Volume of Metric Balls in High-Dimensional Complex Grassmann Manifolds

Renaud-Alexandre Pitaval, Lu Wei, Olav Tirkkonen, and Jukka Corander

Abstract—Volume of metric balls relates to rate-distortion theory and packing bounds on codes. In this paper, the volume of balls in complex Grassmann manifolds is evaluated for an arbitrary radius. The ball is defined as a set of hyperplanes of a fixed dimension with reference to a center of possibly different dimension, and a generalized chordal distance for unequal dimensional subspaces is used. First, the volume is reduced to a one-dimensional integral representation. The overall problem boils down to evaluating a determinant of a matrix of the same size as the subspace dimensionality. Interpreting this determinant as a characteristic function of the Jacobi ensemble, an asymptotic analysis is carried out. The obtained asymptotic volume is moreover refined using moment-matching techniques to provide a tighter approximation in finite-size regimes. Lastly, the pertinence of the derived results is shown by rate-distortion analysis of source coding on Grassmann manifolds.

Index Terms—Grassmann manifold, volume, metric ball, high-dimension, rate-distortion analysis, source coding.

I. INTRODUCTION

A Grassmann manifold is the collection of subspaces of a given dimension in a vector space. Grassmann manifolds find many applications due to their relation to eigenspaces of matrices, see e.g. in [1–3]. In the context of multi-antenna transmissions, complex Grassmannian codes have notably been used for non-coherent space-time coding [4–6] and channel-aware precoding [7, 8].

A metric ball in the Grassmann manifold is analogous to a spherical cap on a sphere and evaluating its volume is critical

Renaud-Alexandre Pitaval was with the Department of Communications and Networking and the Department of Mathematics and Systems Analysis, Aalto University, FI-00076 Aalto, Finland. He is now with Huawei Technologies Sweden, Kista SE-164 94, Sweden (email: renaud-alexandre.pitaval@alumni.aalto.fi).

Lu Wei was with the Department of Mathematics and Statistics, University of Helsinki, FI-00014 Helsinki, Finland. He is now with the School of Engineering and Applied Sciences, Harvard University, MA 02138 Cambridge, USA (email: lwei@g.harvard.edu).

Olav Tirkkonen is with the Department of Communications and Networking, Aalto University, FI-00076 Aalto, Finland (email: olav.tirkkonen@aalto.fi).

Jukka Corander is with the Department of Mathematics and Statistics, University of Helsinki, FI-00014 Helsinki, Finland, and the Department of Biostatistics, University of Oslo, 0317 Oslo, Norway (email: jukka.corander@helsinki.fi).

for several performance measures of Grassmannian codes. When constructing codes as point-sets with the largest possible minimum distance, the volume of a metric ball directly applies to packing bounds on codes [9, 10]. Moreover, rate-distortion theory on Grassmann manifolds has been extensively applied to channel quantization analysis of precoded MIMO systems [11–14]. In this source-coding context, the volume of metric balls is closely related to the cumulative distribution of quantization errors for a uniformly distributed source.

Computing the volume of a metric ball in manifolds is recognized to be a difficult task. For the Grassmann manifold, this was addressed in several previous works. The volume of a metric ball was derived for Grassmannian line packing in [8], whereas asymptotic evaluations for arbitrary subspace dimension were provided in [9], and in [15] for balls with centers of different dimension. A small ball approximation was considered in [10] which was later refined to be exact in [13] for balls of radius less than one. More recently, an exact volume formula for packing (2D) planes has been derived with application to massive MIMO [16]. On the one hand, in large dimensions, the range of validity of the small ball result is becoming limited to exponentially large codebooks. On the other hand, the known high-dimensional asymptotics show slow convergence, providing only asymptotic scaling laws.

The goal of this paper is to provide an accurate but simple volume approximation for regimes not covered in the literature. The obtained asymptotic form is complementary to the result in [13] and with faster convergence than in [9, 15]; it is however limited to the complex case and its derivation is not straightforwardly generalizable to real Grassmannians. We consider the case of possibly unequal dimension between elements in the ball and the center of the ball as in [13, 15]. Related problems with subspaces of non-equal dimensions arise for example in [6, 17–19]. We start by discussing generalizations of the well-known Grassmann chordal distance [1] for subspaces of different dimensions, as well as relevant symmetries of the volume of ball with our chosen distance. Then, from the known representation of the volume of a ball as a multi-dimensional integration over principal angles [9, 13, 20, 21], we reduce the problem to a one-dimensional integral related to a Fourier transform. The formulation is valid for any radius, and reduces the problem to the evaluation of a determinant that only depends of the dimension parameters. Accordingly, this integral can be computed exactly with fixed parameters and we provide several examples illustrating its versatility. The exact volumes have different polynomial representations in different ranges of the integer part of the squared radius. Radius less than one [13] is itself a specific regime. From this, it can be

anticipated that even though an exact generic formula could be derived, it will most probably be a linear combination of special functions similarly as in [16]. Such representations, may not be convenient from application perspective, as for example one often needs to invert the volume to compute bounds on codes.

To provide a good and relevant approximation for large-dimensional Grassmann manifolds, an asymptotic analysis is further carried out. As all the dimension-related parameters are concentrated in a determinant inside the one-dimensional integral reformulation, the problem reduces to a study of the asymptotic behavior of the determinant. This determinant is the partition function of the so-called time-dependent Jacobi ensemble. Interpreting it as a characteristic function of a linear spectral statistic, its asymptotic Gaussianity can be leveraged from random matrix theory [22]. This leads to an asymptotic formulation of the volume of a metric ball in terms of Gauss error functions, which however appears loose in some finite regimes. We provide a finite-size correction to the asymptotic formula via the exact moments of the considered linear statistic, leading to a tighter volume approximation while preserving the simplicity of the asymptotic form.

Finally, the derived asymptotic formula of the volume of a metric ball (with its finite-size correction) is applied to rate-distortion theory on Grassmann manifolds. With increasing dimensions, it is shown to provide a good estimate of the rate-distortion trade-off in a source coding problem. The correction is notable compared with using a small ball approximation outside of its regime of validity.

The rest of this paper is organized as follows. Pertinent definitions and properties are given in Section II. The volume of metric ball is reduced in Section III to a one-dimensional integral leading to some examples of exact derivations. In Section IV, the asymptotic behavior of volume of metric ball is derived and corrected by moment-matching techniques for finite-size applications. In Section V, the derived expression is applied to source-coding on the Grassmann manifold. The paper is concluded in Section VI.

II. PRELIMINARIES

A. Grassmann Manifolds and Chordal Distance

The complex Grassmann manifold $\mathcal{G}_{n,p}^{\mathbb{C}}$ is the collection of p-dimensional subspaces in an ambient n-dimensional complex vector space $\mathbb{C}^n$. This is a homogeneous space of the unitary group $\mathcal{U}_n$ as there is a left action of $\mathcal{U}_n$ on $\mathcal{G}_{n,p}^{\mathbb{C}}$ that acts transitively. A plane $P \in \mathcal{G}_{n,p}^{\mathbb{C}}$ can be described by infinitely many orthogonal bases leading to a non-unique semi-unitary matrix representation $\boldsymbol{P} \in \mathbb{C}^{n \times p}$, such that $\boldsymbol{P}^\dagger \boldsymbol{P} = \boldsymbol{I}_p$, where $()^\dagger$ is the conjugate transpose of a matrix.

