Approximation to Distribution of Product of Random Variables Using Orthogonal Polynomials for Lognormal Density

Zhong Zheng, *Student Member, IEEE,* Lu Wei, *Student Member, IEEE,* Jyri Hämäläinen, *Member, IEEE,* and Olav Tirkkonen, *Member, IEEE*

***Abstract*—We derive a closed-form expression for the orthogonal polynomials associated with the general lognormal density. The result can be utilized to construct easily computable approximations for probability density function of a product of random variables, when the considered variates are either independent or correlated. As an example, we have calculated the approximative distribution for the product of Nakagami-m variables. Simulations indicate that accuracy of the proposed approximation is good with small cross-correlations under light fading condition.**

***Index Terms*—Product of random variables, central limit theorem, lognormal distribution, orthogonal polynomials, Nakagami-m.**

I. Introduction

STATISTICAL properties of products of Random Variables (RVs) are essential in performance analysis of contemporary wireless communication systems. For example, the fading amplitudes of cascaded fading channels follow the distribution of the product of Nakagami-m RVs [1]. In addition, the cascaded-keyhole channel can be modeled using the product of individual keyhole channels [2].

The exact Probability Density Function (PDF) and Cumulative Distribution Function (CDF) of the product of independent Beta, Gamma and Gaussian RVs can be represented in terms of the Meijer-G function [3]. A more general framework involving the Fox H-function was proposed in [4] for the distribution of product of almost any non-negative independent RVs. Although being theoretically interesting, these G- or H-function representations are difficult to evaluate on a low-complexity computing platform, where the calculation of the product statistics is required [5]. The numerical calculations of G- and H-functions require a numerical solution of contour integrals or a construction of look-up tables that cover all parameter combinations. Both approaches set stringent requirements for the considered platform either in the computing capability or the memory capacity. Recently, the authors in [6], [7] proposed approximations to the PDFs and CDFs of products of independent Rayleigh, Gamma, Nakagami-m and Gaussian RVs by exploiting the structure of the Mellin transforms. However, when the RVs in product are correlated, exact distributions are known only in some special cases (e.g. bivariate Nakagami-m in [8]). For arbitrary number of RVs with generic correlations, none of the aforementioned methodologies are suitable to give a tractable solution. This is the main issue to be addressed in this paper.

The authors are with the Department of Communications and Networking, Aalto University, 00076 Aalto, Finland (e-mail: {zhong.zheng, lu.wei, jyri.hamalainen, olav.tirkkonen}@aalto.fi). J. Hämäläinen is also with Ericsson Oulu R&D Center.

Motivated by the Central Limit Theorem (CLT), the approximation for the product of RVs can be constructed by using a lognormal density and its associated orthogonal polynomials. We apply the approach of [9] where the first few moments of the product and the approximated density function are set equal. The resulting approximative distribution involves only a finite sum of polynomials and the elementary lognormal density function. Therefore, implementing the proposed expressions on practical computing platforms is straightforward compared with the G- or H-function representations. We note that the proposed framework is suitable for cases of both independent and correlated variables provided that moments of a product can be computed in closed-form.

In this paper we deduce a closed-form expression for the orthogonal polynomials associated with a general lognormal density. The result is obtained by taking advantage of the determinant representation of orthogonal polynomials. To give a specific example, we derive an approximative PDF and CDF for the product of both independent and correlated Nakagami-m RVs. The proposed approximations are especially useful in performance analysis of wireless communication systems operating in cascaded fading channels [1], [7]. Numerical results are compared with simulations in terms of the Complementary CDF (CCDF). The overall approximation accuracy is measured by the Mean Square Error (MSE) between the approximative CDF and the empirical CDF.

