for all $\delta \in D_L$, $\text{def}(\delta, E'')$ implies $\text{def}(\delta, E)$ for all $\delta \in D_L$. □

Lemma 71 *Let G be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let E and E' be extensions of Δ_G. Let $\mathcal{P}_G$ be the strict partial order given in Definition 46. Let a be a node in G such that $E(a)$ is better than $E'(a)$, and $E(b) = E'(b)$ for all b such that $r(b) < r(a)$. Then $\text{def}(\delta, E', E)$ for all $\mathcal{P}_G$-minimal $\delta \in E(a) \triangle E'(a)$.*

PROOF. Assume $\delta \in E(a) \triangle E'(a)$ and $\text{def}(\delta, E, E')$. We show that δ is not $\mathcal{P}_G$-minimal. Because $E(a)$ is better than $E'(a)$, there is $\delta' \in E'(a)$ and a non-empty positive path from the prerequisite of δ' to the prerequisite δ. As the prerequisite of δ is in E, by Lemma 68 it is also in E'. Hence $\text{appl}(\delta, E')$ and $defeated \delta' E'$, and the conclusions of δ and δ' are complementary. Therefore by definition $\delta' \mathcal{P}_G \delta$. Hence $\delta' \notin E(a)$ and $\delta' \in E(a) \triangle E'(a)$. Therefore δ is not $\mathcal{P}_G$-minimal. □

Lemma 72 *Let $\mathcal{P}$ be a strict partial order on a finite set U. Let S be a subset of U such that $\mathcal{P} \cap (S \times S) = \emptyset$. Let P_1 and P_2 be sets such that $P_1 \cup P_2 = S$ and $P_1 \cap P_2 = \emptyset$. Then the transitive closure of $\mathcal{P} \cup (P_1 \times P_2)$ is asymmetric, and therefore a strict partial order.*

Theorem 73 *Let $G = \langle V, E_d, P, N \rangle$ be a good inheritance network and $\Delta_G = \langle D, W \rangle$ the translation of G as given in Definition 42. Let $\mathcal{P}_G$ be the strict partial order on D given in Definition 46. Then for every extension E of Δ_G, E is $\mathcal{P}_G$-preferred$_d$ ($\mathcal{P}_G$-preferred$_{2d}$) if and only if E is best.*

PROOF. Let E be a $\mathcal{P}_G$-preferred$_{2d}$ extension of Δ_G and assume that there is an extension E' of Δ_G that is preferable to E. By Definition 64 there is n such that $E(c) = E'(c)$ for all nodes c such that $r(c) < n$, and $E'(c)$ is better than $E(c)$ or $E'(c) = E(c)$ for all c such that $r(c) = n$, and for some a such that $r(a) = n$, $E'(a)$ is better than $E(a)$. Let $B = \{b \in V | E'(b)$ is better than $E(b), r(b) = n\}$. This set is non-empty because $a \in B$. Let $D_B = \{p{:}\sigma/\gamma \in D | b \in B, \gamma \in \{b, \neg b\}\}$. By Lemma 70 there is an extension E'' of $\langle D, W \rangle$ such that $E'(b) = E''(b)$ for all $b \in B$, and $\operatorname{def}(\delta, E'')$ implies $\operatorname{def}(\delta, E)$ for all $\delta \in D_L$ where $D_L = \{\delta \in D \backslash D_B | \delta' \mathcal{P}_G \delta$ for no $\delta' \in D_B\}$.

Let $\mathcal{T}$ be any strict total order on D such that $\mathcal{P}_G \subseteq \mathcal{T}$. Let δ be the $\mathcal{T}$-least element of D such that $\operatorname{def}(\delta, E, E'')$ or $\operatorname{def}(\delta, E'', E)$. If $\delta \in D_L$, then $\operatorname{def}(\delta, E, E'')$ as $\operatorname{def}(\delta, E'')$ implies $\operatorname{def}(\delta, E)$ for members of D_L. If $\delta \in E(b) \triangle E''(b)$ for some $b \in B$, then by Lemma 71 $\operatorname{def}(\delta, E, E'')$. This exhausts all possible $\mathcal{T}$-least $\delta \in D$ such that $\operatorname{def}(\delta, E, E'')$ or $\operatorname{def}(\delta, E'', E)$, because for all $\delta' \in D \backslash (D_L \cup D_B)$ there is $\delta'' \in D_B$ such that $\delta'' \mathcal{P}_G \delta'$. The conclusion that $\operatorname{def}(\delta, E, E'')$ and $\operatorname{def}(\delta', E)$ iff $\operatorname{def}(\delta', E'')$ for all $\delta' \mathcal{T} \delta$, contradicts the fact that E is $\mathcal{P}_G$-preferred$_{2d}$. Hence the assumption that there is an extension that is preferable to E is false, and all $\mathcal{P}_G$-preferred$_{2d}$ extensions are best extensions. By Lemma 14 all $\mathcal{P}_G$-preferred$_d$ extensions are $\mathcal{P}_G$-preferred$_{2d}$ extensions and therefore best extensions.

The second direction, that is, every best extension is a $\mathcal{P}_G$-preferred$_d$ extension, is as follows.

Assume E is a best extension of Δ_G. A node p is a *d-node* (with respect to E'), if for an extension E', $E(q) = E'(q)$ for all q such that $r(q) < r(p)$ and $E(p) \neq E'(p)$ and $E'(p)$ is not better than $E(p)$. We construct a strict partial order $\mathcal{P}$ on D such that $\mathcal{P}_G \subseteq \mathcal{P}$ and $p{:}\beta/\beta \mathcal{P} p'{:}\beta'/\beta'$, where $\beta \in \{c, \neg c\}$ and $\beta' \in \{c', \neg c'\}$, whenever

(1) $r(c) < r(c')$,
(2) $r(c) = r(c')$ and c is a d-node and c' is not, or
(3) $p{:}\beta/\beta$ and $p'{:}\beta'/\beta'$ are $\mathcal{P}_G$-minimal in $\{p{:}\beta/\beta \in D | p \in E, \beta \in \{c, \neg c\}\}$ and $\operatorname{appl}(p{:}\beta/\beta, E)$ and $\operatorname{def}(p'{:}\beta'/\beta', E)$.

Let $\mathcal{P}_1$ be the transitive closure of $\mathcal{P}_G \cup \{\langle \delta, \delta' \rangle \in D \times D | r(\delta) < r(\delta')\}$. The relation $\mathcal{P}_1$ is asymmetric because $\delta' \mathcal{P}_G \delta$ for no defaults such that $r(\delta) < r(\delta')$. Let $\mathcal{P}_2$ be the transitive closure of $\mathcal{P}_1 \cup \{\langle \delta, \delta' \rangle \in D \times D | \delta = p{:}\beta/\beta, \delta' = p'{:}\beta'/\beta', r(\delta) = r(\delta'), \beta \in \{b, \neg b\}, \beta' \in \{b', \neg b'\}, b$ is a d-node , b' is not a d-node$\}$.

The relation $\mathcal{P}_2$ is asymmetric by Lemma 72 and because $\mathcal{P}_1$ does not relate defaults of the same rank that have different propositional variables in their conclusions. Define $\Lambda_c = \{p{:}\beta/\beta \in D | p \in E, \beta \in \{c, \neg c\}\}$ for every $c \in V$. Let $\mathcal{P}$ be the transitive closure of $\mathcal{P}_2 \cup \{\langle \delta, \delta' \rangle | c \in V, \delta \text{ and } \delta' \text{ are } \mathcal{P}_G\text{-minimal in } \Lambda_c, \text{appl}(\delta, E), \text{def}(\delta', E)\}$. Because $\mathcal{P}_2$ does not relate $\mathcal{P}_G$-minimal members of Λ_c, $\mathcal{P}$ is asymmetric by repeated applications of Lemma 72 with $P_1 = \{\delta \text{ is } \mathcal{P}_G\text{-minimal in } \Lambda_c | \text{appl}(\delta, E)\}$ and $P_2 = \{\delta \text{ is } \mathcal{P}_G\text{-minimal in } \Lambda_c | \text{def}(\delta, E)\}$. Let $\mathcal{T}$ be any strict total order on D such that $\mathcal{P} \subseteq \mathcal{T}$.

