On the domination number of 2-dimensional torus graphs

Simon Crevals[*] Patric R. J. Östergård[†]

Department of Communications and Networking
Aalto University School of Electrical Engineering
P.O. Box 13000, 00076 Aalto, Finland

Abstract

The domination number of the $m\times n$ torus graph is denoted by $\gamma(C_m\Box C_n)$. Here, an algorithm based on dynamic programming is presented which can be used to determine $\gamma(C_m\Box C_n)$ as a function of n when m is fixed. The value of $\gamma(C_m\Box C_n)$ has previously been determined for $m \le 10$ and arbitrary n. These results are here extended to $m \le 20$ and arbitrary n.

1 Introduction

The domination number is a frequently studied graph invariant [9]. The domination number has been studied both for arbitrary graphs and specific subclasses of graphs. In the current study, we focus on the class of torus graphs.

A *torus graph* is the Cartesian product of two cycles, denoted by $C_m\Box C_n$, where C_i is the cycle with i vertices. The vertex set and edge set of $C_m\Box C_n$ are defined as follows:

$$\begin{aligned} V(C_m\Box C_n) &= \{v_{i,j} : 0 \le i < m, 0 \le j < n\}, \\ E(C_m\Box C_n) &= \{\{v_{i,j}, v_{i+1,j}\}, \{v_{i,j}, v_{i,j+1}\} : 0 \le i < m, 0 \le j < n\}, \end{aligned}$$

where the operations in the subindices are carried out modulo m and modulo n, respectively. The vertices can be ordered based on the colexicographic order of the subindices: $v_{i,j} < v_{i',j'}$ if $j < j'$ or $j = j'$ and $i < i'$. If one takes the Cartesian product of two paths instead of two cycles, then one gets a *grid graph*. For Cartesian products it is known that $G\Box H \cong H\Box G$.

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Given a graph $G = (V, E)$ and two vertices $v, w \in V$, we say that v *dominates* w if $v = w$ or v is adjacent to w, that is, if w is in the closed neighborhood of v, $N_G[v]$. A *dominating set* of G is a subset $D \subseteq V$, such that each vertex in V is dominated by some vertex in D. The *domination number* of G is the minimum cardinality of a dominating set of G and is denoted by $\gamma(G)$. A dominating set D is called *perfect* if each vertex in V is dominated by exactly one vertex in D.

The domination number of grid graphs has been studied extensively. This work culminated in the determination of the domination number for arbitrary grid graphs [7]. For torus graphs, however, there are still open questions regarding the domination number. Earlier results include the determination of $\gamma(C_m \Box C_n)$ for $m \leq 10$ and arbitrary n [4, 6, 11, 14] and general upper and lower bounds on $\gamma(C_m \Box C_n)$ [5, 15]. The best known general lower bounds are

$$\gamma(C_m \Box C_n) \geq \begin{cases} \frac{mn}{5}, & \text{if } m \equiv 0 \pmod 5 \\ \frac{mn}{5} + \frac{2n}{15}, & \text{if } m \equiv 1 \pmod 5 \\ \frac{mn}{5} + \frac{n}{10}, & \text{if } m \equiv 2 \pmod 5 \\ \frac{mn}{5} + \frac{3n}{20}, & \text{if } m \equiv 3 \pmod 5 \\ \frac{mn}{5} + \frac{n}{5}, & \text{if } m \equiv 4 \pmod 5. \end{cases}$$

For each m, $m \not\equiv 3 \pmod 5$, there are infinitely many n for which the corresponding bound is tight. Perfect dominating sets exist exactly when both m and n are divisible by 5.