II. Orthogonal Polynomials Associated with General Lognormal Density

Assume that Y is a Gaussian RV with mean μ and variance σ^2. Then $X = e^Y$ follows the lognormal distribution

$$f_{\mathrm{LN}}(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}, \quad x \in (0, \infty) \tag{1}$$

and the i-th moment, ν_i ($i \in \mathbb{N}$), of X is given by

$$\nu_i = \int_0^\infty x^i f_{\mathrm{LN}}(x)\,\mathrm{d}x = e^{i\mu + \frac{1}{2} i^2 \sigma^2}. \tag{2}$$

Let $\pi_n(x)$ be the n-th degree polynomial

$$\pi_n(x) = \sum_{k=0}^{n} c_{n,k} x^k, \tag{3}$$

where $c_{n,n} \neq 0$. The polynomials $\{\pi_n(x)\}$, $n \in \mathbb{N}$, are said to be orthogonal with respect to the general lognormal density if

$$\int_0^\infty \pi_j(x)\pi_k(x) f_{\mathrm{LN}}(x)\mathrm{d}x = h_j\delta_{jk}, \quad j,k \in \mathbb{N}, \tag{4}$$

where $h_j = \sum_{i=0}^{j}\sum_{k=0}^{j} c_{j,i}c_{j,k}\nu_{i+k}$ is a normalization factor. The symbol δ_{jk} is the Kronecker delta symbol, which is defined as $\delta_{jk} = 1$ if $j = k$ and zero otherwise.

Due to (2) any arbitrary moment of $f_{\mathrm{LN}}(x)$ exists. Thus, functions x^i ($i \in \mathbb{N}$) belong to the space of square integrable functions with respect to the weight function $f_{\mathrm{LN}}(x)$. There exists a unique set of orthogonal polynomials $\{\pi_n(x)\}$ which admits the explicit determinant representation $\pi_n(x) = (\Delta_n)^{-1}\Delta_n(x)$ [10, Th. 2.1.1], where $\Delta_n(x)$ denotes the n-th degree polynomial

$$\Delta_n(x) = \begin{vmatrix} \nu_0 & \cdots & \nu_{n-1} & \nu_n \\ \vdots & \ddots & \vdots & \vdots \\ \nu_{n-1} & \cdots & \nu_{2n-2} & \nu_{2n-1} \\ 1 & \cdots & x^{n-1} & x^n \end{vmatrix}, \tag{5}$$

and the constant Δ_n is formed by deleting the last row and column from $\Delta_n(x)$, i.e. $\Delta_n = |\nu_{i+j}|_{i,j=0,\cdots,n-1}$ with $\Delta_0 = 1$. The determinant (5) can be expanded using the cofactors with respect to the last row. Then $\pi_n(x)$ becomes

$$\pi_n(x) = \sum_{k=0}^{n}(-1)^{n+k}\frac{\Delta_{n,k}}{\Delta_n}x^k, \tag{6}$$

where the cofactor $\Delta_{n,k}$ with respect to x^k is obtained by removing the $(k+1)$-th column and the last row from (5), $\Delta_{0,0} = 0$ and $\Delta_{1,0} = 1$. After comparing (6) with (3), we obtain

$$c_{n,k} = (-1)^{n+k}\frac{\Delta_{n,k}}{\Delta_n}, \quad k = 0,\cdots,n. \tag{7}$$

Let us calculate an explicit expression for $c_{n,k}$. First, by the definition of $\Delta_{n,k}$ and Δ_n it is observed that $c_{n,n} = 1$, i.e. the orthogonal polynomials (6) are monic[1]. Furthermore, after inserting (2) into $\Delta_{n,k}$ and denoting $q = e^{\sigma^2}$, we obtain (see Appendix for details)

$$\Delta_{n,k} = E(n)\frac{\nu_n\prod_{i=0}^{n-1}\prod_{j=i+1}^{n}(q^j - q^i)}{\nu_k\prod_{j=k+1}^{n}(q^j - q^k)\prod_{i=0}^{k-1}(q^k - q^i)}, \tag{8}$$

where $E(n) = e^{n(n-1)\mu + \frac{\sigma^2}{6}n(2n^2-3n+1)}$. For $k = n$, $\Delta_n = \Delta_{n,n}$ and (8) becomes

$$\Delta_n = E(n)\prod_{i=0}^{n-2}\prod_{j=i+1}^{n-1}\left(q^j - q^i\right). \tag{9}$$

After substituting (8) and (9) into (7) we find that

$$c_{n,k} = (-1)^{n+k}e^{(n-k)\mu}q^{(n-1/2)(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{10}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$ is the generalized binomial coefficient.