Claim A Let p be a member of V, and E, E' and E'' consistent extensions of Δ_G. If $E(p) \neq E'(p)$ and $E(p) \neq E''(p)$ and $E(q) = E'(q) = E''(q)$ for all q such that $r(q) < r(p)$, then $E'(p) = E''(p)$. *Proof:* Assume $a{:}\pi/\pi \in E''(p)$. Hence $a \wedge \overline{\pi} \in E''$. Hence by Lemma 69 $\pi \in E$ and $\overline{\pi} \in E'$. By Lemma 68 $a \in E'$. Hence $a{:}\pi/\pi \in E'(p)$. Therefore $E''(p) \subseteq E'(p)$. By symmetry $E'(p) \subseteq E''(p)$. Hence $E'(p) = E''(p)$. Q.E.D.

Claim B Let p be a d-node with respect to an extension E'. Then for all $\delta \in E(p) \backslash E'(p)$ there is $\delta' \in E'(p) \backslash E(p)$ such that $\delta' \mathcal{P} \delta$. *Proof:* Because $E'(p)$ is not better than $E(p)$, there is $\delta_0 \in E'(p) \backslash E(p)$ such that there is no $\delta_1 \in E(p) \backslash E'(p)$ with a non-empty positive path from the prerequisite of δ_1 to the prerequisite of δ_0. Because $\text{def}(\delta_0, E')$, $a_0 \in E'$ where $\delta_0 = a_0{:}\pi/\pi$. By Lemma 68 $a_0 \in E$. If there is $\delta_2 \in E'(p) \backslash E(p)$ and a non-empty positive path from the prerequisite of δ_2 to a_0 (let the path be maximal), then let $\delta_3 = \delta_2$, otherwise let $\delta_3 = \delta_0$. Clearly δ_3 is $\mathcal{P}_G$-minimal in Λ_p. Hence $\delta_3 \mathcal{P} \delta$ by the definition of $\mathcal{P}$, and $\delta_3 \in E'(p) \backslash E(p)$. Q.E.D.

Claim C Let E' be any extension of Δ_G such that $E' \neq E$. There is $\delta = a{:}\pi/\pi \in D$ such that $\text{def}(\delta, E', E)$ and π is a d-node. *Proof:* Let n be the least number such that $E(p) = E'(p)$ for all p such that $r(p) < n$. Because E' is not preferable to E, $E(p) \neq E'(p)$ and $E'(p)$ is not better than $E(p)$ for some p such that $r(p) = n$. Now p is a d-node for E with respect to E'. By the definition of *better than* there is $\delta \in E'(p) \backslash E(p)$, that is, $\text{def}(\delta, E', E)$. Q.E.D.

Let E' be any extension of Δ_G such that $E' \neq E$. By Claim **C** there is $\delta = a{:}\pi/\pi$ with the given properties. Assume $\text{def}(\delta', E, E')$ for some $\delta' = a'{:}\pi'/\pi' \in D$ such that $\pi' \in \{p, \neg p\}$. If $r(\pi') > r(\pi)$, then by definition $\delta \mathcal{P} \delta'$. Otherwise $r(\pi') = r(\pi)$. If p is not a d-node, then by definition $\delta \mathcal{P} \delta'$. Hence assume that p is a d-node. By Claim **A** p is a d-node with respect to E'. By Claim **B** there is $\delta'' \mathcal{P} \delta'$ such that $\delta'' \in E'(p) \backslash E(p)$. Therefore for all δ' such that $\text{def}(\delta', E, E')$ there is $\delta'' \mathcal{P} \delta'$ such that $\text{def}(\delta'', E', E)$. Because $\mathcal{P} \subseteq \mathcal{T}$, also $\delta'' \mathcal{T} \delta'$. As this holds for any E', $\mathcal{T}$ is a $\Delta_G, \mathcal{P}_G\text{-ordering}_d$ for E. Because E is a $\mathcal{P}_G\text{-preferred}_d$ extension of Δ_G, by Lemma 14 E is a $\mathcal{P}_G\text{-preferred}_{2d}$ extension of Δ_G. $\square$

The theorem shows that Gelfond and Przymusinska's inheritance theory co-incides with the inheritance theory presented in Section 5 both with $\mathcal{P}_G$-preferred$_d$ and $\mathcal{P}_G$-preferred$_{2d}$ extensions.

6.2   An inheritance theory by Brewka

Brewka [4] presents a translational inheritance theory that is based on predicate logic and circumscription [17]. The theory is about the class of inheritance networks with the following properties. Defeasible links may not be followed by any links and a defeasible link and a strict link may not end in the same node. All strict links are positive. These restrictions correspond to a division of nodes to classes and properties. All links between classes are strict and positive, and all links from classes to properties are defeasible. No link starts from a property node.

Definition 74 (Class-property network) *Let $G = \langle V, E, P, N \rangle$ be a good inheritance network. Then G is a* class-property network, *if there is a set $C \subseteq V$ such that $E \subseteq (C \times C \times \{2\}) \cup (C \times (V \backslash C) \times \{0, 1\})$. The members of C are the* classes *and the members of $V \backslash C$ are the* properties *in G.*

Brewka allows non-Boolean properties, that is, instead of only two kinds of links – positive and negative – there may be arbitrarily many different kinds of links from classes to properties. In this section we show that Brewka's inheritance theory coincides with the one given in Section 5.3. We consider only the Boolean case, and assume that inheritance networks have only one individual U. We give a translation from our definition of inheritance networks to Brewka's translation. Brewka's representation of inheritance networks uses a Lisp-like notation that is not used here.

Let $G = \langle V, E, P, N \rangle$ be a class-property network. Let C be the classes in G. Links in E are translated as follows [4].

$$\begin{aligned}
\Phi_c = &\{\text{HasSlot}(x, y, T) | \langle x, y, 0 \rangle \in E\} \\
&\cup \{\text{HasSlot}(x, y, F) | \langle x, y, 1 \rangle \in E\} \\
&\cup \{\text{Specializes}(x, y) | \langle x, y, 2 \rangle \in E\}
\end{aligned}$$

An individual may be a member of a class, and an individual may have or not have a property. We have only one individual U.

$$\begin{aligned}
\Phi_i = &\{\text{Holds}(p, U, T) | p \in P \backslash C\} \\
&\cup \{\text{Holds}(p, U, F) | p \in N \backslash C\} \\
&\cup \{\text{Is}(U, c) | c \in P \cap C\}
\end{aligned}$$

No two symbols refer to the same class or property.

$$\Phi_u = \{\neg(s = s')|\{s, s'\} \subseteq V \cup \{U, T, F\}, s \neq s'\}$$

In addition to the above formulae that depend on the inheritance networks, the set Φ_b of formulae below is needed. These formulae state the following. The class-inclusion relation Specializes propagates class-membership. Class-inclusion is transitive. If an individual that is a member of a class is not exceptional with respect to a property associated with the class, then the individual has the default value of the property. An individual is exceptional with respect to a property associated with a class if there is a subclass of the class with a different value for the same property and the individual belongs to that subclass. An individual may not have two different values for a property.