There are several possible choices to define a distance on the Grassmann manifold. We consider the chordal distance [1, 2] which is related to an embedding of the Grassmannian into an Euclidean sphere, and has been prominently used in the literature [4, 7, 13, 23, 24]. The chordal distance is well-defined between subspaces with equal dimensions. However, one can find slight variations for its generalization to subspaces of unequal dimensions. We will use the same definition as

e.g. [13, 17, 19] arising from the concept of the principal angles; discussions on its theoretical foundation can be found in the recent work [3].

In [3], it is shown that any measure of distance that only depends on the relative position between two subspaces must be a function of the principal angles. The collection of principal angles provides the relative position between subspaces which is transitive under the action of the unitary group. However, compressing this "vector-like distance" to a classical scalar distance d, one loses transitivity. Grassmann manifolds are not in general two-point homogeneous spaces [25]: one cannot necessarily find a unitary mapping between two pairs of equidistant points, i.e., a pair (P, Q) cannot always be mapped to a pair (P', Q') even if $d(P, Q) = d(P', Q')$.

Consider two integers p and q satisfying $p, q \leq n$ and $m = \min(p, q)$. Given $P \in \mathcal{G}_{n,p}^{\mathbb{C}}$ and $Q \in \mathcal{G}_{n,q}^{\mathbb{C}}$ with respective orthonormal bases $\boldsymbol{P} \in P$, $\boldsymbol{Q} \in Q$ one can define m principal angles [20] between these two subspaces. We denote the principal angles by $\theta_1 \ldots \theta_m \in [0, \frac{\pi}{2}]$. They are independent of the choice of coordinates and can be computed via the singular value decomposition of $\boldsymbol{P}^\dagger \boldsymbol{Q}$ whose singular values are $\{\cos \theta_i\}_{i=1}^m$. The considered square chordal distance is given by

$$
\begin{aligned}
d_c^2(P, Q) &= \sum_{k=1}^{\min(p,q)} \sin^2(\theta_k) \\
&= \min(p, q) - \sum_{k=1}^{\min(p,q)} \cos^2(\theta_k) \\
&= \min(p, q) - \|\boldsymbol{P}^\dagger \boldsymbol{Q}\|_F^2. \quad (1)
\end{aligned}
$$

B. Relation to Other Chordal Distances

It is noted in [3] that d_c gives a notion of distance in a sense of a distance from a point to a set, but it does not give a metric between subspaces of different dimensions since two distinct subspaces of different dimensions can have distance zero. Nevertheless, a simple variation leading to a properly-defined metric function is given in [3]. This is obtained by assigning $|p - q|$ additional principal angles with value $\frac{\pi}{2}$ for the dimensions mismatched between P and Q. One can then define

$$
d_{c\#}^2(P, Q) = \max(p, q) - \sum_{k=1}^{\min(p,q)} \cos^2(\theta_k). \quad (2)
$$

The chordal distance has also been generalized to subspaces of different dimensions from their corresponding projection operators in [6]. This corresponds to the Euclidean distance of a spherical embedding into $\mathcal{S}^{n^2-1}(\frac{\sqrt{n}}{2})$, each Grassmannian being itself embedded in a different cross-sectional sphere, specifically the p-dimensional subspaces to $\mathcal{S}^{n^2-2}(\sqrt{\frac{p(n-p)}{n}})$ and the q-dimensional subspaces to $\mathcal{S}^{n^2-2}(\sqrt{\frac{q(n-q)}{n}})$ [1, 10]. This gives a proper metric which can be expressed in term of principal angles as

$$
\begin{aligned}
d_{c*}^2(P, Q) &= \|\boldsymbol{P}\boldsymbol{P}^\dagger - \boldsymbol{Q}\boldsymbol{Q}^\dagger\|_F^2 \\
&= p + q - 2 \sum_{k=1}^{\min(p,q)} \cos^2(\theta_k). \quad (3)
\end{aligned}
$$

Finally, we suggest a third metric for subspaces of unequal dimensions which provides a slight reduction in the dimension of the embedding. The main observation is that all Grassmannians in n dimensions can be embedded in a single sphere $\mathcal{S}^{n^2-2}$. This holds in fact for any flag manifold as well [26]. To obtain this, one must detrace the projectors and rescale them as $\tilde{P} = \sqrt{\frac{n}{p(n-p)}}(PP^\dagger - \frac{p}{n}I)$ and $\tilde{Q} = \sqrt{\frac{n}{q(n-q)}}(QQ^\dagger - \frac{q}{n}I)$, then $\tilde{P}$ and $\tilde{Q}$ lie on the same unit sphere $\mathcal{S}^{n^2-2}$. The corresponding Euclidean distance is

$$
\begin{aligned}
d_{c\star}^2(P,Q) &= \|\tilde{P} - \tilde{Q}\|_F^2 \\
&= K_1 - K_2\|P^\dagger Q\|_F^2 \\
&= K_1 - K_2 \sum_{k=1}^{\min(p,q)} \cos^2(\theta_k)
\end{aligned}
\tag{4}
$$

with $K_1 = 2 + 2\sqrt{\frac{pq}{(n-p)(n-q)}}$ and $K_2 = \frac{2n}{\sqrt{pq(n-p)(n-q)}}$.

All the distances d_c, $d_{c\#}$, $\frac{1}{\sqrt{2}}d_{c*}$ and $\frac{1}{\sqrt{K_2}}d_{c\star}$ reduce to the classical definition of chordal distance for $p = q$. In the rest of this paper, we will keep the definition of the distance d_c as in (1) due to its compactness in term of principal angles and for consistency with [13, 17, 19]. Corresponding results can be easily extended to the other distances considered above since d_c includes the main information of interest, and differs only by constant factors from the other metrics.

C. Metric Ball and Normalized Volume

Define the metric ball of q-dimensional subspaces with distance at most r from the p-dimensional center $P \in \mathcal{G}_{n,p}^{\mathbb{C}}$ by

$$
B_{P,q}(r) = \left\{ Q \in \mathcal{G}_{n,q}^{\mathbb{C}} : \quad d_c(P,Q) \leq r \right\}.
\tag{5}
$$

The ball $B_{P,q}(r)$ is a subset of $\mathcal{G}_{n,q}^{\mathbb{C}}$ though it is defined with reference to a point in $\mathcal{G}_{n,p}^{\mathbb{C}}$.

We consider the uniform measure μ on $\mathcal{G}_{n,q}^{\mathbb{C}}$ inherited from the invariant Haar measure on the unitary group $\mathcal{U}_n$. For any measurable set $\mathcal{S} \subset \mathcal{G}_{n,p}^{\mathbb{C}}$ and any $U \in \mathcal{U}_n$, the uniform measure satisfies

$$
\mu(U\mathcal{S}) = \mu(\mathcal{S}).
\tag{6}
$$

The quantity $\mu(B_{P,q}(r))$ is independent of the center P except for its dimensionality p, and we will simply write $\mu(B_{p,q}(r))$ or even $\mu(B(r))$ when there is no ambiguity.