[1]Monic polynomial is defined as the polynomial where the coefficient of highest degree term is unity.

III. Approximation to the Density of Product of Nakagami-m Random Variables

A. General Framework

According to the CLT, the distribution of a product of RVs can be approximated by a lognormal distribution when the number of RVs is large. Motivated by this, we choose $f_{\mathrm{LN}}(x)$ as an initial approximation for the distribution of a product of RVs in the context of the moment based density approximation, which is derived in [9]. Let $M(k)$ denote the k-th moment of a density function $f(x)$, the N-th order approximation to $f(x)$ reads

$$f(x) \simeq f_N(x) = c_w f_{\mathrm{LN}}(x)\sum_{i=0}^{N}\eta_i\pi_i(x), \tag{11}$$

where the constant $\eta_i = \frac{1}{h_i}\sum_{k=0}^{i}c_{i,k}M(k)$. Denote the k-th moments of $f_N(x)$ as

$$M_N(k) = \int_0^\infty x^k f_N(x)\,\mathrm{d}x = c_w\sum_{i=0}^{N}\eta_i\sum_{j=0}^{i}c_{i,j}\nu_{j+k}, \tag{12}$$

where $c_w = \frac{1}{M_N(0)}$ normalizes $f_N(x)$ to unity. Note that the first N moments of the approximation (11) are matched with the corresponding moments of $f(x)$ [9].

Alternatively, (11) can be rearranged by combining the coefficients of $\pi_i(x)$ with the same power of x, resulting in $f_N(x) = c_w f_{\mathrm{LN}}(x)\sum_{i=0}^{N}\xi_i x^i$ where $\xi_j = \sum_{k=j}^{N}c_{k,j}\eta_k$. After direct integration the approximated CDF attains the form

$$F(x) \simeq c_w\sum_{i=0}^{N}\xi_i\nu_i\Phi\left(\frac{\log(x)-\mu}{\sigma} - i\sigma\right), \tag{13}$$

where $\Phi(\cdot)$ is the CDF of a standard normal RV. The CCDF $\bar{F}(x) = 1 - F(x)$ is approximated by replacing $F(x)$ with the expression (13).

Due to the orthogonality of the set $\{\pi_i(x)\}$, addition of the k-th term $f_{\mathrm{LN}}(x)\eta_k\pi_k(x)$ to the $(k-1)$-th order approximation $f_{k-1}(x)$ does not alter the first $(k-1)$ moments of $f_{k-1}(x)$. Therefore, to find a suitable order N of approximation, we propose an iterative construction for the moment-based approximation (11): given a set $\{x \in \mathbf{x}\}$ of interest, evaluate the maximum relative improvement of the k-th order approximation $\tau = \max_{x\in\mathbf{x}}\left|\frac{f_k(x)-f_{k-1}(x)}{f_{k-1}(x)}\right| = \max_{x\in\mathbf{x}}\left|\frac{f_{\mathrm{LN}}(x)\eta_k\pi_k(x)}{f_{k-1}(x)}\right|$. If τ is less than a pre-defined threshold τ_T, use $f_k(x)$ as the approximation for $f(x)$, otherwise continue to evaluate the $(k+1)$-th term until $\tau < \tau_T$ or exceeding a maximum order $N_{\max}$.

B. Moment Based Approximation to Distribution of Product of Nakagami-m RVs

Let RV $P = \prod_{i=1}^{K}R_i$ be a product of K Nakagami-m RVs each with the PDF

$$f_{R_i}(x) = \frac{2m_i^{m_i}x^{2m_i-1}}{\Omega_i^{m_i}\Gamma(m_i)}\exp\left(-\frac{m_i}{\Omega_i}x^2\right), \quad x \in (0,\infty). \tag{14}$$

We first consider the case when the variables R_i are correlated with each other. In the literature, there exists different representations for the joint PDF of multivariate Nakagami-m