There is a correspondence between dominating sets of torus graphs and covering codes in the Lee metric. Consider codes as subsets of the words in the space $\mathbb{Z}_q^n$, where $\mathbb{Z}_q$ is the ring of integers modulo q. The *Lee distance* between two codewords, $c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$, is defined as

$$\sum_{i=1}^{n} \min(|c_i - d_i|, q - |c_i - d_i|).$$

A code $C \subseteq \mathbb{Z}_q^n$ with the property that each word in $\mathbb{Z}_q^n$ is within Lee distance R from one (resp., exactly one) codeword in C is said to be a Lee code (resp., perfect Lee code) with covering radius R. By letting a vertex $v_{i,j}$ in $C_q \Box C_q$ correspond to a word $(i, j) \in \mathbb{Z}_q^2$, one can see that the distance between two vertices in the graph $C_q \Box C_q$ equals the Lee distance between the corresponding codewords in $\mathbb{Z}_q^2$. Moreover, a (perfect) dominating set of $C_q \Box C_q$ gives a (perfect) Lee code in $\mathbb{Z}_q^2$ with covering radius 1, and vice versa. For general results on Lee codes and perfect Lee codes, see [10, 12].

Analogously, there is a correspondence between dominating sets in the graph $C_q \Box C_q \Box \cdots \Box C_q$ (n times) and Lee codes in $\mathbb{Z}_q^n$ with covering radius 1. Also torus graphs $C_m \Box C_n$ for which $m \neq n$—and their generalizations to products of more cycles—can be viewed in the framework of Lee codes. Then the space and codes simply have different alphabets in different coordinates.

In this paper we consider a computational approach for obtaining $\gamma(C_m \square C_n)$ for a fixed m and arbitrary n, continuing work on related graphs in [1] and [3]. We present the main algorithm in Section 2, and determine $\gamma(C_m \square C_n)$ for $m \leq 20$ and arbitrary n in Section 3. These results are not enough to form a conjecture about $\gamma(C_m \square C_n)$, but it does appear that Van Wieren's conjecture [15] claiming that $\lim_{n\to\infty} \gamma(C_m \square C_n)/n = (6k^2 + 9k + 3)/(6k + 4)$, when $m = 5k + 3$, holds true.

2 Algorithm

The general idea of determining the domination number of an infinite family of product graphs using dynamic programming and finite calculations (in practice, computations) was presented in the 1990s by Livingston and Stout [13]. See also [2]. After the seminal work of Livingston and Stout, work has been done to find fast practical algorithms for $G \square H$ with different G and H. In [1] and [3], Cartesian products $P_m \square P_n$ are considered (for the variant of total domination in [3]). The general approach of the current study is analogous to that of [1] and [3], which can be consulted for more details.

The main focus in [13] is on Cartesian products $G \square P_n$ for fixed G, where P_n is the path with n vertices. The presence of P_n gives a natural ordering of the vertices and thereby starting points for dynamic programming, which we do not have in the case of $C_m \square C_n$. This is one reason why the details of the algorithms for $P_m \square P_n$ [1] and $C_m \square C_n$ differ.

2.1 Sketch of algorithm

The algorithm builds up dominating sets of $G = C_m \square C_n$ by iteratively adding vertices to dominate undominated vertices. We assume that m is fixed but not n (n will be fixed at later stages). Actually, since it then suffices to know the set of vertices that are dominated, the algorithm considers such sets rather than sets of dominating vertices. We call a set of dominated vertices a *dominated set*.

We define $\mathcal{S}_k$, a collection of sets, as

$$\begin{aligned} \mathcal{S}_0 &= \{\{\}\}, \\ \mathcal{S}_{k+1} &= \{S \cup N_G[v] \mid S \in \mathcal{S}_k, v \in N_G[v(S)]\}, \end{aligned}$$

where $v(S) := \min\{v : v \in V(G) \setminus S\}$ assuming colexicographic order on the vertices.

At any moment, we can further reduce the number of sets in $\mathcal{S}_k$ by removing all sets in $\mathcal{S}_k$ that are a subset of another set in $\mathcal{S}_k$, cf. [8] and [1, Theorem 2.2].

Let $C^j = \{v_{i,j} : 0 \leq i \leq m-1\}$. We can partition each collection $\mathcal{S}$ based on the dominated vertices in C^{n-1} and C^{n-2} (considering these as C^{-1} and C^{-2}

it is not necessary to be specific about the exact value of n, which is not fixed at this stage). This allows us to restrict the removal of subsets to each part of the partition, which greatly reduces the number of comparisons needed. Since the dominated vertices in C^{n-1} and C^{n-2} do not change while the dominated sets are built, only one partition is considered over all collections. All sets in a specific part of $\mathcal{S}_{k+1}$ come from the same part in $\mathcal{S}_k$.