The invariant measure can be interpreted as a normalized volume

$$
\mu(B_{p,q}(r)) = \frac{\text{vol}(B_{p,q}(r))}{\text{vol}(\mathcal{G}_{n,q}^{\mathbb{C}})}
\tag{7}
$$

where with our choice of distance the corresponding volume of the Grassmann manifold is [27]

$$
\text{vol}(\mathcal{G}_{n,q}^{\mathbb{C}}) = \pi^{q(n-q)} \prod_{i=1}^{q} \frac{(q-i)!}{(n-i)!}.
\tag{8}
$$

It is shown in [3] that distances constructed from principal angles as considered here are independent of the dimension n of the ambient space. The choice of ambient space, however, has an impact on the maximum possible value of d_c and thus the range of μ. From [13, Lem. 2] it can be deduced that the
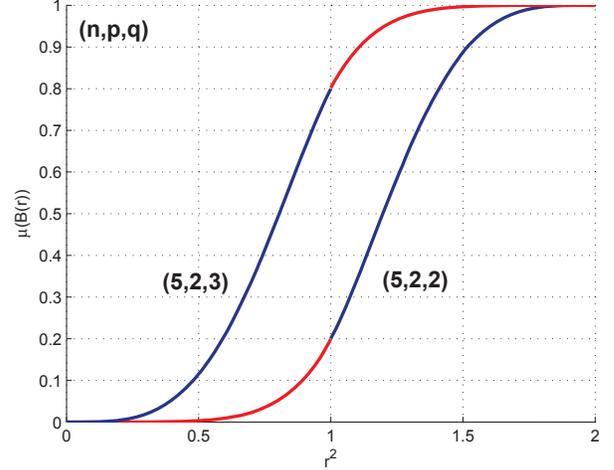

Fig. 1: Illustration of the symmetry (11) between $\mu(B_{p,q}(r))$ and $\mu(B_{p,n-q}(r))$: one curve is the rotation of the other by $180°$ around the median.

chordal distance must satisfy $d_c(P,Q) \leq d_{\max}$ with $d_{\max}^2 = \min(p,q,n-p,n-q)$. As a consequence $\mu(B_{p,q}(r))$ is defined on the range $[0, d_{\max}]$, its maximum is $\mu(B_{p,q}(d_{\max})) = 1$, and the volume depends on n. The dependence on the choice of ambient space can be understood by considering e.g. the problem of packing 2D real planes. It is impossible to find two fully orthogonal planes in $\mathbb{R}^3$ (i.e. planes that intersect only in the zero vector), while it is possible in $\mathbb{R}^4$.

D. Symmetries and Complementary Balls

Without loss of generality we will assume in this paper that $p \leq q < n$ and $p + q \leq n$, implying that $p \leq n/2$. Results in other parameter ranges can be reproduced using the chordal distance and the canonical isomorphism $\mathcal{G}_{n,p}^{\mathbb{C}} \cong \mathcal{G}_{n,n-p}^{\mathbb{C}}$:

$$
\mu(B_{p,q}(r)) = \mu(B_{q,p}(r))
\tag{9}
$$
$$
\mu(B_{p,q}(r)) = \mu(B_{n-p,n-q}(r)).
\tag{10}
$$

These symmetries were used for volume computations in [13] as for $p+q \geq n$, one can evaluate $\mu(B_{p',q'}(r))$ with $p' = n-p$ and $q' = n-q$ satisfying $p' + q' \leq n$.

An additional symmetry than can be used for extending results from one Grassmannian to another with a different range of radius values is as follows. Let p, q satisfy $p \leq q < n$ and $p + q \leq n$. Then we have

$$
\mu\left(B_{p,q}(r)\right) = 1 - \mu\left(B_{p,n-q}\left(\sqrt{p - r^2}\right)\right).
\tag{11}
$$

The proof is in Appendix A. For the specific case $q = \frac{n}{2}$, one sees that the volume is a symmetric function in $r = \sqrt{p/2}$ since $\mu(B(r)) = 1 - \mu(B(\sqrt{p - r^2}))$. As an example, combining (11) and the result of [13] directly leads to the following elementary exact evaluation of volume of balls for any radius with $n = 4$, $p = q = 2$:

$$
\mu(B(r)) = \begin{cases} \frac{1}{2}r^8 & \text{for } r \leq 1 \\ 1 - \frac{1}{2}(2 - r^2)^4 & \text{for } r \geq 1 \end{cases}.
\tag{12}
$$

The symmetry (11) is illustrated in Figure 1. The exact evaluation for $p = q$ and $r \leq 1$ in [13] is highlighted in red. It can be directly used for computing the volume for $q = n - p$ and $r \geq \sqrt{p-1}$.

E. Sphere-Covering/Packing Bounds

When subspaces have the same dimension $p = q$, a direct application of the volume of a metric ball occurs in the evaluation of fundamental coding bounds. The Gilbert-Varshamov and Hamming bounds, derived from a sphere-covering and sphere-packing arguments, respectively, relate the code's cardinality to its minimum distance. Namely, for any distance δ, there exists a code $\mathcal{C} \subset \mathcal{G}_{n,p}^{\mathbb{C}}$ with cardinality $|\mathcal{C}|$ such that

$$\frac{1}{\mu(B(\delta))} \leq |\mathcal{C}| \tag{13}$$

while for any $(|\mathcal{C}|, \delta)$-code $\mathcal{C} \subset \mathcal{G}_{n,p}^{\mathbb{C}}$ one must have

$$|\mathcal{C}| \leq \frac{1}{\mu(B(\frac{\delta}{2}))}. \tag{14}$$

III. EXACT INTEGRAL FORMULATIONS

The volume of a metric ball in the Grassmann manifold can be expressed as a multivariate integration over principal angles. Here we reduce the problem to a one-dimensional integral related to a Fourier transform. It is assumed without loss of generality that $p \leq q < n$ and $p + q \leq n$ and complementary cases can be treated by symmetry. It is worth noting that as a consequence the range of radii of balls is $r \in [0, \sqrt{p}]$.

A. Multi-dimensional Integration

An integration of the volume element on the Grassmann manifold can be split in three parts including two densities on Stiefel manifolds that can be fully integrated. The overall calculation reduces then to an integral over the marginal distribution of the principal angles [9, 20]. With the cosines of the principal angles $c_i = \cos\theta_i$, $i = 1 \ldots p$, the volume of a metric ball $\mu(B(r))$ in complex Grassmann manifolds can be written as a p-dimensional integral of the form [13, 20, 21]

$$\mu(B(r)) = v_{n,p,q} \int_{\substack{0 \leq c_i \leq 1, \\ \sum_{i=1}^{p}(1-c_i^2) \leq r^2}} \Delta^2(\boldsymbol{c}^2) \prod_{j=1}^{p} c_j^{2(q-p)} \left(1 - c_j^2\right)^{n-p-q} dc_i^2, \tag{15}$$

where the normalization constant $v_{n,p,q}$ is given by

$$v_{n,p,q} = \prod_{j=1}^{p} \frac{\Gamma(n-j+1)}{\Gamma(j+1)\Gamma(n-q-j+1)\Gamma(q-j+1)}, \tag{16}$$

and

$$\Delta(\boldsymbol{c}) = \det\left(c_j^{2(i-1)}\right) = \prod_{1 \leq i < j \leq p} \left(c_i^2 - c_j^2\right) \tag{17}$$

denotes a Vandermonde determinant. We note here that the volume element is unique up to a scaling factor (which is included into the overall normalization) and the choice of distance affects only the domain of integration.

Applying the change of variables $x_j = 1 - c_j^2$, Eq. (15) simplifies to

$$\mu(B(r)) = v_{n,p,q} \int_{\substack{0 \leq x_i \leq 1, \\ \sum_{i=1}^{p} x_i \leq r^2}} \Delta^2(\boldsymbol{x}) \prod_{j=1}^{p} x_j^{n-p-q} (1-x_j)^{q-p} dx_j \tag{18}$$

where $\Delta(\boldsymbol{x}) = \det\left(x_j^{(i-1)}\right) = \prod_{1 \leq i < j \leq p} (x_i - x_j)$ similarly as in (17). The constraint $\sum_{i=1}^{p} x_i \leq r^2$ in the integral (18) presents the main challenge to obtain an explicit expression.