RVs, which are either limited in parameter values or cross-correlation structures. In this paper, we adopt a recent result derived in [11], which gives the joint PDF as a single integral. In the proposed approach, the RVs R_i are assumed to have the same fading parameter m and the joint PDF is valid for integer and half-integer values of m. The power cross-correlation coefficient between R_i^2 and R_j^2 is of the form

$$\rho_{R_i^2,R_j^2} = \frac{\mathbb{E}[R_i^2 R_j^2] - \mathbb{E}[R_i^2]\mathbb{E}[R_j^2]}{\sqrt{\mathrm{Var}[R_i^2]\mathrm{Var}[R_j^2]}} = \lambda_i^2 \lambda_j^2. \quad (15)$$

Note that the derived approximation framework is not limited by the above multivariate model.

Based on the joint PDF [11, eq. (20)], the k-th moment of the RV P is calculated by using [12, eq. (6.643/2)] as

$$M(k) = \frac{\Gamma(m+k/2)^K}{m^{Kk/2}\Gamma(m)^{K+1}} \prod_{i=1}^{K} \left[\Omega_i(1-\lambda_i^2)\right]^{\frac{k}{2}} \times \int_0^\infty t^{m-1} e^{-t} \prod_{i=1}^{K} {}_1\mathrm{F}_1\left(-\frac{k}{2}, m, \frac{\lambda_i^2 t}{\lambda_i^2 - 1}\right) dt, \quad (16)$$

where ${}_1\mathrm{F}_1(a,b,c)$ denotes the Kummer confluent hypergeometric function. The parameters μ and σ^2 of $f_{\mathrm{LN}}(x)$ can be obtained by equating them with the mean and variance of $\log(P)$ respectively as

$$\mu = \sum_{i=1}^{K} \mathbb{E}[\log(R_i)] = \frac{1}{2}\sum_{i=1}^{K}\left[\Psi_0(m) - \log\left(\frac{m}{\Omega_i}\right)\right], \quad (17)$$

$$\sigma^2 = \sum_{i=1}^{K} \mathrm{Var}[\log(R_i)] + 2\sum_{i<j} \mathrm{Cov}[\log(R_i), \log(R_j)]$$
$$= \frac{1}{4}\sum_{i=1}^{K}\Psi_1(m) + 2\sum_{i<j}\left(\int_0^\infty \frac{I_i(t) I_j(t) t^{m-1}}{4\Gamma(m)e^t} dt - \zeta_i\zeta_j\right), \quad (18)$$

where $\Psi_n(x)$ is the polygamma function that is defined as the $(n+1)$-th derivative of the logarithm of $\Gamma(x)$, and $\zeta_i = \frac{1}{2}\left[\Psi_0(m) - \log\left(\frac{m}{\Omega_i}\right)\right]$. Here the function $I_i(t)$ is

$$I_i(t) = \Psi_0(m) + \log\left(\frac{\Omega_i(1-\lambda_i^2)}{m}\right) - {}_1\mathrm{F}_1{}'\left(a, m, \frac{\lambda_i^2 t}{\lambda_i^2 - 1}\right)\Bigg|_{a=0},$$

where ${}_1\mathrm{F}_1{}'(a,b,c)$ refers to the derivative of ${}_1\mathrm{F}_1(a,b,c)$ with respect to the parameter a.

Next consider the case where the RVs R_i are independent. We notice that with $\lambda_i = 0$, the joint PDF [11, eq. (20)] is in the form of a product of individual PDFs of R_i ($i = 1, 2, \cdots, K$), which implies statistical independence. Thus, we let $\lambda_i = 0$ and calculate the moments of P from (16)

$$M(k) = \prod_{i=1}^{K} \frac{\Gamma(m+k/2)}{\Gamma(m)}\left(\frac{\Omega_i}{m}\right)^{k/2}. \quad (19)$$