To speed up the algorithm, we only allow specific parts based on the symmetries of the torus. For each part in the partition we build the dominated sets for the entire torus graph. For every possible n we need a separate calculation to dominate the undominated vertices in C^{n-1} and C^{n-2}.

Just as in [3] we detect periodicity in the dominated sets. For some k and l, $k \neq l$, there will be an easy bijection between $\mathcal{S}_k$ and $\mathcal{S}_l$ from which we can derive a formula for an upper bound for $\gamma(C_m \Box C_n)$ for each part of the partition. After merging these upper bounds we know $\gamma(C_m \Box C_n)$ for all n.

2.2 Generating all parts

Given D, a dominating set for $C_m \Box C_n$, we can represent the vertices in C^0 with a ternary number $t = (t_0, t_1, \ldots, t_{m-1})$, such that

$$t_i = \begin{cases} 0, & \text{if } v_{i,0} \in D \\ 1, & \text{if } v_{i,0} \notin D \text{ and } \{v_{i-1,0}, v_{i+1,0}, v_{i,n-1}\} \cap D \neq \emptyset \\ 2, & \text{otherwise.} \end{cases}$$

Since t represents the vertices of a cycle, t_0 and t_{m-1} are considered adjacent and operations on the subindices of t are carried out modulo m. From t we derive a set T of vertices (from C^{n1}, C^0, C^1) that have to be in D. The algorithm for this part will start with exactly the vertices from T as dominating vertices. The dominated vertices for this part will be the vertices in C^{n-1} and C^{n-2} dominated by a vertex from T and t will serve as a representative for the part. We denote a collection $\mathcal{S}$ restricted to the sets in the part represented by t as $\mathcal{S}^t$.

We restrict the number of parts by imposing the following seven properties on t. For each of these properties we argue why t can be restricted in that way.

The first property follows from the definition of t. If $t_i = 0$ then $v_{i,0} \in D$. Now either $v_{i-1,0} \in D$ and $t_{i-1} = 0$, or $v_{i-1,0} \notin D$ and $t_{i-1} = 1$ because $v_{i-1,0}$ is dominated by $v_{i,0}$. A similar argument can be made for $v_{i+1,0}$.

Property 2.1. *No 0 in t is adjacent to a 2.*

We can reject a certain dominating pattern in our dominating sets.

Lemma 2.1. *Let T be a dominating set of $C_m \Box C_n$, with $m \leq n$. There exists a dominating set T', with $|T'| \leq |T|$, such that there is no C^j with four out of five consecutive vertices in T'.*

Proof. We start from $T' = T$ and modify T' until the requirements are met. If there is a C^j with five consecutive vertices of which at least four are in T', then we replace those vertices in T' by four vertices as shown in Figure 1. Dominating vertices are coloured black and vertices they dominate are coloured gray. After replacing the vertices we still have that $|T'| \leq |T|$ and that T' is a dominating set.

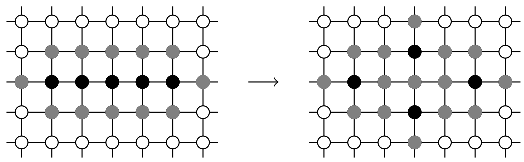

Figure 1: First replacement

If replacing vertices from C^j leads to five consecutive vertices in C^{j+1} of which at least four are in T' then we replace those dominating vertices from C^{j+1} by three dominating vertices as shown in Figure 2. Figure 2 shows a superset of the dominating vertices on the left and a replacement with three dominating vertices on the right. There are at least four dominating vertices from C^{j+1} being replaced, which means the number of vertices in T' decreases. This can happen only a finite number of times. A similar replacement can be done when the first replacement leads to a pattern in C^{j-1}, but not to a pattern in C^{j+1}.