$$\forall x \forall y (\mathrm{Specializes}(x, y) \rightarrow (\mathrm{Is}(U, x) \rightarrow \mathrm{Is}(U, y)))$$

$$\forall x \forall y \forall z (\mathrm{Specializes}(x, y) \wedge \mathrm{Specializes}(y, z) \rightarrow \mathrm{Specializes}(x, z))$$

$$\forall x \forall y \forall v (\mathrm{HasSlot}(x, y, v) \wedge \mathrm{Is}(U, x) \wedge \neg \mathrm{Exceptional}(U, x, y) \rightarrow \mathrm{Holds}(y, U, v))$$

$$\forall x_1 \forall x_2 \forall v_1 \forall v_2 \forall z (\mathrm{Is}(U, x_1) \wedge \mathrm{HasSlot}(x_1, z, v_1) \wedge \mathrm{Specializes}(x_1, x_2) \wedge$$

$$\mathrm{HasSlot}(x_2, z, v_2) \wedge v_1 \neq v_2 \rightarrow \mathrm{Exceptional}(U, x_2, z))$$

$$\forall x \forall v_1 \forall v_2 (\mathrm{Holds}(x, U, v_1) \wedge \mathrm{Holds}(x, U, v_2) \rightarrow v_1 = v_2)$$

Brewka says that Exceptional is circumscribed in $\Phi = \Phi_c \cup \Phi_i \cup \Phi_u \cup \Phi_b$ and all other predicates in Φ are varied. The notation $M(S)$ denotes the interpretation of the predicate symbol S in the model M. A definition of circumscription with varied predicates is given in [15]. Let $Z_1, \ldots, Z_n$ be the predicates to be varied while circumscribing P in Φ. The result Φ_{circ} of circumscribing P in Φ with other predicates varied is the second-order formula $\Phi \wedge \neg \exists p z_1 \cdots z_n (\Phi(p, z_1, \ldots, z_n) \wedge p < P)$ where $\Phi(p, z_1, \ldots, z_n)$ is like Φ but occurrences of P have been replaced by p and occurrences of Z_i have been replaced by z_i for all $i \in \{1, \ldots, n\}$, and $p < P$ is defined as $(p \leq P) \wedge \neg(p = P)$ where $p = P$ stands for $\forall x(p(x) \equiv P(x))$ and $p \leq P$ stands for $\forall x(p(x) \rightarrow P(x))$.

Lemma 75 *Let $G = \langle V, E, P, N \rangle$ be a class-property network. Let Φ be the translation of G as defined by Brewka. Let M be a model such that $M \models \Phi$. Then there is a model M' such that $M' \models \Phi$ and $M'(\mathrm{Exceptional}) = M(\mathrm{Exceptional})$ and $M'(\mathrm{Holds}) = M(\mathrm{Holds})$, and for all x and y and z, $M' \models \mathrm{Specializes}(x, y, z)$ if and only if $\Phi \models \mathrm{Specializes}(x, y, z)$, and for all x and y, $M' \models \mathrm{Is}(x, y)$ if and only if $\Phi \models \mathrm{Is}(x, y)$, and for all x and y and z, $M' \models \mathrm{HasSlot}(x, y, z)$ if and only if $\Phi \models \mathrm{HasSlot}(x, y, z)$,*

Lemma 76 *Let $G = \langle V, E, P, N \rangle$ be a class-property network with classes*

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$C \subseteq V$. Let Φ be the translation of G as defined by Brewka. Let $\Delta_G = \langle D, W \rangle$ be the translation of G as given in Definition 42. Then for all $p \in C$, $W \cup I_s(D) \models p$ if and only if $\Phi \models \mathrm{Is}(U, p)$ (I_s is defined in Section 5.3.)

The following two lemmata both prove a half of the equivalence between membership respectively of p and $\neg p$ in an extension and the truth of $\mathrm{Holds}(p, U, T)$ and $\mathrm{Holds}(p, U, F)$ in a model of Φ_{circ}.

Lemma 77 *Let $G = \langle V, E_d, P, N \rangle$ be a class-property network with classes $C \subseteq V$. Let Φ be the translation of G as defined by Brewka. Let Φ_{circ} be the result of circumscribing* Exceptional *in Φ with other predicates varied. Let $\Delta_G = \langle D, W \rangle$ be the translation of G as given in Definition 42 and let $\mathcal{P}_G$ be the priorities for Δ_G as given in Definition 46. Let $\Delta = \langle I(D), W \cup I_s(D) \rangle$ and $\mathcal{P} = \{ \langle :p \to \beta/p \to \beta, :p' \to \beta'/p' \to \beta' \rangle \mid p{:}\beta/\beta \mathcal{P}_G p'{:}\beta'/\beta' \}$ (I and I_s are defined in Section 5.3.) Let M be a model of Φ_{circ}. Then there is a $\mathcal{P}$-preferred$_d$ extension E of Δ such that for all $p \in V \backslash C$, if $p \in E$ then $M \models \mathrm{Holds}(p, U, T)$, and if $\neg p \in E$ then $M \models \mathrm{Holds}(p, U, F)$.*

PROOF. Because G is good and therefore acyclic, by Lemma 47 $\mathcal{P}_G$ is a strict partial order. Assume there is model M such that $M \models \Phi_{\mathrm{circ}}$. Because of the unique names axioms Φ_n, we can assume that constant symbols are interpreted as themselves in M. Let M' be the model given in Lemma 75. Define $\Lambda_b = \{ :q \to \gamma/q \to \gamma \in I(D) \mid \gamma \in \{b, \neg b\}, \Phi \models \mathrm{Is}(U, q) \}$ for all $b \in V$. Define $\mathcal{P}_1$ as the transitive closure of the relation $\mathcal{P}_0$ defined as $:p \to \beta/p \to \beta \mathcal{P}_0 :p' \to \beta'/p' \to \beta'$ whenever $:p \to \beta/p \to \beta \mathcal{P} :p' \to \beta'/p' \to \beta'$ or $\beta = \overline{\beta'}$ and $:p \to \beta/p \to \beta$ and $:p' \to \beta'/p' \to \beta'$ are $\mathcal{P}$-minimal elements in Λ_b and $M' \not\models \mathrm{Exceptional}(U, p, b)$ and $M' \models \mathrm{Exceptional}(U, p', b')$. By Lemma 72 $\mathcal{P}_1$ is asymmetric and therefore a strict partial order.

Let $\mathcal{T}$ be any strict total order on $I(D)$ such that $\mathcal{P}_1 \subseteq \mathcal{T}$. Let $:\phi_1/\phi_1, \ldots, :\phi_n/\phi_n$ be the ordering $\mathcal{T}$ on $I(D)$. Construct $S_0, \ldots, S_n$ and the preferred subtheory $S = S_n$ of $\langle \{\phi | :\phi/\phi \in I(D)\}, W \cup I_s(D), \{\langle \phi, \psi \rangle | :\phi/\phi \mathcal{P} :\psi/\psi\} \rangle$ by using Definition 27 with the strict total order $\phi_1, \ldots, \phi_n$.

Induction hypothesis:

(1) For all $p \in C$, $p \in \mathrm{Cn}(S_i)$ if and only if $\Phi \models \mathrm{Is}(U, p)$.
(2) For all $j \in \{1, \ldots, i\}$, if $:p_j \to \beta_j/p_j \to \beta_j$ is $\mathcal{P}$-minimal in Λ_{b_j} for $b_j \in V$, then $p_j \to \beta_j \in S_i$ if and only if $M' \not\models \mathrm{Exceptional}(U, p_j, b_j)$.
(3) For all $p \in V \backslash C$, if $S_i \models p$ then $M' \models \mathrm{Holds}(p, U, T)$, and if $S_i \models \neg p$ then $M' \models \mathrm{Holds}(p, U, F)$.

Base case $i = 0$:

(1) By Lemma 76 for all $p \in C$, $p \in \mathrm{Cn}(S_0)$ if and only if $\Phi \models \mathrm{Is}(U, p)$.