B. Time-dependent Jacobi Ensemble

To address the issue raised above, we rewrite the integral (18) by using an indicator function:

$$\mu(B(r)) = v_{n,p,q} \times$$
$$\int_0^{r^2} \int_{0 \leq x_i \leq 1} \delta\left(t - \sum_{j=1}^{p} x_j\right) \Delta^2(\boldsymbol{x}) \prod_{j=1}^{p} x_j^{n-p-q} (1-x_j)^{q-p} dx_j \, dt. \tag{19}$$

Here $\delta(\cdot)$ is the Dirac delta function, which admits the following Fourier representation

$$\delta(t - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(t-a)\nu} d\nu. \tag{20}$$

Note that a similar idea of using an indicator function was considered in [28, 29] for evaluating volumes in the unitary group.

Inserting (20) into (19) and performing the integration over t, we arrive at

$$\mu(B(r)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\nu} \left(1 - e^{ir^2\nu}\right) D_p(\nu) d\nu, \tag{21}$$

where

$$D_p(\nu) = v_{n,p,q} \times$$
$$\int \cdots \int_{0 \leq x_j \leq 1} \Delta^2(\boldsymbol{x}) \prod_{j=1}^{p} x_j^{n-p-q} (1-x_j)^{q-p} e^{-i\nu x_j} dx_j. \tag{22}$$

Comparing (18) to (21) and (22), we see that the reformulation amounts to eliminating the constraint $\sum_{j=1}^{p} x_j \leq r^2$ at the expense of introducing a deformation $e^{-i\nu x_j}$ in the p-dimensional integral (22). As such the main difficulty is now concentrated in evaluating $D_p(\nu)$ which is independent of the radius and only depends on dimension parameters. The integral in (22) is the partition function of the so-called time-dependent Jacobi ensemble [30], which is the classical Jacobi ensemble deformed by $e^{-i\nu \sum x_j}$.

C. One-dimensional Integral Formula and Exact Evaluations

It is possible to further simplify the volume formula to a one-dimensional integral. To proceed, we use the Andréief

integral identity [31, 32], see Appendix B, as well as the symmetry of the integrand to simplify the integral (22) to

$$D_p(\nu) = p! v_{n,p,q} \times$$
$$\det\left(\int_0^1 x^{i+j-2+n-p-q}(1-x)^{q-p}e^{-i\nu x}\,dx\right) \quad (23)$$
$$= p! v_{n,p,q} \det\left(B(\alpha,\beta)\ {}_1F_1(\alpha,\alpha+\beta;-i\nu)\right), \quad (24)$$

where the last equality is obtained by [33, Eq. 3.383] with

$$\alpha = i+j+n-p-q-1, \quad (25)$$
$$\beta = q-p+1. \quad (26)$$

Here

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (27)$$

is the Beta function and

$$_1F_1(\alpha,\beta;x) = \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\beta)_k k!}x^k \quad (28)$$

defines the hypergeometric function, where $(\alpha)_k = \Gamma(\alpha+k)/\Gamma(\alpha)$ is Pochhammer symbol. Note that the step from (22) to (23) using Andréief identity could not be generalized to compute volumes in real Grassmann manifolds. This is because the Vandermonde determinant in (22) is not squared in the real case [13].

Putting everything together, we obtain an integral representation of $\mu(B(r))$ for any radius,

$$\mu(B(r)) = \frac{p! v_{n,p,q}}{2\pi} \times$$
$$\int_{-\infty}^{\infty}\frac{i}{\nu}\left(1-e^{ir^2\nu}\right)\det\left(B(\alpha,\beta)\ {}_1F_1(\alpha,\alpha+\beta;-i\nu)\right)\,d\nu. \quad (29)$$

For the special case $p = q = n/2$, the above general result simplifies further to

$$\mu(B(r)) = \frac{p! v_{n,p,q}}{2\pi} \times$$
$$\int_{-\infty}^{\infty}\frac{e^{ir^2\nu}-1}{(i\nu)^{p^2+1}}\det\left(\Gamma(i+j-1)\left(1-e^{-i\nu}\sum_{k=0}^{i+j-2}\frac{(i\nu)^k}{k!}\right)\right)\,d\nu. \quad (30)$$

The main technical difficulty in the one-dimensional formulation (29) and (30) lies in computing a determinant of a $p \times p$ matrix. This is tractable and can be further carried out for specific small values of the parameters n, p and q. We list some of the results in Appendix C. The obtained expressions are verified to match Monte Carlo simulations in Fig. 2 for different values of n, p and q. Fig. 2 also includes the asymptotic approximation derived and discussed in the next section. As can be seen, balls of radius less than one cover a very limited portion of the space with large dimensions. For example with $(n, p, q) = (8, 4, 4)$ a ball with radius $r \leq 1$ covers a maximum of 0.000042% of the space which corresponds, according to the Gilbert-Varshamov bound, to codes with at least 2.4×10^4 elements, or equivalently 14.5 bits.
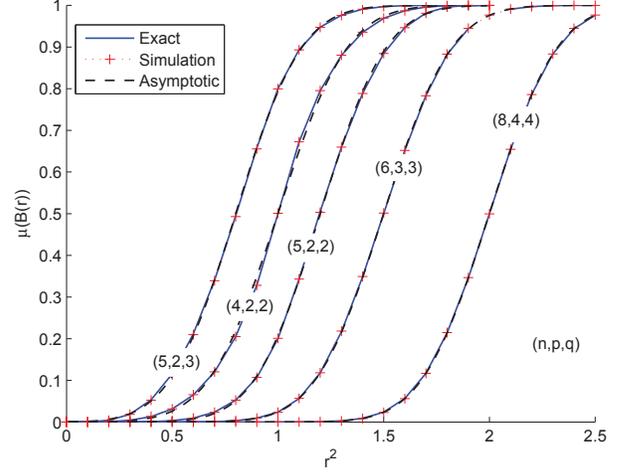

Fig. 2: Volume formulas (29) and (30) (Appendix C) for $\mu(B(r))$ versus simulations and asymptotic finite-size approximation (55).

IV. ASYMPTOTIC ANALYSIS

As shown in the previous section, it is possible, for any radius, to derive exactly the volume of metric balls with specific values of n, p, q. However, the resulting expressions are cumbersome in large dimensions. In this section, starting from the reformulation (21), the volume is analyzed through asymptotics of the determinant $D_p(\nu)$, providing good approximations for large-dimensional Grassmann manifolds. Again, we assume $p \leq q < n$ with $p + q \leq n$ and other cases can be treated by symmetry.

A. Asymptotic Volume via Random Matrix Theory

For reasons that will become clear later, we consider the linear transforms $y_j = 2x_j - 1$, $j = 1, \ldots, p$ in the integral (22). After calculating the Jacobians associated with the transforms, we have

$$\widetilde{D}_p(\nu) = e^{i\frac{\nu}{2}p}D_p(\nu) \quad (31)$$
$$= \tilde{v}_{n,p,q}\int\cdots\int_{-1\leq y_j\leq 1}\Delta^2(\boldsymbol{y})\times$$
$$\prod_{j=1}^{p}(1-y_j)^{q-p}(1+y_j)^{n-p-q}e^{-i\frac{\nu}{2}y_j}\,dy_j, \quad (32)$$

where

$$\tilde{v}_{n,p,q} = 2^{-p(n-p)}v_{n,p,q}. \quad (33)$$

One can interpret $\widetilde{D}_p(\nu)$ as the characteristic function of the random variable

$$Y = \sum_{j=1}^{p}\frac{y_j}{2} \quad (34)$$

over the so-called Jacobi ensemble with probability density

$$f(\boldsymbol{y}) = \tilde{v}_{n,p,q}\Delta^2(\boldsymbol{y})\prod_{j=1}^{p}(1-y_j)^{q-p}(1+y_j)^{n-p-q}. \quad (35)$$