When using the first replacement on a pattern in C^j, the number of dominating vertices inside C^j decreases. Therefore the first replacement can only be done a finite number of times within C^j before using the second replacement or running out of patterns within C^j.

We continue applying these changes until T' is such that there is no C^j that has five consecutive vertices of which at least four are in T'. Now T' is still dominating and we have that $|T'| \leq |T|$. □

Based on this lemma we demand the following three properties in t.

If t has four 0s in five consecutive digits then the dominating set would have the pattern we can exclude due to Lemma 2.1 in C^0.

Property 2.2. *No five consecutive digits of t contain four 0s.*

If t has four 2s in five consecutive digits then the dominating set would have the pattern we can exclude due to Lemma 2.1 in C^1.

Property 2.3. *No five consecutive digits of t contain four 2s.*

If t has four 1s that are not adjacent to a 0 in five consecutive digits then the dominating set would have the pattern we can exclude due to Lemma 2.1 in C^{n-1}.

Property 2.4. *No five consecutive digits of t contain four 1s that are not adjacent to a 0.*

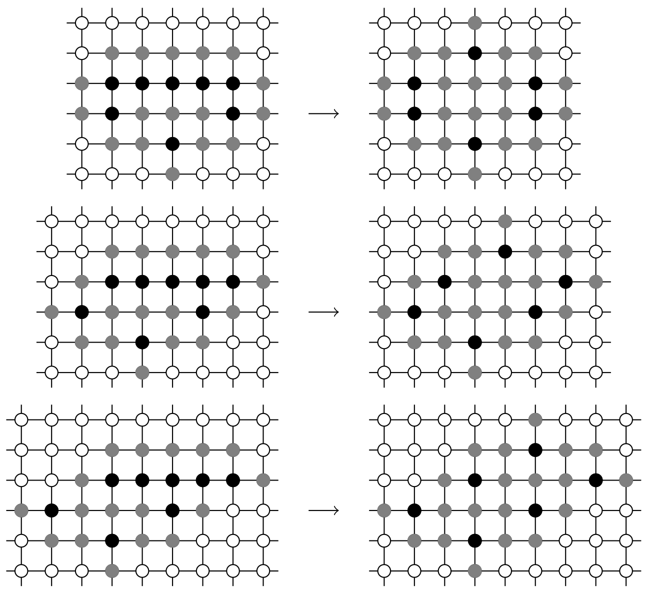

Figure 2: Second replacement

Finally, we require t to have some properties based on the symmetries of the torus. Due to the rotational symmetry of the torus, we can select any cycle as C^0. Due to the pigeonhole principle, there has to be at least one C^j with at most $\gamma(C_m \Box C_n)/n$ dominating vertices. We choose this cycle as C^0 and we get the following property.

Property 2.5. *There are at most $\gamma(C_m \Box C_n)/n$ 0s in t.*

Because we do not know $\gamma(C_m \Box C_n)$ when we start the algorithm, we start with $c = 0$ dominating vertices in C^0 and gradually increase c until the algorithm obtains an upper bound on $\gamma(C_m \Box C_n)$ with fewer than $(c + 1)n$ dominating vertices.

Property 2.5 takes care of the rotational symmetries of C_n. Now we can still consider the reflectional symmetry in C_n. We can choose the direction in which we start dominating the torus. In other words, we can change the labels of the cycles such that we change C^j with C^{n-j} for $0 < j < n/2$. We choose the direction that leads to the least number of 2s. Only 1s not adjacent to a 0 can become a 2 when changing C^2 with C^n. If t has more 2s than 1s not adjacent to a 0, reflecting the torus will lead to fewer 2s.

Property 2.6. *There are at most as many 2s in t as 1s that are not adjacent to a 0.*

Lastly, we also take the symmetries of C_m into account. When we rotate the torus k positions along C^0, the vertices $v_{i,j}$ from the dominating set are replaced by $v_{i+k,j}$ and in the corresponding reflection they are replaced by $v_{n-1-i-k,j}$. Therefore, we only have to consider one of the rotations and corresponding reflections of t.