While the calculation of μ is not affected by the independency and it is given by expression (17), the variance σ^2 is reduced to the form

$$\sigma^2 = \frac{1}{4}\sum_{i=1}^{K}\Psi_1(m) \quad (20)$$

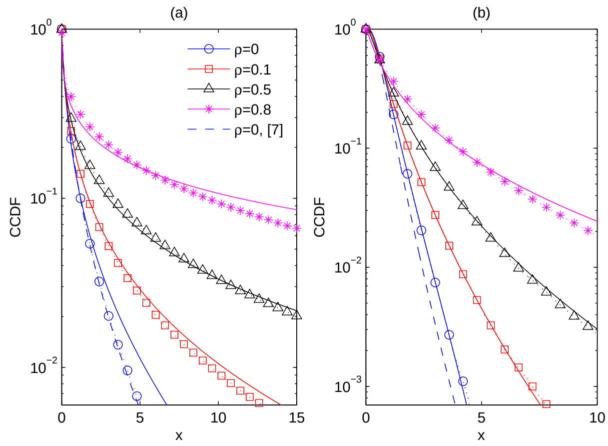

Fig. 1. CCDF of products of six Nakagami-m RVs when $\Omega = 1$ and $\tau_T = 10^{-4}$. (a) $m = 1$; (b) $m = 4$. Solid lines: approximative PDFs; markers: simulated PDFs.

with all the covariances $\mathrm{Cov}[\log(R_i), \log(R_j)]$ in (18) equal to zero. It is noted that (19) and (20) agree with the results given by [1].

IV. Numerical Results

In Fig. 1, we compare simulations with the approximative CCDFs for the product of six Nakagami-m RVs with parameters $m = 1,\ 4$ and $\Omega = 1$, using $\tau_T = 10^{-4}$. Here we consider equal cross-correlations, i.e. $\rho_{R_i^2,R_j^2} = \rho$ ($i \neq j$), and calculate the parameters of the approximation using (16), (17) and (18) with $\rho = 0.1,\ 0.5$ and 0.8 for correlated Nakagami-m RVs. In case of independent RVs (with $\rho = 0$), equations (19), (17) and (20) are applied. When $m = 1$ the RVs R_i are Rayleigh distributed, while with $m = 4$ the R_i are approximately Rician distributed with the Rician κ-factor given by $\kappa = 6.46$ [5]. For each simulated CCDF curve, we generate 10^6 realizations of the Nakagami-m RVs R_i ($i = 1, \cdots, 6$) using Sim's method [13]. As a comparison, we also plot the approximative CCDFs calculated from [7, eq. (36)] for the cases with independent Nakagami-m RVs. Fig. 1(b) shows that differences between the approximative CCDFs and simulations are less than 10^{-3} when $\rho = 0$ and 0.1, and less than 10^{-2} when $\rho = 0.5$. In addition, when $m = 4$ and $\rho = 0$, the proposed approximation yields improved accuracy compared with the results given in [7]. However, as $m = 1$, the approximations with the same correlation coefficients are noticeably inaccurate due to slow convergence of the series in (13). Note that in both cases the approximative CCDFs with $\rho = 0.8$ deviate from the simulations since the CLT fails under high correlation condition.

Table I summarizes the MSE $\epsilon^2 = \int_0^\infty |F^*(x) - F(x)|^2\, dF(x)$ between the approximation $F(x)$ and the empirical distribution function $F^*(x)$ of the product P as a function of the number of RVs K. When K is small ($K \leq 10$), Table I shows that the approximated CDF achieves considerably smaller MSE in low-correlation cases ($\rho = 0.1$ and 0.5) compared with the independent case ($\rho = 0$). As K increases, the accuracy for the independent case becomes superior to