(2) Immediate as there are no $j \in \{1, \ldots, 0\}$.

(3) Because G is good, the union of W and the consequents of the implications in $I_s(D)$ is consistent by Lemma 44. Therefore $S_0 = W \cup I_s(D)$ is consistent. Assume $S_0 \models p$ and $p \in V \backslash C$. If $p \in W$, then by definition $p \in P$. By definition $\text{Holds}(p, U, T) \in \Phi$. Hence $M' \models \text{Holds}(p, U, T)$. Because there are no occurrences of $p \in V \backslash C$ in $I_s(D)$ and S_0 is consistent, there is no p such that $p \notin W$ and $S_0 \models p$. The case $S_0 \models \neg p$ is similar.

Inductive case $i \geq 1$:

(1) Assume $p \in \text{Cn}(S_i) \cap C$. If $S_{i-1} \cup \{p_i \to \beta_i\}$ is inconsistent and hence $S_i = S_{i-1}$, then the result is immediate by the induction hypothesis. If $S_i = S_{i-1} \cup \{p_i \to \beta_i\}$, then by Lemma 90 β_i is the only literal that may be in $\text{Cn}(S_i) \backslash \text{Cn}(S_{i-1})$. Because the propositional variable b_i in β_i is in $V \backslash C$, $p \neq b_i$. Hence $p \in \text{Cn}(S_{i-1})$. Hence by induction hypothesis $\Phi \models \text{Is}(U, p)$.

Assume $\Phi \models \text{Is}(U, p)$. By induction hypothesis $p \in \text{Cn}(S_{i-1})$. Because $S_{i-1} \subseteq S_i$, $p \in \text{Cn}(S_i)$ by the monotonicity of Cn.

(2) We prove the contrapositions of both halves of the equivalence. For $j \in \{1, \ldots, i-1\}$ the result is immediate by the induction hypothesis. It remains to consider the case $i = j$. So assume $:p_i \to \beta_i / p_i \to \beta_i$ is $\mathcal{P}$-minimal in Λ_{b_i}. Assume $p_i \to \beta_i \notin S_i$. Hence $S_i = S_{i-1}$ and $S_{i-1} \models p_i \wedge \neg \beta_i$. Hence by item 1 of the induction hypothesis $\Phi \models \text{Is}(U, p_i)$ and hence $M' \models \text{Is}(U, p_i)$, and by item 3 $M' \models \text{Holds}(b_i, U, T)$ if $\beta_i = \neg b_i$ and $M' \models \text{Holds}(b_i, U, F)$ if $\beta_i = b_i$. Because $:p_i \to \beta_i / p_i \to \beta_i \in I(D)$, $\langle p_i, b_i, 0 \rangle \in E_d$ if $\beta_i = b_i$ and $\langle p_i, b_i, 1 \rangle \in E_d$ if $\beta_i = \neg b_i$. Therefore $\text{HasSlot}(p_i, b_i, T) \in \Phi$ if $\beta_i = b_i$ and $\text{HasSlot}(p_i, b_i, F) \in \Phi$ if $\beta_i = \neg b_i$. Because $\forall x \forall y \forall v (\text{HasSlot}(x, y, v) \wedge \text{Is}(U, x) \wedge \neg \text{Exceptional}(U, x, y) \to \text{Holds}(y, U, v)) \in \Phi$ and $M' \models \Phi$, finally $M' \models \text{Exceptional}(U, p_i, b_i)$.

Assume that $M' \models \text{Exceptional}(U, p_i, b_i)$. In the following we assume that $\beta_i = b_i$; the proof for the case $\beta_i = \neg b_i$ is almost identical.

Claim A $\Phi \models \text{Is}(U, p_i) \wedge \text{HasSlot}(p_i, b_i, T)$. *Proof:* By assuming the opposite and deriving the non-minimality of M'. Q.E.D.

We analyze by cases.

(a) Consider the case where for some $q \in V$, $:q \to \overline{\beta_i} / q \to \overline{\beta_i}$ is $\mathcal{P}$-minimal in Λ_{b_i} and $M' \not\models \text{Exceptional}(U, q, b_i)$. Because $:p_i \to \beta_i / p_i \to \beta_i$ is $\mathcal{P}$-minimal in Λ_{b_i}, by definition $:q \to \overline{\beta_i} / q \to \overline{\beta_i} \mathcal{P}_1 : p_i \to \beta_i / p_i \to \beta_i$. Hence $q \to \overline{\beta_i} = p_j \to \beta_j$ for some $j < i$. Because $M' \not\models \text{Exceptional}(U, q, b_i)$, by item 2 of the induction hypothesis $q \to \overline{\beta_i} \in S_{i-1}$. Because $S_{i-1} \models q$ by the definition of Λ_{b_i} and item 1 of the induction hypothesis, $S_{i-1} \models \overline{\beta_i}$. By Claim **A** and item 1 of the induction hypothesis, $p_i \in \text{Cn}(S_{i-1})$. Therefore $S_{i-1} \cup \{p_i \to \beta_i\}$ is inconsistent and $p_i \to \beta_i \notin S_i$.

(b) Consider the case where $\Phi \models \text{Holds}(b_i, U, F)$. Now $b_i \in N$ and $\neg b_i \in W \subseteq S_{i-1}$. By Claim **A** and item 1 of the induction hypothesis $p_i \in \text{Cn}(S_{i-1})$. Therefore $S_{i-1} \cup \{p_i \to \beta_i\}$ is inconsistent and $p_i \to \beta_i \notin S_i$.

(c) We show that the assumptions of (a) and (b) cannot both be false.

So we assume they are and derive a contradiction with the fact $M' \models \Phi_{\text{circ}}$. We construct a model M'' such that $M'' \models \Phi$ and $M''(\text{Exceptional}) \subset M'(\text{Exceptional})$. Let the universe of M'' be $V \cup \{U, T, F\}$ and constant symbols are interpreted as themselves. Let

$$M''(\text{Exceptional}) = M'(\text{Exceptional}) \setminus \{\langle U, p_i, b_i \rangle\},$$
$$M''(\text{Holds}) = M'(\text{Holds}) \setminus \{\langle b_i, U, F \rangle\} \cup \{\langle b_i, U, T \rangle\}, \text{ and}$$
$$M''(P) = M'(P) \text{ for all other predicate symbols } P.$$

The proof that $M'' \models \Phi$ is by analyzing all formulae in Φ.

(3) Assume $p \in V \setminus C$ and $S_i \models p$ (the proof for $\neg p$ is similar.) Assume $S_{i-1} \cup \{p_i \to \beta_i\}$ is inconsistent. Hence $S_i = S_{i-1}$ and the result is immediate by induction hypothesis. So assume $S_i = S_{i-1} \cup \{p_i \to \beta_i\}$. By Lemma 90 β_i is the only propositional variable possibly in $\text{Cn}(S_i) \setminus \text{Cn}(S_{i-1})$. Hence the result for all p such that $p \neq \beta_i$ is immediate by the induction hypothesis, and it suffices to consider the case $p = \beta_i$. If $\beta_i \in \text{Cn}(S_{i-1})$, then the result is immediate by the induction hypothesis. Otherwise $\beta_i \in \text{Cn}(S_i) \setminus \text{Cn}(S_{i-1})$, and $p_i \in \text{Cn}(S_{i-1})$ by Lemmata 89 and 90. By item 1 of the induction hypothesis $M' \models \text{Is}(U, p_i)$. Because $:p_i \to \beta_i / p_i \to \beta_i \in I(D)$, $\langle p_i, b_i, 0 \rangle \in E_d$. Hence $\text{HasSlot}(p_i, b_i, T) \in \Phi$. 1. If $:p_i \to \beta_i / p_i \to \beta_i$ is $\mathcal{P}$-minimal in Λ_{b_i}, then by item 2 above $M' \not\models \text{Exceptional}(U, p_i, b_i)$. Because $M' \models \Phi$ and $M' \models \text{Is}(U, p_i)$ and $M' \models \forall x \forall y \forall v (\text{HasSlot}(x, y, v) \wedge \text{Is}(U, x) \wedge \neg\text{Exceptional}(U, x, y) \to \text{Holds}(y, U, v))$, $M' \models \text{Holds}(b_i, U, T)$. 2. If $:p_i \to \beta_i / p_i \to \beta_i$ is not $\mathcal{P}$-minimal in Λ_{b_i} then there is $:p_j \to \beta_j / p_j \to \beta_j \mathcal{P} :p_i \to \beta_i / p_i \to \beta_i$ in Λ_{b_i}. By the definition of Λ_{b_i} and item 1 of the induction hypothesis $p_j \in \text{Cn}(S_{i-1})$. By definition of $\mathcal{P}_G$, $\beta_j = \overline{\beta_i}$. Now $p_j \to \beta_j \notin S_{i-1}$ because otherwise $S_{i-1} \cup \{p_i \wedge \beta_i\}$ would be inconsistent. Because $p_j \to \beta_j \notin S_{i-1}$, $S_{i-1} \models \beta_i$. By item 3 of the induction hypothesis $M' \models \text{Holds}(b_i, U, T)$.