Property 2.7. *The ternary number t is lexicographically the smallest of all its rotations and their reflections.*

For every ternary number generated this way, we create a dominated set S, such that

$$S = \begin{array}{l} \{v_{i,0} \mid 0 \leq i < m\} \cup \\ \{v_{i,1} \mid t_i = 0, t_{i-1} = 2, t_i = 2, \text{ or } t_{i+1} = 2\} \cup \\ \{v_{i,2} \mid t_i = 2\} \cup \\ \{v_{i,n-1} \mid t_i = 0, t_{i-1} = *, t_i = *, \text{ or } t_{i+1} = *\} \cup \\ \{v_{i,n-2} \mid t_i = *\}, \end{array}$$

where $t_i = *$ if $t_i = 1$ and $t_{i-1} \neq 0$ and $t_{i+1} \neq 0$. If $T = \{t_i \mid t_i = 0, t_i = *, \text{ or } t_i = 2\}$, the set of dominating vertices that has to be in D due to the definition of t, then S is the set of vertices dominated by a vertex in T. Therefore S is considered the first set of the part represented by t and S is the only set in $\mathcal{S}_k^t$, where $k = |T|$.

2.3 Building dominated sets

Let t be a ternary number, generated according to Section 2.2, with corresponding dominated set, $S \in \mathcal{S}_k$. Within its part of the partition we have that $\mathcal{S}_k^t = \{S\}$ and $\mathcal{S}_{l+1}^t = \{S \cup N_G[v] \mid S \in \mathcal{S}_l^t, v \in N_G[v(S)]\}$, for $l \geq k$, with $G = C_m \Box C_n$.

Whenever $C^j \not\subset S$, with $S \in \mathcal{S}_l^t$, and $C^j \subset S \cup N_G[v]$ an upper bound for $\gamma(C_m \Box C_{j+3})$ is determined as follows. First, the dominated vertices from C^{n-2} and C^{n-1} are added as dominated vertices in C^{j+1} and C^{j+2}, respectively. Then x, the minimum number of dominating vertices needed to dominate the remaining undominated vertices, is computed exhaustively. We then have that $l + x$ is an upper bound for $\gamma(C_m \Box C_{j+3})$. For some l all sets in $\mathcal{S}_l^t$ will dominate all vertices in C^j. Then the minimum number of dominating vertices needed to dominate $C_m \Box C_{j+3}$ starting from t is determined by taking the minimum over all computed upper bounds for $\gamma(C_m \Box C_{j+3})$.

To find the formula for the upper bounds for $\gamma(C_m \Box C_n)$ the method presented in [3] is used. After adding some finite number of dominating vertices there will be integers $a > 0$, $b > 0$, and d such that $|\mathcal{S}_d^t| = |\mathcal{S}_{d+a}^t|$ and for every set $S \in \mathcal{S}_d^t$ there is a set $S' \in \mathcal{S}_{d+a}^t$ with the following two properties. Let q, r be the indices of the lexicographically smallest undominated vertex of S, that is, $v(S) = v_{q,r}$. The first property considers the lexicographically smallest vertex of

S': $v(S') = v_{q,r+b}$. This means that $v(S) \in C^r$ and the only vertices for which it remains unknown whether or not they are in S are elements of C^r, C^{r+1}, or C^{r+2}. The second property specifies which of those vertices are in S: for every i and j we have that $v_{i,j} \in S \cap (C^r \cup C^{r+1} \cup C^{r+2})$ if and only if $v_{i,j+b} \in S' \cap (C^{r+b} \cup C^{r+b+1} \cup C^{r+b+2})$. From this repetition a formula for the upper bounds for $\gamma(C_m \Box C_n)$ can be derived as follows. If $l = \min\{j \mid \exists S \in \mathcal{S}_d^t : C^j \subset S \text{ and } \nexists S \in \mathcal{S}_d^t : C^{j+1} \subset S\}$, then for all $n \geq l+3$ the derived upper bound has the form $\gamma(C_m \Box C_n) \leq an/b + c_i$ where c_i, $i \equiv n (\text{mod } b)$, is a constant.