TABLE I
MEAN SQUARE ERROR ϵ^2 OF APPROXIMATIVE DISTRIBUTION

K	2	4	6	8	10	12	14	16	18	20
$m=1$, $\rho=0$	1.14e-3	1.09e-3	6.28e-4	3.78e-4	2.72e-4	2.02e-4	1.67e-4	1.45e-4	1.28e-4	1.14e-4
$m=1$, $\rho=0.1$	1.06e-3	5.00e-4	1.05e-4	1.66e-5	1.73e-5	5.35e-5	1.03e-4	1.72e-4	2.40e-4	3.21e-4
$m=1$, $\rho=0.5$	9.99e-4	1.68e-4	2.10e-5	9.54e-5	2.07e-4	3.18e-4	4.16e-4	5.36e-4	6.24e-4	6.98e-4
$m=4$, $\rho=0$	8.13e-6	2.29e-5	2.43e-5	1.64e-5	2.24e-5	3.15e-5	4.34e-5	4.78e-5	5.14e-5	5.02e-5
$m=4$, $\rho=0.1$	8.31e-6	1.16e-5	2.65e-6	2.33e-6	1.77e-5	4.95e-5	8.93e-5	1.15e-4	1.31e-4	1.47e-4
$m=4$, $\rho=0.5$	7.03e-6	1.34e-5	7.44e-6	9.73e-6	1.74e-5	2.48e-5	3.36e-5	4.21e-5	4.99e-5	5.49e-5

those of correlated ones. When $m = 4$, all approximations are much more accurate compared with the corresponding Rayleigh cases ($m = 1$), and the MSEs ϵ^2, except for those where $K > 16$ and $\rho = 0.1$, is less than 10^{-4} in case of independent RVs or when correlation is low.

To understand the rate of convergence of (11) and (13) as N increases, we adopt the error analysis method using higher moment comparisions as suggested in [14]. The utilized method is straightforward to compare the relative differences of approximation errors under different parameter settings. Specifically, the deviations between the approximation $f_N(x)$ and the exact PDF $f(x)$ is given by the moment series expansion [14, eq. (2)] as $(-1)^{N+1}\frac{M(N+1)-M_N(N+1)}{(N+1)!}f_N^{(N+1)}(x)+O\left(f_N^{(N+2)}(x)\right)$, where the coefficient of the leading order term is used to quantify the approximation error. Numerical results show that the magnitude of the coefficient $\left|\frac{M(N+1)-M_N(N+1)}{(N+1)!}\right|$ is much larger when $m = 1$ compared with the case when $m = 4$. It evidences that the approximation (11) is less accurate in deep fading conditions due to the increased deviations between unmatched higher order moments.

V. CONCLUSION

Knowledge of the distributions of product of random variables is important for understanding the performance of various communication systems. In this work, we first derived a closed-form expression for the orthogonal polynomials associated with general lognormal density. The derived result was subsequently applied in approximating the distribution of the product of random variables. As an example, we calculated closed-form approximations for the distributions of product of Nakagami-m variates. Under light fading conditions, the resulting expressions achieve a good trade-off between computation complexity and approximation accuracy for small cross-correlations. It is still an open problem to find a closed-form (approximated) distribution for the product of correlated Nakagami-m RVs under deep fading condition.

APPENDIX

We note that the determinant

$$\Delta_{n,k} = \left| e^{(i+j)\mu + \frac{(i+j)^2}{2}\sigma^2} \right|_{i=0,\cdots,n-1;\; j=0,\cdots,n;\; j\neq k} \tag{21}$$

remains unchanged by factoring out the term $e^{i\mu+i^2\sigma^2/2}$ from the i-th row and the term $e^{j\mu+j^2\sigma^2/2}$ from the j-th column. Namely, we have

$$\Delta_{n,k} = E(n)\frac{\nu_n}{\nu_k}\left| e^{ij\sigma^2} \right|_{i=0,\cdots,n-1;\; j=0,\cdots,n;\; j\neq k} \tag{22}$$

$$= E(n)\frac{\nu_n \prod_{i=0}^{n-1}\prod_{j=i+1}^{n}(q^j - q^i)}{\nu_k \prod_{j=k+1}^{n}(q^j - q^k)\prod_{i=0}^{k-1}(q^k - q^i)}, \tag{23}$$

where $E(n) = e^{n(n-1)\mu + \frac{\sigma^2}{6}n(2n^2-3n+1)}$ and the second equality follows from the fact that $\left|e^{ij\sigma^2}\right|$ is a Vandermonde determinant.

ACKNOWLEDGMENT

This work was supported in part by the Academy of Finland (grant 133562) and in part by the Finnish Funding Agency for Technology and Innovation (TEKES).

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