In this form, the asymptotic behavior of the linear spectral statistic (34) is a well-investigated subject in random matrix theory. Specifically, by using the result [22, Th. 3.2] straightforward manipulations[1] show that (32) converges to

$$
\begin{aligned}
\widetilde{D}_p(\nu) &= \mathbb{E}\left[e^{-i\nu Y}\right] \\
&\simeq e^{-i\nu\left(\frac{n-2q}{4}\right)-\frac{\nu^2}{32}}
\end{aligned} \tag{36}
$$

in the regime

$$
n, p, q \to \infty, \quad \text{with fixed } q - p \text{ and } n - p - q. \tag{37}
$$

To wit, in the asymptotic regime (37) the random variable (34) follows a Gaussian distribution with mean and variance read off from (36) as

$$
\mathbb{E}\left[Y\right] = \frac{n-2q}{4}, \quad \mathbb{V}\left[Y\right] = \frac{1}{16}. \tag{38}
$$

This is a central limit theorem for the linear statistic (34) of the Jacobi ensemble (35). By the relation (31), we have

$$
D_p(\nu) \simeq e^{-i\nu\left(\frac{n+2p-2q}{4}\right)-\frac{\nu^2}{32}}. \tag{39}
$$

Inserting this into (21) an asymptotic representation of the volume of metric balls is obtained as

$$
\mu\left(B\left(r\right)\right) \simeq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{i}{\nu}\left(1 - e^{ir^2\nu}\right) e^{-i\nu\left(\frac{n+2p-2q}{4}\right)-\frac{\nu^2}{32}} \, d\nu. \tag{40}
$$

The imaginary part of the integrand is an odd function which integrates to zero. The real part is even, from which an asymptotic volume formula is obtained as

$$
\mu\left(B\left(r\right)\right) \simeq \frac{1}{2}\mathrm{erf}\left(2\sqrt{2}\alpha\right) - \frac{1}{2}\mathrm{erf}\left(2\sqrt{2}\left(\alpha - r^2\right)\right), \tag{41}
$$

where

$$
\alpha = \frac{1}{4}(n + 2p - 2q) \tag{42}
$$

and

$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt \tag{43}
$$

is the Gauss error function.

Although the derived volume formula (41) is asymptotically tight in the regime (37), it may not be very accurate when used at finite size. The asymptotic (36) is obtained by letting the size p of the product in the Jacobi ensemble (35) grow to infinity, while $n, q \to \infty$ keeping the exponents $a = q - p$ and $b = n - p - q$ fixed in (35). As will be observed below (see Figures 3 and 4), the convergence to the asymptotic distribution is slow when either $|a-b|$, a or b is large, leading to poor approximations with small p. This fact motivates us to find finite-size corrections to the asymptotic mean and variance (38) while preserving the simplicity of (41).

[1]Namely, with the notations in [22], $g(x) = -i\frac{\nu}{2}x$ is a linear combination of only the first-order Chebyshev polynomial, then by identifications $a = q - p$, $b = n - p - q$, and $c_1 = -i\frac{\nu}{2}$, Theorem 3.2 from [22] gives $\log \mathbb{E}\left[\exp(\sum g(y_j))\right] \to \frac{1}{8}c_1^2 - \frac{1}{2}(a-b)c_1$ as $p \to \infty$, where the expectation is over $f(\boldsymbol{y})$ in (35).

B. Finite-size Corrections via Exact Moments

The idea here is to use exact moments of the linear statistic Y to construct a volume approximation instead of using the asymptotic ones (38). In consistence with the asymptotic Gaussianity in (36), we consider a Gaussian approximation of the random variable Y using the first two moments. The exact moments of Y can be recursively obtained via the connection between the moment generating function and an ordinary differential equation. Specifically, the moment generating function of $Y = \sum_{j=1}^{p} y_j/2$ is given by

$$
\begin{aligned}
M_p(\nu) &= \tilde{v}_{n,p,q} \int \cdots \int_{-1 \le y_j \le 1} \Delta^2(\boldsymbol{y}) \times \\
&\quad \prod_{j=1}^{p} (1 - y_j)^a (1 + y_j)^b e^{-\frac{\nu}{2}y_j} \, dy_j,
\end{aligned} \tag{44}
$$

where

$$
a = q - p, \quad b = n - p - q \tag{45}
$$

and $\tilde{v}_{n,p,q}$ is as defined in (33). By definition, we have

$$
\log M_p(\nu) = \sum_{j=1}^{\infty} \kappa_j \frac{(-1)^j \nu^j}{j!}, \tag{46}
$$

where κ_j denotes the j-th cumulant of Y. The derivative of cumulant generating function (46) satisfies, up to a translation in ν, a nonlinear second-order differential equation [30], which in our notations reads

$$
\begin{aligned}
(2\nu\sigma''(\nu))^2 &= (\sigma(\nu) - \nu\sigma'(\nu) + 2(2p + a + b)\sigma'(\nu))^2 \\
&+ 4(\sigma(\nu) - \nu\sigma'(\nu) - p(p + b)) \left((2\sigma'(\nu))^2 - 2a\sigma'(\nu)\right),
\end{aligned} \tag{47}
$$

where

$$
\sigma(\nu) = \sum_{j=1}^{\infty} \kappa_j \frac{(-1)^j \nu^j}{2^j (j-1)!} - \frac{p\nu}{4} + p(p + b). \tag{48}
$$

Inserting this into (47), the cumulants can be calculated in a recursive manner. The first three cumulants are

$$
\kappa_1 = \frac{p(n - 2q)}{2n}, \tag{49}
$$

$$
\kappa_2 = \frac{pq(n - p)(n - q)}{n^2 (n^2 - 1)}, \tag{50}
$$

$$
\kappa_3 = -\frac{2pq(n - 2p)(n - 2q)(n - p)(n - q)}{n^3 (n^4 - 5n^2 + 4)}, \tag{51}
$$

where we have substituted the parameters according to (45). Now we approximate the random variable Y by a Gaussian with mean and variance

$$
\mathbb{E}\left[Y\right] = \kappa_1, \quad \mathbb{V}\left[Y\right] = \kappa_2, \tag{52}
$$

so that the corresponding moment generating function is approximated by

$$
M_p(\nu) \approx e^{-\kappa_1 \nu + \frac{\kappa_2}{2}\nu^2}. \tag{53}
$$

Comparing the moment generating function (44) and the characteristic function (32), we have

$$
\widetilde{D}_p(\nu) \approx e^{-i\kappa_1 \nu - \frac{\kappa_2}{2}\nu^2}. \tag{54}
$$

Following similar steps that led from (39) to (41), we arrive at a finite-size volume approximation

$$\mu\left(B\left(r\right)\right) \simeq \frac{1}{2}\mathrm{erf}\left(\frac{\beta}{\sqrt{2\kappa_2}}\right) - \frac{1}{2}\mathrm{erf}\left(\frac{\beta - r^2}{\sqrt{2\kappa_2}}\right), \qquad (55)$$

where

$$\beta = \frac{p(n-q)}{n}, \qquad \kappa_2 = \frac{pq(n-p)(n-q)}{n^2\left(n^2-1\right)}. \qquad (56)$$

Recall that (55) is guaranteed to be asymptotically tight in the regime (37) according to the asymptotic Gaussianity of Y. One can also verify with the change of variables $a = q - p$, $b = n - p - q$ and by letting $p \to \infty$ that the mean and variance in (52) become

$$\mathbb{E}\left[Y\right] = \frac{(b-a)p}{2(a+b+2p)} \longrightarrow \frac{(b-a)}{4} = \frac{n-2q}{4} \qquad (57)$$

$$\mathbb{V}\left[Y\right] = \frac{p(a+p)(b+p)(a+b+p)}{(a+b+2p)^2((a+b+2p)^2-1)} \longrightarrow \frac{1}{16} \qquad (58)$$

matching (38) as expected[2]. From (57) and (58), one sees that the larger a, b, the slower the convergence of the mean and the variance to their asymptotic values would be.