For each part in the partition one formula for the upper bound of $\gamma(C_m \Box C_n)$ is obtained. These upper bounds are of the form $\gamma(C_m \Box C_n) \leq a_t n/b_t + c_{t,i}$, for $n > N_t$, with $i \equiv n (\text{mod } b_t)$, where t is a representative for the part. For each n, the exact value of $\gamma(C_m \Box C_n)$ is obtained by taking the minimum value of the upper bounds for $\gamma(C_m \Box C_n)$ taken over all parts.

The upper bounds for all parts can be merged into one formula of the form $\gamma(C_m \Box C_n) = an/b + c_i$, for $n > N$, with $i \equiv n (\text{mod } b)$. Obviously, $a_t/b_t = a/b$ is constant throughout all parts and we let b be the least common multiple of all b_t. The constant $c_i = \min_t(c_{t,j})$, with $j \equiv n (\text{mod } b_t)$, gives the minimum of the upper bounds for each $i \equiv n (\text{mod } b)$, and $N = \max_t(N_t)$. For $n \leq N$ the upper bound is the minimum of all calculated values.

3 Results

The formulae for $\gamma(C_m \Box C_n)$ are listed below for $m \leq 20$ and arbitrary $n \geq m$. For $m \leq 5$ the formulae were first proved by Klavžar and Seifter [11]. El-Zahar and Saheen proved the formulae for $m \leq 10$ [4, 6, 14]. All these formulae were confirmed by the algorithm described in this paper and furthermore formulae for $m \leq 20$ were determined. Running times for $m \geq 16$ can be found in Table 1 (using a 3-GHz Intel Core2 Duo CPU E8400).

Table 1: Running time for $m \geq 16$

m	16	17	18	19	20
time	32 m	1 h 25 m	2 d 5 h	3 d 6 h	12 d 12 h

$$\gamma(C_3 \Box C_n) = \left\lceil \frac{3n}{4} \right\rceil,$$

$$\gamma(C_4 \Box C_n) = n,$$

$$\gamma(C_5 \Box C_n) = \begin{cases} n, & \text{if } n \equiv 0 \pmod 5 \\ n+1, & \text{if } n \equiv 1,2,4 \pmod 5 \\ n+2, & \text{if } n \equiv 3 \pmod 5), \end{cases}$$

$$\gamma(C_6\square C_n) = \begin{cases} \left\lceil \frac{4n}{3} \right\rceil, & \text{if } n \equiv 0,1,4 \pmod 6 \text{ or } n \equiv 5 \pmod{18} \\ \left\lceil \frac{4n}{3} \right\rceil + 1, & \text{if } n \equiv 2,3,5 \pmod 6 \text{ but } n \not\equiv 5 \pmod{18}, \end{cases}$$

$$\gamma(C_7\square C_n) = \begin{cases} \left\lceil \frac{3n}{2} \right\rceil, & \text{if } n \equiv 0,5,9 \pmod{14} \\ \left\lceil \frac{3n}{2} \right\rceil + 1, & \text{if } n \equiv 1,3,4,6,7,10,11,13 \pmod{14} \\ \left\lceil \frac{3n}{2} \right\rceil + 2, & \text{if } n \equiv 2,8,12 \pmod{14}, \end{cases}$$

$$\gamma(C_8\square C_n) = \begin{cases} \left\lceil \frac{9n}{5} \right\rceil, & \text{if } n \equiv 0,4,9 \pmod{10} \text{ or } n \equiv 13,18,22,23,31,32 \pmod{40} \\ \left\lceil \frac{9n}{5} \right\rceil + 1, & \text{otherwise}, \end{cases}$$

$$\gamma(C_9\square C_n) = \begin{cases} 2n, & \text{if } n \notin \{11,13\} \\ 2n+1, & \text{if } n \in \{11,13\}, \end{cases}$$

$$\gamma(C_{10}\square C_n) = \begin{cases} 2n, & \text{if } n \equiv 0 \pmod 5 \\ 2n+2, & \text{if } n \equiv 1,2,4 \pmod 5 \\ 2n+4, & \text{if } n \equiv 3 \pmod 5, \end{cases}$$