By definition $S = S_n$ is a preferred subtheory of $\langle \{\phi|:\phi/\phi \in I(D)\}, W \cup I_s(D), \{\langle \phi, \psi \rangle|:\phi/\phi \mathcal{P}:\psi/\psi\} \rangle$. By Theorem 33 and the fact that for prerequisite-free defaults $\text{appl}(\delta, E)$ is equivalent to not $\text{def}(\delta, E)$, $E = \text{Cn}(S)$ is a $\mathcal{P}$-preferred$_d$ extension of $\Delta = \langle I(D), W \cup I_s(D) \rangle$. By item 3 of the induction hypothesis and because $M(\text{Holds}) = M'(\text{Holds})$, for all $p \in V \setminus C$, if $p \in E$ then $M \models \text{Holds}(p, U, T)$, and if $\neg p \in E$ then $M \models \text{Holds}(p, U, F)$. $\square$

Lemma 78 *Let $G = \langle V, E_d, P, N \rangle$ be a class-property network with classes $C \subseteq V$. Let Φ be the translation of G as defined by Brewka. Let Φ_{circ} be the result of circumscribing Exceptional in Φ with other predicates varied. Let $\Delta_G = \langle D, W \rangle$ be the translation of G as given in Definition 42 and let $\mathcal{P}_G$ be the priorities for Δ_G as given in Definition 46. Let $\Delta = \langle I(D), W \cup I_s(D) \rangle$ and $\mathcal{P} = \{\langle :p \to \beta/p \to \beta, :p' \to \beta'/p' \to \beta' \rangle | p:\beta/\beta \mathcal{P}_G p':\beta'/\beta'\}$. Let E be a $\mathcal{P}$-preferred$_d$ extension of Δ. Then there is a model M such that $M \models \Phi_{\text{circ}}$*

and for all $p \in V \backslash C$, if $p \notin E$ then $M \not\models \mathrm{Holds}(p, U, T)$.

PROOF. Because G is acyclic, by Lemma 47 $\mathcal{P}_G$ is a strict partial order. We construct a model M such that $M \models \Phi_{\mathrm{circ}}$ and directly by construction for all $p \in V \backslash C$, if $p \notin E$ then $M \not\models \mathrm{Holds}(p, U, T)$. The universe of M is $V \cup \{U, T, F\}$, the constant symbols $V \cup \{U, T, F\}$ are interpreted as themselves, and the predicate symbols are interpreted as follows.

$$M(\mathrm{Specializes}) = \{\langle x, y \rangle \in C \times C | \Phi \models \mathrm{Specializes}(x, y)\}$$
$$M(\mathrm{HasSlot}) = \{\langle x, y, v \rangle \in C \times (V \backslash C) \times \{T, F\} | \Phi \models \mathrm{HasSlot}(x, y, v)\}$$
$$M(\mathrm{Is}) = \{\langle U, x \rangle | x \in C, \Phi \models \mathrm{Is}(U, x)\}$$
$$M(\mathrm{Exceptional}) = \left\{ \langle U, p, b \rangle \left| \frac{: p \to \beta}{p \to \beta} \in I(D), \beta \in \{b, \neg b\}, \mathrm{def}(\frac{: p \to \beta}{p \to \beta}, E) \right. \right\}$$
$$\cup \{\langle U, p, b \rangle | p \in C, b \in V \backslash C, \Phi \models \mathrm{Exceptional}(U, p, b)\}$$
$$M(\mathrm{Holds}) = \{\langle x, U, T \rangle | x \in E, x \in V \backslash C\} \cup \{\langle x, U, F \rangle | \neg x \in E, x \in V \backslash C\}$$

The triples $\{\langle U, p, b \rangle | p \in C, b \in V \backslash C, \Phi \models \mathrm{Exceptional}(U, p, b)\}$ in $M(\mathrm{Exceptional})$ are needed because Φ_b makes exceptional all links that are less specific than contradicting links the antecedents of which are entailed by Φ, even when the contradicting links are exceptional.

We first show that $M \models \Phi_c \cup \Phi_i \cup \Phi_b \cup \Phi_u$ and then that $M \models \Phi_{\mathrm{circ}}$. Formulae in $\Phi_c \cup \Phi_i \cup \Phi_u$ are immediate. Formulae $\forall x \forall y (\mathrm{Specializes}(x, y) \to (\mathrm{Is}(U, x) \to \mathrm{Is}(U, y))), \forall x \forall y \forall z (\mathrm{Specializes}(x, y) \wedge \mathrm{Specializes}(y, z) \to \mathrm{Specializes}(x, z))$, and $\forall x_1 \forall x_2 \forall v_1 \forall v_2 \forall z (\mathrm{Is}(U, x_1) \wedge \mathrm{HasSlot}(x_1, z, v_1) \wedge \mathrm{Specializes}(x_1, x_2) \wedge \mathrm{HasSlot}(x_2, z, v_2) \wedge v_1 \neq v_2 \to \mathrm{Exceptional}(U, x_2, z))\}$ are true in M directly because their antecedents are true if and only if they are logical consequences of Φ, and then so are the consequents.

Consider $\forall x \forall y \forall v (\mathrm{HasSlot}(x, y, v) \wedge \mathrm{Is}(U, x) \wedge \neg \mathrm{Exceptional}(U, x, y) \to \mathrm{Holds}(y, U, v)) \in \Phi_b$. Assume $M \models \mathrm{HasSlot}(x, y, v) \wedge \mathrm{Is}(U, x) \wedge \neg \mathrm{Exceptional}(U, x, y)$ for some $x \in C$, $y \in V \backslash C$ and $v \in \{T, F\}$ (we give the proof for $v = F$, proof for $v = T$ is similar.) Hence $\mathrm{HasSlot}(x, y, F) \in \Phi_c$ and $\langle x, y, 1 \rangle \in E_d$. By definition $\Phi \models \mathrm{Is}(U, x)$ and hence by Lemma 76 $x \in E$. By definition of $M(\mathrm{Exceptional})$ not $\mathrm{def}(:x \to \neg y / x \to \neg y, E)$. Now $\neg y \in E$, and by definition $M \models \mathrm{Holds}(y, U, F)$.