C. Simulations

In figures 4 and 5, we plot the volume of metric ball (15) calculated by the random matrix theory (RMT) approximation (41) as well as the finite-size approximation (55). As a benchmark, we also provide volume curves by Monte-Carlo simulations. The finite-size correction curves are also included in Fig. 2 for comparison with exact expressions. In Fig. 4, asymptotic behavior with the regime (37) can be observed, i.e. letting n, p, q grow with fixed $q - p$ and $n - p - q$. One verifies that convergence occurs for both approximations, while it is much faster for the finite-size correction curves in all cases. The convergence rate of the RMT approximation desponds on the constant values $a = q - p$ and $b = n - p - q$.

To evaluate the convergence of the two asymptotic approximations, we compute the divergence between the Gaussian distributions Y_1 and Y_2 with means μ_1, μ_2 and variance σ_1^2, σ_2^2 as given in (38) and (52), respectively. The Hellinger distance is a type of f-divergence which is a frequently-used metric for the spaces of probability distributions [34]. The corresponding distance between two Gaussian distributions is [35]

$$H(Y_1, Y_2) = \sqrt{1 - \rho(Y_1, Y_2)} \qquad (59)$$

where

$$\rho(Y_1, Y_2) = \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}} \, e^{-\frac{1}{4}\frac{(\mu_1-\mu_2)^2}{\sigma_1^2+\sigma_2^2}} \qquad (60)$$

is the corresponding Bhattacharyya coefficient. The distance (59) is displayed in Fig. 3 as a function of p for fixed values of a and b. For all cases, the Hellinger distance is converging to zero as expected. A slower rate of convergence is seen for larger a, b, or $|a - b|$. Even for the maximum considered value $p = 30$, the RMT approximation is at more than half of the maximum distance from the finite-size approximation in the highest dimensional cases.

---

[2]It can be as well verified that the third cumulant is asymptotically vanishing; $\kappa_3 \to 0$.
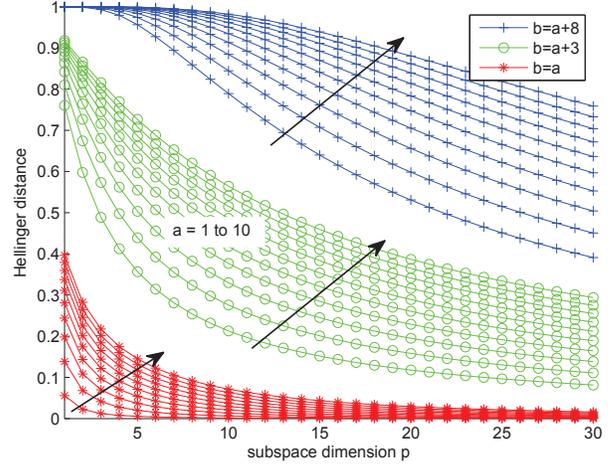

Fig. 3: Hellinger distance between Gaussian distributions with moments given in (38) and (52), representing RMT and finite-size approximations, for different fixed values of $a = q - p$ and $b = n - p - q$ as a function of p.

In Fig. 5 we consider cases of fixed p and growing values of n and q. In Fig. 5a $p = q = 4$ and $n = 8, 9, 10$. It is seen that the RMT curves shift horizontally away from the simulated ones as $b = n - 2p$ increases. This means a shift in the mean value if we interpret the curves as CDFs of volume density. Simulations indicate that for $p = q$ the RMT approximation (41) incurs a nontrivial loss in the mean when the difference $b = n - 2p$ is greater than zero. In Fig. 5b, we consider the case when $n = 2q$ with $q = 2, 4, 6$ and 8 in clock-wise order for a fixed $p = 2$. The RMT based curves rotate away from the simulated ones as $a = q - p$ increases. While here $a = b$ and the means of both approximations are equal in (57), the RMT approximation (41) fails to capture the variance of volume density as q increases. For all the cases considered in Fig. 5, the finite-size approximation (55) matches the simulation almost exactly.

Intuitively, the RMT approximation (41) fails to capture the volume curves since the corresponding asymptotic mean and variance (38) do not involve all the possible parameters p, q and n. In particular, the variance (38) obtained by RMT is a constant. On the contrary, the mean and variance (52) used to construct the finite-size approximation (55) are functions of all the parameters.

V. Application to Source Coding on Grassmann Manifolds

The asymptotic volume of a ball with its finite size-correction can be applied to evaluate the rate-distortion trade-off of source quantization in large-dimensional Grassmann manifolds.

Given a code $\mathcal{C} = \{C_1, \ldots, C_N\} \subset \mathcal{G}_{n,p}^{\mathbb{C}}$ with size $N = |\mathcal{C}|$, consider a uniformly distributed source on $\mathcal{G}_{n,q}^{\mathbb{C}}$ quantized to $\mathcal{C}$ using the chordal distance d_c as a quantization map:

$$\begin{aligned} \mathcal{G}_{n,q}^{\mathbb{C}} &\to \mathcal{C} \\ Q &\mapsto \arg\min_{C_k \in \mathcal{C}} d_c(Q, C_k). \end{aligned} \qquad (61)$$
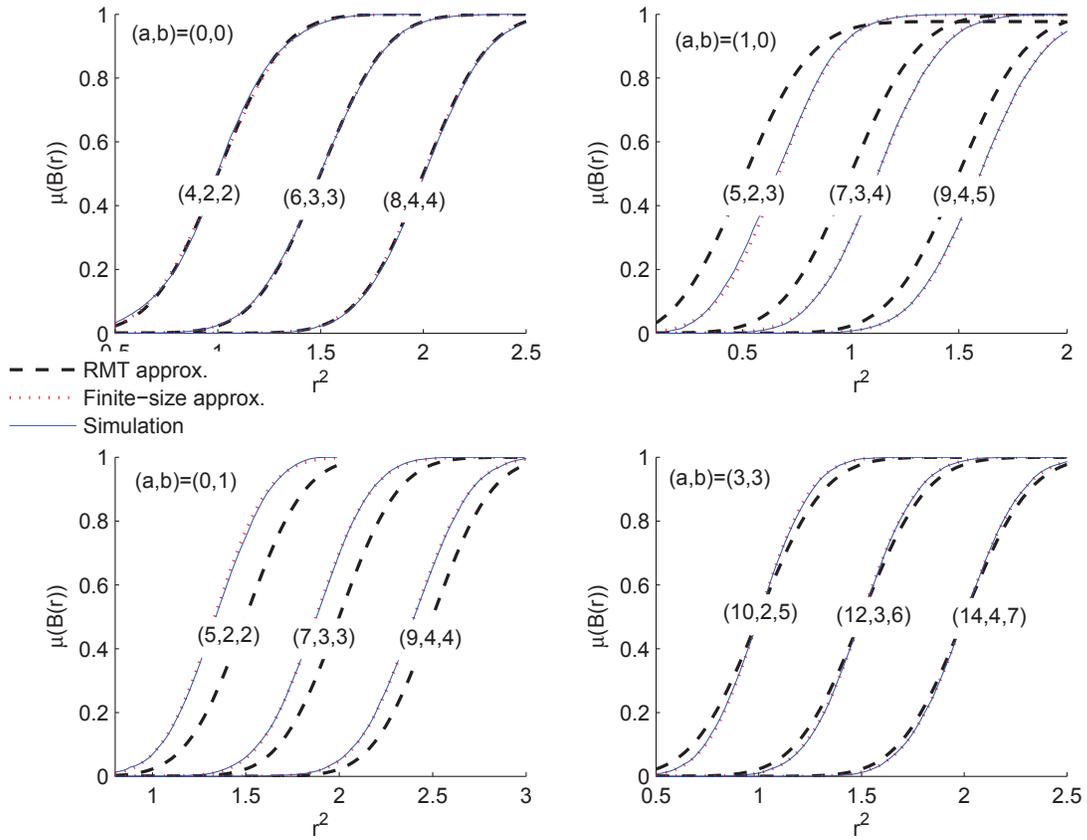
Fig. 4: Volumes of metric balls with RMT approx. (41) versus finite-size approx. (55) in the regime (37). Each graph corresponds to a fixed value of $a = q - p$ and $b = n - q - p$: $(a, b) = (0, 0), (1, 0), (0, 1)$ and $(3, 3)$. In each graph, curves from left to right correspond to $p = 2, 3, 4$.