$$\gamma(C_{11}\square C_n) = \begin{cases} \left\lceil \frac{7n}{3} \right\rceil, & \text{if } n \equiv 0,5,10,28 \pmod{33} \\ \left\lceil \frac{7n}{3} \right\rceil + 1, & \text{if } n \in \{11,16\} \text{ or } n \equiv 1,4,7,12,13,14,15,17,19,20,22,23,25,29 \pmod{33} \\ \left\lceil \frac{7n}{3} \right\rceil + 2, & \text{if } n \equiv 2,3,6,8,9,11,16,18,24,26,27,30,31,32 \pmod{33} \text{ and } n \notin \{11,16\} \\ \left\lceil \frac{7n}{3} \right\rceil + 3, & \text{if } n \equiv 21 \pmod{33}, \end{cases}$$

$$\gamma(C_{12}\square C_n) = \begin{cases} \left\lceil \frac{5n}{2} \right\rceil, & \text{if } n \equiv 0,5,19 \pmod{24} \\ \left\lceil \frac{5n}{2} \right\rceil + 1, & \text{if } n \equiv 9,10,11,14,15 \pmod{24} \\ \left\lceil \frac{5n}{2} \right\rceil + 2, & \text{if } n \in \{12\} \text{ or } n \equiv 1,3,4,6,13,16,17,20,21,23 \pmod{24} \\ \left\lceil \frac{5n}{2} \right\rceil + 3, & \text{if } n \equiv 2,7,8,12,18,22 \pmod{24} \text{ and } n \notin \{12\}, \end{cases}$$

$$\gamma(C_{13}\square C_n) = \begin{cases} \left\lceil \frac{45n}{16} \right\rceil, & \text{if } n \equiv 0,5,14,19,21,26,31,36,37,41,46,51,52,65,69,79,83,84,97,101,106 \\ & 111,116,117,130,133,138,143,148,149,157,162,180,181,194,195 \pmod{208} \\ \left\lceil \frac{45n}{16} \right\rceil + 1, & \text{if } n \equiv 1,2,3,4,6,7,8,9,10,11,12,13,15,16,17,18,20,22,23,24,25 \\ & 27,28,30,32,33,35,38,39,40,42,45,47,50,53,55,56,57,58,60,61 \\ & 62,63,66,67,68,70,71,72,73,74,76,77,78,81,82,85,86,87,88,89 \\ & 90,91,92,93,94,95,96,98,99,100,102,103,104,105,108,109,110 \\ & 113,114,115,118,119,120,121,122,123,124,125,126,127,128,129 \\ & 131,132,134,135,136,137,141,142,144,146,147,151,152,153,154 \\ & 156,158,159,161,163,164,165,166,167,168,169,170,171,173,174 \\ & 175,176,178,179,182,183,184,185,186,188,189,190,191,193,196 \\ & 197,198,199,200,201,202,203,204,205,206,207 \pmod{208} \\ \left\lceil \frac{45n}{16} \right\rceil + 2, & \text{if } n \equiv 29,34,43,44,48,49,54,59,64,75,80,107,112,139,140,145 \\ & 150,155,160,172,177,187,192 \pmod{208}, \end{cases}$$

$$\gamma(C_{14}\square C_n) = \begin{cases} 3n, & \text{if } n \notin \{16, 18, 23\} \\ 3n+1, & \text{if } n \in \{16, 18, 23\}, \end{cases}$$

$$\gamma(C_{15}\square C_n) = \begin{cases} 3n, & \text{if } n \equiv 0 \pmod 5 \\ 3n+3, & \text{if } n \equiv 1, 2, 4 \pmod 5 \\ 3n+6, & \text{if } n \equiv 3 \pmod 5, \end{cases}$$