Consider $\forall x \forall v_1 \forall v_2 (\mathrm{Holds}(x, U, v_1) \wedge \mathrm{Holds}(x, U, v_2) \to v_1 = v_2) \in \Phi_b$. Assume $M \models \mathrm{Holds}(x, U, v_1) \wedge \mathrm{Holds}(x, U, v_2)$ for some x, v_1 and v_2. By definition $\{v_1, v_2\} \subseteq \{T, F\}$. Assume $v_1 \neq v_2$. Then $x \in E$ and $\neg x \in E$ by the definition of M, which contradicts the consistency of E (Lemmata 44 and 80.) Hence $v_1 = v_2$ and $M \models v_1 = v_2$.

It remains to show that the second conjunct of Φ_{circ} is true in M. This formula says that there are no models M' such that $M' \models \Phi$ and $M'(\text{Exceptional}) \subset M(\text{Exceptional})$. Assume there is such a model M'. Define $Q = W \cup I_s(D) \cup \{p \to \beta |: p \to \beta / p \to \beta \in I(D), \beta \in \{b, \neg b\}, \langle U, p, b \rangle \notin M'(\text{Exceptional})\}$.

Claim A Q is consistent. *Proof:* Let M'' be a propositional model such that $M'' \models p$ if and only if $p \in C$ and $M' \models \text{Is}(U, p)$ or $p \in V \backslash C$ and $M' \models \text{Holds}(p, U, T)$. We show that $M'' \models Q$. Assume $\neg p \in W$. Hence $p \in N$, $\text{Holds}(p, U, F) \in \Phi$, $M' \models \text{Holds}(p, U, F)$, $M' \not\models \text{Holds}(p, U, T)$, and $M'' \not\models p$. Assume $p \in W$. Hence $p \in P$, $\text{Holds}(p, U, T) \in \Phi$ if $p \in V \backslash C$ and $\text{Is}(U, p) \in \Phi$ if $p \in C$, $M' \models \text{Holds}(p, U, T)$ if $p \in V \backslash C$ and $M' \models \text{Is}(U, p)$ if $p \in C$, and finally $M'' \models p$. Assume $q \to q' \in I_s(D)$. Hence $\langle q, q', 2 \rangle \in E_d$ and $\text{Specializes}(q, q') \in \Phi$. Assume $M'' \models q$. Hence $M' \models \text{Is}(U, q)$. Because $M' \models \text{Specializes}(q, q')$ and $M' \models \forall x \forall y (\text{Specializes}(x, y) \to (\text{Is}(U, x) \to \text{Is}(U, y)))$, $M' \models \text{Is}(U, q')$, and hence $M'' \models q'$ and $M'' \models q \to q'$. Assume $p \to \beta \in Q \backslash I_s(D)$. Hence $M' \not\models \text{Exceptional}(U, p, b)$ for $\beta \in \{b, \neg b\}$ and $M' \models \text{HasSlot}(p, b, T)$ (we assume that $\beta = b$; the case $\beta = \neg b$ is similar.) Assume $M'' \models p$. Hence $M' \models \text{Is}(U, p)$. Because $M' \models \forall x \forall y \forall v (\text{HasSlot}(x, y, v) \wedge \text{Is}(U, x) \wedge \neg \text{Exceptional}(U, x, y) \to \text{Holds}(y, U, v))$, $M' \models \text{Holds}(b, U, T)$ and $M'' \models \beta$. Q.E.D.

Define $\Lambda_b = \{: q \to \beta / q \to \beta \in I(D) | \beta \in \{b, \neg b\}, q \in E\}$ for all $b \in V \backslash C$.

Claim B Let b be a member of $V \backslash C$. For all $\mathcal{P}$-minimal $: q \to \beta / q \to \beta \in \Lambda_b$, $\langle U, q, b \rangle \in M(\text{Exceptional})$ if and only if $q \to \beta \notin E$.

Claim C If $b \in E$ for $b \in V \backslash (C \cup P)$, then there is a $\mathcal{P}$-minimal default $: q \to b / q \to b \in \Lambda_b$ such that $q \to b \in E$.

Claim D $E = \text{Cn}(W \cup I_s(D) \cup F)$ where $F = \{p \to \beta |: p \to \beta / p \to \beta \in I(D), \beta \in \{b, \neg b\}, \langle U, p, b \rangle \notin M(\text{Exceptional})\}$. *Proof:* First we show $\supseteq$. By definition $W \cup I_s(D) \subseteq E$. Assume that $: p \to \beta / p \to \beta \in I(D)$ and $\beta \in \{b, \neg b\}$ and $\langle U, p, b \rangle \notin M(\text{Exceptional})$. Because of the last fact, not $\text{def}(: p \to \beta / p \to \beta, E)$. Hence by definition of extensions $p \to \beta \in E$. Then we show $\subseteq$. By Lemma 81 $E = \text{Cn}(S)$ where $S = W \cup I_s(D) \cup \{p \to \beta |: p \to \beta / p \to \beta \in I(D), \text{not def}(: p \to \beta / p \to \beta, E)\}$. We show that $W \cup I_s(D) \cup F \models \phi$ for all $\phi \in S$. For formulae in $W \cup I_s(D)$ the result is immediate. Assume $: p \to \beta / p \to \beta \in I(D)$ and not $\text{def}(: p \to \beta / p \to \beta, E)$. If $\Phi \not\models \text{Exceptional}(U, p, b)$, then $\langle U, p, b \rangle \notin M(\text{Exceptional})$, and by definition $p \to \beta \in F$. Otherwise $\Phi \models \text{Exceptional}(U, p, b)$. The following formulae in Φ_b have occurrences of Exceptional: $\forall x \forall y \forall v (\text{HasSlot}(x, y, v) \wedge \text{Is}(U, x) \wedge \neg \text{Exceptional}(U, x, y) \to \text{Holds}(y, U, v))$ and $\forall x_1 \forall x_2 \forall v_1 \forall v_2 \forall z (\text{Is}(U, x_1) \wedge \text{HasSlot}(x_1, z, v_1) \wedge \text{Specializes}(x_1, x_2) \wedge \text{HasSlot}(x_2, z, v_2) \wedge v_1 \neq v_2 \to \text{Exceptional}(U, x_2, z))$. Assume that $\beta = b$ (the case $\beta = \neg b$ is similar.) Therefore $\Phi \models \text{HasSlot}(p, b, T) \wedge ((\text{Is}(U, p) \wedge \neg \text{Holds}(b, U, T)) \vee (\text{Is}(U, x) \wedge \text{HasSlot}(x, b, F) \wedge \text{Specializes}(x, p)))$ for some x.

Because $M \not\models \neg\mathrm{Holds}(b, U, T)$, $M \models \mathrm{Is}(U, x) \wedge \mathrm{HasSlot}(x, b, F) \wedge \mathrm{Specializes}(x, p)$ for some x. Therefore 1. $\Phi \models \mathrm{Is}(U, x)$ and by Lemma 76 $W \cup I_s(D) \models x$, 2. $\Phi \models \mathrm{HasSlot}(x, b, F)$, and because HasSlot occurs only positively in Φ_c, by definition $\langle x, y, 1 \rangle \in E_d$ and $:x \to \neg b / x \to \neg b \in I(D)$, and 3. $\Phi \models \mathrm{Specializes}(x, p)$ and hence $:x \to \neg b / x \to \neg b \mathcal{P} : p \to b / p \to b$. Because $\{x, p\} \subseteq \mathrm{Cn}(W \cup I_s(D)) \subseteq E$, $:p \to b / p \to b$ is in Λ_b but not $\mathcal{P}$-minimal. By Claim **C** there is $\mathcal{P}$-minimal $:q \to b / q \to b \in \Lambda_b$ such that $q \to b \in E$. By Claim **B** $\langle U, q, b \rangle \notin M(\mathrm{Exceptional})$. Hence $q \to b \in F$ by definition. Because $q \in E$ and hence in $\mathrm{Cn}(W \cup I_s(D))$, finally $\mathrm{Cn}(W \cup I_s(D) \cup F) \models p \to b$. Q.E.D.