(a) $(n, p, q) = (n, 4, 4)$
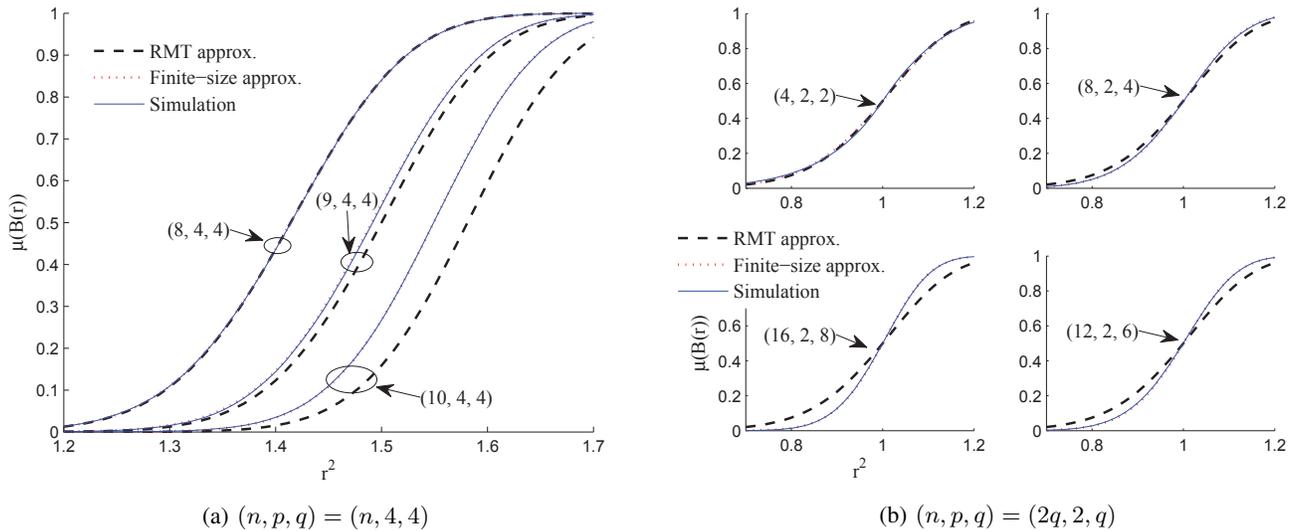
(b) $(n, p, q) = (2q, 2, q)$

Fig. 5: Volumes of metric balls with RMT approx. (41) versus finite-size approx. (55). Comparison with fixed values of p and growing values of n, q.

A source code is considered optimal if it minimizes the average distortion of the process, i.e. the average square quantization error

$$
\begin{aligned}
D(\mathcal{C}) &= \mathbb{E}_{Q \in \mathcal{G}_{n,q}^{\mathbb{C}}}[\min_k d_c^2(C_k, Q)] \\
&= \int_0^p z \, dF_{\mathcal{C}}(z) \\
&= \int_0^p (1 - F_{\mathcal{C}}(z)) dz, \qquad (62)
\end{aligned}
$$

where $F_{\mathcal{C}}(z) = \Pr\{Q \mid \min_k d_c^2(C_k, Q) \leq z\}$ is the CDF of quantization error. The distortion-rate function is the infimum of all possible distortions for a given codesize,

$$
D(N) = \inf_{|\mathcal{C}| = N} D(\mathcal{C}). \qquad (63)
$$

The quantization map leads to the partition of the Grassmann manifold into Voronoi cells centered around the codewords which are defined as

$$
V_k = \{Q \mid d_c^2(C_k, Q) \leq d_c^2(C_j, Q), \forall j\}. \qquad (64)
$$

It follows that the probability of quantization error being less or equal to a value z is given by

$$
\begin{aligned}
F_{\mathcal{C}}(z) &= \Pr\left\{Q \mid \min_k d_c^2(C_k, Q) \leq z\right\} \\
&= \Pr\left\{\cup_{k=1}^N \left\{Q \in (B_{C_k}(\sqrt{z}) \cap V_k)\right\}\right\}. \qquad (65)
\end{aligned}
$$

As points on the manifold belonging to the Voronoi cells' borders appear with probability zero, we further have

$$
\begin{aligned}
F_{\mathcal{C}}(z) &= \sum_{k=1}^N \Pr\left\{B_{C_k}(\sqrt{z}) \cap V_k\right\} \qquad (66) \\
&= \sum_{k=1}^N \mu\left(B_{C_k}(\sqrt{z}) \cap V_k\right). \qquad (67)
\end{aligned}
$$

The CDF of quantization errors (67) is actually equal to the volume of balls until some border effect, i.e. $F_{\mathcal{C}}(z) = N\mu(B(\sqrt{z}))$ on the interval $[0, \varrho^2]$ where ϱ is the *kissing radius* of the code [36]: the shortest distance from a codeword to the border of a Voronoi cell. In general, for any k, $(B_{C_k}(\sqrt{z}) \cap V_k) \subseteq B_{C_k}(\sqrt{z})$, and so for any z one has $F_{\mathcal{C}}(z) \leq N\mu(B(\sqrt{z}))$.

As shown in [13], by using this upper bound to design an ideal distribution $F_{\mathcal{C}}^*(z) = N\mu(B(\sqrt{z}))$ on $[0, z^*]$ such that $N\mu(B(\sqrt{z^*})) = 1$ leads to a lower bound on distortions, i.e. $D(N) \geq \int_0^{z^*} z \, dF_{\mathcal{C}}^*(z)$. Following this principle, after estimating z^* by inverting the derived volume (55) from the previous section and by direct integration we obtain the following approximation to the rate-distortion trade-off

$$
D(N) \gtrsim
$$
$$
\beta - N\sqrt{\frac{\kappa_2}{2\pi}}\left(e^{-\left(\mathrm{erf}^{-1}\left(\mathrm{erf}\left(\frac{\beta}{\sqrt{2\kappa_2}}\right) - \frac{2}{N}\right)\right)^2} - e^{-\frac{\beta^2}{2\kappa_2}}\right),
$$
$$
\qquad (68)
$$

which is asymptotically a lower bound for large-dimensional codes as $n, p, q \to \infty$ with fixed $q - p$ and $n - p - q$. The parameters β and κ_2 are given in (56).

The approximation (68) is compared to simulations in Fig. 6 for different values of n, p, and with $q = p$. Codes with cardinality between 2 and 256, i.e., between one and eight bits, are considered. Rate-distortion trade-offs have been numerically minimized by applying vector quantization based on Lloyd's algorithm. In addition to the numerical vector quantization results, averaged distortions of random codes are shown, which by construction provide an upper bound on $D(N)$. It is visible that the derived approximation (68) is asymptotically a lower bound in large dimensions, while obviously not for the smallest case $(n, p) = (2, 1)$.