$$\gamma(C_{16}\square C_n) = \begin{cases} \left\lceil \frac{10}{3} \right\rceil, & \text{if } n \equiv 0, 5, 10, 43 \pmod{48} \\ \left\lceil \frac{10n}{3} \right\rceil + 1, & \text{if } n \equiv 15, 17, 20, 22, 25, 38 \pmod{48} \\ \left\lceil \frac{10n}{3} \right\rceil + 2, & \text{if } n \in \{16, 23\} \text{ or } n \equiv 1, 4, 7, 12, 13, 14, 19, 24, 27 \\ & 28, 29, 30, 32, 33, 34, 35, 37, 39, 40, 44 \pmod{48} \\ \left\lceil \frac{10n}{3} \right\rceil + 3, & \text{if } n \in \{21, 26\} \text{ or } n \equiv 2, 3, 6, 8, 9, 11, 16, 18, 23 \\ & 31, 41, 42, 45, 46, 47 \pmod{48} \text{ and } n \notin \{16, 23\} \\ \left\lceil \frac{10n}{3} \right\rceil + 4, & \text{if } n \equiv 21, 26, 36 \pmod{48} \text{ and } n \notin \{21, 26\}, \end{cases}$$

$$\gamma(C_{17}\square C_n) = \begin{cases} \left\lceil \frac{7}{2} \right\rceil, & \text{if } n \equiv 0, 5, 29 \pmod{34} \\ \left\lceil \frac{7n}{2} \right\rceil + 1, & \text{if } n \equiv 10, 15, 19, 24 \pmod{34} \\ \left\lceil \frac{7n}{2} \right\rceil + 2, & \text{if } n \equiv 9, 11, 14, 16, 20, 21, 25 \pmod{34} \\ \left\lceil \frac{7n}{2} \right\rceil + 3, & \text{if } n \equiv 1, 3, 4, 6, 13, 23, 26, 27, 30, 31, 33 \pmod{34} \\ \left\lceil \frac{7n}{2} \right\rceil + 4, & \text{if } n \equiv 2, 7, 8, 12, 17, 18, 22, 28, 32 \pmod{34}, \end{cases}$$

$$\gamma(C_{18}\square C_n) = \begin{cases} \left\lceil \frac{42}{11} \right\rceil, & \text{if } n \equiv 0, 5, 24, 47, 48 \pmod{66} \\ \left\lceil \frac{42n}{11} \right\rceil + 1, & \text{if } n \equiv 4, 6, 9, 10, 14, 16, 19, 21, 26, 27, 28, 29, 30, 31, 32, 36 \\ & 37, 38, 41, 42, 43, 46, 51, 52, 53, 54, 56, 60, 61, 65 \pmod{66} \\ \left\lceil \frac{42n}{11} \right\rceil + 2, & \text{if } n \equiv 1, 2, 3, 7, 8, 11, 12, 13, 15, 17, 18, 20, 22, 23, 25, 33 \\ & 34, 35, 40, 45, 49, 50, 55, 57, 58, 59, 62, 63, 64 \pmod{66} \\ \left\lceil \frac{42n}{11} \right\rceil + 3, & \text{if } n \equiv 39, 44 \pmod{66}, \end{cases}$$

$$\gamma(C_{19}\square C_n) = \begin{cases} 4n, & \text{if } n \notin \{21, 23, 26, 28, 33\} \\ 4n+1, & \text{if } n \in \{26, 33\} \\ 4n+2, & \text{if } n \in \{21, 23, 28\}, \end{cases}$$

$$\gamma(C_{20}\square C_n) = \begin{cases} 4n, & \text{if } n \equiv 0 \pmod 5 \\ 4n+4, & \text{if } n \equiv 1, 2, 4 \pmod 5 \\ 4n+8, & \text{if } n \equiv 3 \pmod 5, \end{cases}$$

In particular, the minimum size of a Lee covering code in $\mathbb{Z}_q^2$ with covering radius 1, $K_q^2(2,1) = \gamma(C_q\square C_q)$, can be found in Table 2.

Table 2: Values for $K_q^2(2,1)$, for $q \leq 20$

q	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$K_q^2(2,1)$	3	4	5	8	12	16	18	20	27	32	38	42	45	56	64	71	76	80

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