By Claim **D** $E = \mathrm{Cn}(W \cup I_s(D) \cup F)$. Clearly $W \cup I_s(D) \cup F \subseteq Q$. By Claim **A** Q is consistent. This contradicts the facts that $p \to \beta \in \mathrm{Cn}(Q) \backslash E$ and E is by Theorem 34 a maximal consistent subset of $W \cup I_s(D) \cup \{\phi | : \phi / \phi \in I(D)\}$ that contains $p \to \beta$. Therefore the assumption that there is a model M' of Φ such that $M'(\mathrm{Exceptional}) \subset M(\mathrm{Exceptional})$ is false, and $M \models \Phi_{\mathrm{circ}}$. $\square$

Theorem 79 *Let $G = \langle V, E_d, P, N \rangle$ be a class-property network with classes $C \subseteq V$. Let Φ be the translation of G as defined by Brewka. Let Φ_{circ} be the result of circumscribing* Exceptional *in Φ with other predicates varied. Let $\Delta_G = \langle D, W \rangle$ be the translation of G as given in Definition 42 and let $\mathcal{P}_G$ be the priorities for Δ_G as given in Definition 46. Let $\Delta = \langle I(D), W \cup I_s(D) \rangle$ and $\mathcal{P} = \{\langle :p \to \beta / p \to \beta, :p' \to \beta' / p' \to \beta' \rangle | p : \beta / \beta \mathcal{P}_G p' : \beta' / \beta' \}$. Then for all $p \in V \backslash C$, $\Phi_{\mathrm{circ}} \models \mathrm{Holds}(p, U, T)$ if and only if $\Delta \models_{\mathcal{P}}^d p$.*

PROOF. Because G is good and hence acyclic, Lemmata 77 and 78 can be applied, and by Lemma 47 $\mathcal{P}_G$ is a strict partial order. Assume $\Phi_{\mathrm{circ}} \not\models \mathrm{Holds}(p, U, T)$, that is, there is a model M such that $M \models \Phi_{\mathrm{circ}}$ and $M \not\models \mathrm{Holds}(p, U, T)$. By Lemma 77 there is a $\mathcal{P}$-preferred$_d$ extension E of Δ such that $p \notin E$. Hence $\Delta \not\models_{\mathcal{P}}^d p$. Assume $\Delta \not\models_{\mathcal{P}}^d p$, that is, there is a $\mathcal{P}$-preferred$_d$ extension E of Δ such that $p \notin E$. By Lemma 78 there is a model M such that $M \models \Phi_{\mathrm{circ}}$ and $M \not\models \mathrm{Holds}(p, U, T)$. Hence $\Phi_{\mathrm{circ}} \not\models \mathrm{Holds}(p, U, T)$. $\square$

7 Conclusions

We extended Reiter's default logic [22] with a notion of priorities that is based on lexicographic comparison. It turned out that there are two ways of defining lexicographic comparison for strict partial orders. Also, there are at least two ways to value a default with respect to an extension. These alternatives yield four different definitions of preferred extensions in default logic. We investigated questions concerning the relations between different kinds of preferred extensions, existence of preferred extensions, and other properties. Also, we

established a relationship between default logic with our definition of priorities and those by Brewka [6] and Ryan [26].

In the second part of the work, we gave two translations of inheritance networks to prioritized default logic and showed their equivalence to inheritance theories respectively by Gelfond and Przymusinska [10] and by Brewka [4]. Brewka's theory is expressed as a set of formulae in the first-order logic, and resolution of inheritance conflicts is based on McCarthy's circumscription. Our representation of inheritance networks is simpler than Gelfond and Przymusinska's. First, our translation avoids the reification of inheritance networks, that is, each link of an inheritance network can be represented as a default in Reiter's default logic, whereas Gelfond and Przymusinska have formulae of autoepistemic logic that describe the properties of links that are represented by atomic formulae. Second, the multiple extensions for inheritance networks is simply due to multiple extensions of the corresponding default theories. Gelfond and Przymusinska resolve inheritance conflicts by introducing the notion of explanation that is not present in the definition of autoepistemic logic. Multiple extensions in inheritance networks correspond to the existence of multiple explanations. The explanations correspond to sets of defeated defaults in our translation.

Future research should address the problem of automating default reasoning with priorities, and apply these techniques for solving problems in knowledge representation and related areas. In addition to inheritance reasoning, problems like model-based diagnosis [23] and temporal reasoning can be expressed in default logic. The applicability of priorities in these contexts should be investigated. After a problem has been formalized, effective methods for solving it are needed, and therefore work on automating reasoning in prioritized default logics is important. An investigation on the complexity of prioritized default reasoning [25] points out similarities and differences to default reasoning without priorities, and suggests approaches for automating prioritized default reasoning. Techniques for automating reasoning with restricted forms of prioritized defaults [2] and for defaults without priorities [19] have been presented in earlier research. A natural next step would be to extend these techniques to the more general case of prioritized default theories.

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A Auxiliary results

Lemma 80 *Let $\langle D, W \rangle$ be a default theory in which all defaults in D have at least one justification. Let E be an extension of $\langle D, W \rangle$. Then E is consistent if and only if W is consistent.*

Lemma 81 *Let $\langle D, W \rangle$ be a default theory in which all defaults in D are prerequisite-free. Then for all $E \subseteq \mathcal{L}$, E is an extension of $\langle D, W \rangle$ if and only if*

$$E = \mathrm{Cn}(W \cup \left\{ \gamma \left| \frac{: \beta_1, \ldots, \beta_n}{\gamma} \in D, \{\neg\beta_1, \ldots, \neg\beta_n\} \cap E = \emptyset \right. \right\}).$$

Lemma 82 *Let D and D' be sets of defaults such that $D' \subseteq D$. Then $\mathrm{Cn}_W(D') \subseteq \mathrm{Cn}_W(D)$.*

Lemma 83 *Let E be a set of formulae and D a set of defaults. Let $D^E = \{\alpha{:}\beta_1, \ldots, \beta_n/\gamma \in D | \{\neg\beta_1, \ldots, \neg\beta_n\} \cap E = \emptyset\}$ and $D_E = \{\delta \in D | \mathrm{not}\ \mathrm{def}(\delta, E)\}$. Then $D^E \subseteq D_E$.*

Lemma 84 *Let E be a set of formulae and D a set of defaults. Then $\mathrm{Cn}_W(D_E) = \mathrm{Cn}_W(D^E)$, where $D_E = \{\delta \in D | \mathrm{not}\ \mathrm{def}(\delta, E)\}$ and $D^E = \{\alpha{:}\beta_1, \ldots, \beta_n/\gamma \in D | \{\neg\beta_1, \ldots, \neg\beta_n\} \cap E = \emptyset\}$.*

Lemma 85 *Let D be a set of defaults, W a set of formulae, and $E = \mathrm{Cn}_W(D)$ so that $\neg\beta \notin E$ for all $\alpha{:}\sigma/\gamma \in D$ and β in σ. Let $D' = \{\alpha{:}\sigma/\gamma \in D | \alpha \in \mathrm{Cn}_W(D)\}$. Then E is an extension of $\langle D', W \rangle$ and $D' = \mathrm{GD}(E, \langle D', W \rangle)$.*

PROOF. Because $\neg\beta \in E$ for no $\alpha{:}\sigma/\gamma \in D'$ and β in σ, $D'^E = D'$. Because $D' \subseteq D$, by Lemma 82 $\mathrm{Cn}_W(D') \subseteq \mathrm{Cn}_W(D)$. We show by induction that $\mathrm{Cn}_W(D) \subseteq \mathrm{Cn}_W(D')$. Let $C_0, C_1, \ldots$ be the sets in Definition 2 for $\mathrm{Cn}_W(D)$, and $C'_0, C'_1, \ldots$ for $\mathrm{Cn}_W(D')$. *Induction hypothesis*: $C_i \subseteq C'_i$. *Base case $i = 0$*: $C'_0 = W = C_0$ by definition. *Inductive case $i \geq 1$*: Assume that $\gamma \in C_i$. If $\gamma \in C_{i-1}$ then by induction hypothesis $\gamma \in C'_{i-1} \subseteq C'_i$. If $\gamma \in C_i \backslash C_{i-1}$, then $C_{i-1} \models \alpha$ for some $\alpha{:}\sigma/\gamma \in D$. Because $C_{i-1} \subseteq E$ and E is closed under logical consequence, $\alpha \in E$ and hence $\alpha{:}\sigma/\gamma \in D'$. By induction hypothesis $C_{i-1} \subseteq C'_{i-1}$ and hence $C'_{i-1} \models \alpha$. Therefore $\gamma \in C'_i$.

Hence $\mathrm{Cn}_W(D) = \mathrm{Cn}_W(D')$ and $\mathrm{Cn}_W(D')$ is an extension of $\langle D', W \rangle$. Because

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for all $\alpha{:}\sigma/\gamma \in D'$ $\alpha \in E$ and $\neg\beta \in E$ for no $\beta \in \sigma$, by definition $D' = \text{GD}(E, \langle D', W \rangle)$. $\quad\square$

The extensions of two default theories that differ only slightly are in many cases closely related. The following lemmata describe connections of this kind.

Lemma 86 *Let α and γ be formulae and σ a sequence of formulae that contains β. Let E be a set of formulae such that $\neg\beta \in E$. Then E is an extension of $\langle D \cup \{\alpha{:}\sigma/\gamma\}, W \rangle$ if and only if E is an extension of $\langle D, W \rangle$.*

Lemma 87 *Let E be a set of formulae such that $\alpha \notin E$. Then E is an extension of $\langle D, W \rangle$ if and only if E is an extension of $\langle D \cup \{\alpha{:}\sigma/\gamma\}, W \rangle$.*

Lemma 88 *Let $\langle D, W \rangle$ be a default theory in which prerequisites, justifications and conclusions in D are literals, and every default in D is either normal or has no justifications, and W is a consistent set of literals. Let E be an extension of $\langle D, W \rangle$. Let $\alpha{:}\sigma/\gamma$ be a default such that $\alpha \in E$ and $\neg\gamma \notin E$ and σ is the empty sequence or consists of γ, and $\gamma{:}\sigma'/\gamma' \in D$ for no σ' and γ'. Then $E' = \text{Cn}(E \cup \{\gamma\})$ is an extension of $\langle D \cup \{\alpha{:}\sigma/\gamma\}, W \rangle$.*

PROOF. Define $D_E = \{\delta \in D | \text{not def}(\delta, E)\}$ and $D'_{E'} = \{\delta \in D' | \text{not def}(\delta, E')\}$ where $D' = D \cup \{\alpha{:}\sigma/\gamma\}$. Assume that $\alpha'{:}\sigma'/\gamma' \in D_E$. Hence either $\alpha' \notin E$, or $\alpha' \in E$ and $\neg\beta' \in E$ for no β' in σ'. In the first case, because γ is not the prerequisite of any default in D and E' is the closure of a set of literals under logical consequence, $\alpha' \notin E'$. Hence $\alpha'{:}\sigma'/\gamma' \in D'_{E'}$. In the second case, $\gamma' \in E$, and hence $\gamma' \neq \overline{\gamma}$. Because E' is the closure of a set of literals under logical consequence and σ' is the empty sequence or it consists of γ', $\neg\beta' \in E'$ for no β' in σ'. Hence $\alpha'{:}\sigma'/\gamma' \in D'_{E'}$ and $D_E \subseteq D'_E$. That $D'_{E'} \subseteq D_E \cup \{\alpha{:}\sigma/\gamma\}$ is immediate because $E \subseteq E'$. Because $\neg\gamma \notin E'$, $\alpha{:}\sigma/\gamma \in D'_{E'}$. Hence $D'_{E'} = D_E \cup \{\alpha{:}\sigma/\gamma\}$.

By Definition 2 and Lemma 84 $E = \text{Cn}_W(D_E)$. We show that $E' = \text{Cn}_W(D'_{E'})$. Let $C_0, C_1, \dots$ be the sets in the definition of $\text{Cn}_W(D^E)$ in Definition 2, and $C'_0, C'_1, \dots$ the sets for $\text{Cn}_W(D'_{E'})$.

Induction hypothesis: $C_i \subseteq C'_i \subseteq C_i \cup \{\gamma\}$. *Base case $i = 0$*: $C_0 = W = C'_0$. *Inductive case $i \geq 1$*: Assume that $\gamma' \in C_i$. If $\gamma' \in C_{i-1}$, then by induction hypothesis $\gamma' \in C'_{i-1} \subseteq C'_i$. If $\gamma' \in C_i \backslash C_{i-1}$, then there is $\alpha'{:}\sigma'/\gamma' \in D_E$ such that $C_{i-1} \models \alpha'$. By induction hypothesis $C_{i-1} \subseteq C'_{i-1}$, and because $D_E \subseteq D'_{E'}$, $\gamma' \in C'_i$ by definition. Then we show the second inclusion. Assume that $\gamma' \in C'_i$. If $\gamma' \in C'_{i-1}$, then by induction hypothesis $\gamma' \in C_{i-1} \cup \{\gamma\} \subseteq C_i \cup \{\gamma\}$. If $\gamma' \in C'_i \backslash C'_{i-1}$, then $C'_{i-1} \models \alpha'$ for some $\alpha'{:}\sigma'/\gamma' \in D'_{E'}$. If $\gamma' = \gamma$, then clearly $\gamma \in C_i \cup \{\gamma\}$. Otherwise $\alpha'{:}\sigma'/\gamma' \in D_E$. Because by induction hypothesis $C'_{i-1} \subseteq C_{i-1} \cup \{\gamma\}$, and $C_{i-1} \cup \{\gamma\} \models \alpha'$, $\alpha' \in C_{i-1} \cup \{\gamma\}$. Because γ is not the

prerequisite of any default in D and C'_{i-1} is a set of literals, $\alpha' \in C_{i-1}$. Hence $C_{i-1} \models \alpha'$ and $\gamma' \in C_i$ by definition.

Because $\alpha \in E$, $\alpha \in C_j \subseteq C'_j$ for some $j \geq 0$. Therefore by definition $\gamma \in C'_{j+1} \subseteq \bigcup_{i \geq 0} \mathrm{Cn}(C'_i)$. Therefore $E' = \mathrm{Cn}_W(D'_{E'})$ and $\mathrm{Cn}(E \cup \{\gamma\})$ is an extension of $\langle D \cup \{\alpha{:}\sigma/\gamma\}, W \rangle$. $\quad\square$

The following lemmata on propositional 2-literal Horn clauses are used in Sections 5.3 and 6.2. We define Horn clauses as formulae $p_1 \wedge \cdots \wedge p_n \to l$ where p_i are propositional variables and l is a literal.

Lemma 89 *Let β be a literal and S a consistent set of literals and implications $q \to \beta'$ such that q is a propositional variable and β' is a literal and $\beta \neq \neg q$. If $S \models \beta$, then $\beta \in S$ or $S \models p$ for some $p \to \beta \in S$.*

Lemma 90 *Let $S \cup \{p \to \beta\}$ be a consistent set of literals and formulae $p' \to \beta'$ such that p' is a propositional variable and β' is a literal and $p' \neq \beta$. Then if $S \not\models q$ and $S \cup \{p \to \beta\} \models q$ for a propositional variable q, then $q = \beta$.*