The large-dimension bound (68) is further compared to the high-resolution bounds in [13, Th. 2]. The results in [13] are asymptotics in a different regime; they are valid given a sufficiently large codesize. A necessary condition can be found in [13]. Here, the lower bound for code size where the results [13] start to apply would be 0, 1, 5.4, 14.6, 27.1, 68.6 and 114.1 bits for the cases $(n, p) = (2, 1)$, $(4, 2)$, $(6, 3)$, $(8, 4)$, $(10, 4)$, $(16, 4)$ and $(16, 8)$, respectively. These cases are plotted in Fig. 6 from bottom up. The high-resolution bounds in [13] provide also very good approximations of the rate-distortion trade-off for much smaller codesizes, outside of their given range of validity. Nevertheless, in the large-dimensional regime and with fixed code cardinality, one can observe in Fig. 6 a trend in the slope of the rate-distortion trade-off which is not captured by the high-resolution bounds. The bound (68) provides a good approximation in almost all cases. This is the consequence of the volume evaluation (55) for any radius. Moreover, the bounds in [13] depends on a fastly-decreasing constant $c_{n,p,q} \to 0$ as $n, p, q \to \infty$ which might lead to numerical computation issues in the large-dimension regime. For example, for $(n, p) = (8, 4)$ the value is $c_{n,p,q} = 4.2 \times 10^{-5}$ and for $(n, p) = (16, 8)$, $c_{n,p,q} = 4.5 \times 10^{-35}$. For $(n, p) = (32, 16)$ we faced numerical computation errors due to machine precision.

VI. CONCLUSION

We evaluated the volume of a metric ball in a Grassmann manifold. To analyze the case of a center with mismatched dimension, we discuss generalizations of the chordal distance to subspaces with unequal dimensions. A new symmetry property of the volume of a metric ball is presented. Then, multivariate integration of the volume of a ball with any radius is performed. We reduce the multivariate integration problem to a one-dimensional integral related to the Fourier transform. We provide explicit examples in small dimensions for any radius from the obtained formula. For large dimensions, the derived integral expression provides a tractable starting point for asymptotic analysis. From the asymptotic behavior of the time-dependent Jacobi ensemble and by moment-matching techniques, we provide a simple asymptotic volume formula which provides a tight approximation in finite-size dimensions. This allows us to precisely quantify the rate-distortion trade-off of source coding problems in large-dimensional Grassmann manifolds.
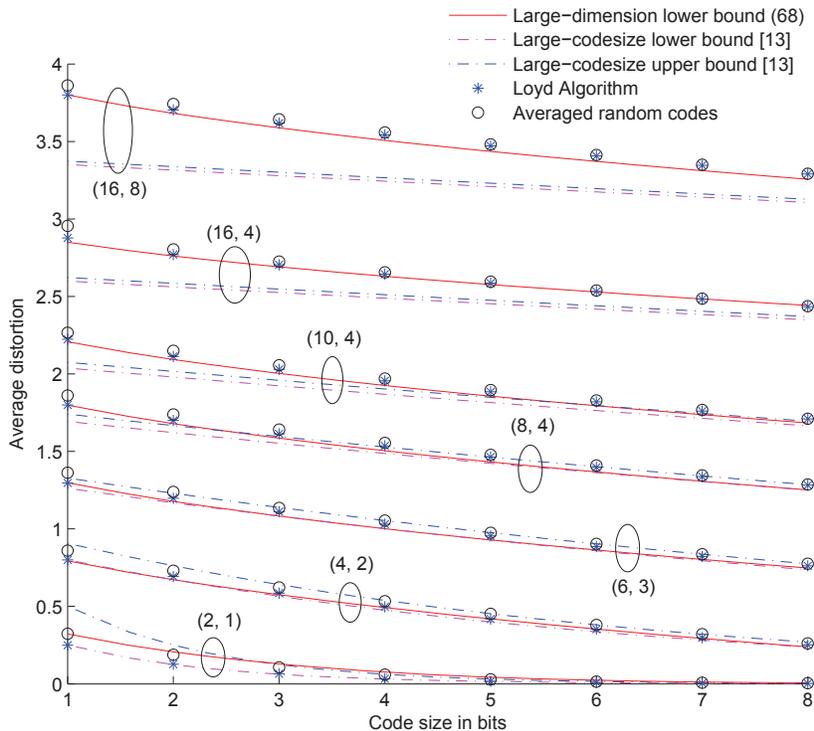
Fig. 6: The large dimension bound (68) compared to simulated average distortions and high-resolution bounds [13].

The results presented in this paper are valid for the Grassman manifold over the complex field. Using the same methodology for a generalization to the real field does not appear trivial. The exact volume formulas were explicitly derived via the Andréief identity. The Andréief identity is intrinsically connected with the asymptotic analysis presented here. By using this identity, one is able to relate the problem to a Hankel or Toeplitz determinant whose asymptotical behaviors have been extensively studied in statistics. In order to proceed with Andréief identity, the square of the Vandermonde determinant in the volume element is instrumental. In contrast, for the real Grassmann manifolds, the Vandermonde determinant in the volume element is not squared and one cannot thus directly reduce the problem in a similar fashion. It remains thus an open problem for future research to identify comparable accurate approximation techniques for the real case.

APPENDIX A
PROOF OF SYMMETRY RELATIONSHIP (11)

Given $P \in \mathcal{G}_{n,p}^{\mathbb{C}}$ with orthogonal complement $P^{\perp} \in \mathcal{G}_{n,n-p}^{\mathbb{C}}$, and similarly given $Q \in \mathcal{G}_{n,q}^{\mathbb{C}}$ with orthogonal complement $Q^{\perp} \in \mathcal{G}_{n,n-q}^{\mathbb{C}}$, one has

$$p = \|\boldsymbol{P}^{\dagger}\boldsymbol{Q}\|_F^2 + \|\boldsymbol{P}^{\dagger}\boldsymbol{Q}^{\perp}\|_F^2, \qquad (69)$$

which leads to

$$d_c^2(P,Q) = p - d_c^2(P,Q^{\perp}). \qquad (70)$$

Given a point $Q \in \mathcal{G}_{n,q}^{\mathbb{C}}$ such that $d_c(P,Q) \geq r$, it follows that $d_c(P^{\perp},Q) \leq \sqrt{p-r^2}$, and thus $Q \notin B_{P,q}(r)$ implies

$Q \in B_{P^{\perp},q}(\sqrt{p-r^2})$. Reciprocally $Q \in B_{P,q}(r)$ implies $Q \notin B_{P^{\perp},q}(\sqrt{p-r^2})$, so that

$$B_{P,q}(r) \cap B_{P^{\perp},q}(\sqrt{p-r^2}) = \emptyset \qquad (71)$$
$$B_{P,q}(r) \cup B_{P^{\perp},q}(\sqrt{p-r^2}) = \mathcal{G}_{n,q}^{\mathbb{C}}. \qquad (72)$$

Finally,

$$\mu(B_{P,q}(r)) + \mu(B_{P^{\perp},q}(\sqrt{p-r^2})) = 1, \qquad (73)$$

and using (9), (10), we obtain (11).

APPENDIX B
ANDRÉIEF INTEGRAL [31]

For two $n \times n$ matrices $\boldsymbol{A}(\boldsymbol{x})$ and $\boldsymbol{B}(\boldsymbol{x})$, with the respective ij-th entry being functions $A_i(x_j)$ and $B_i(x_j)$, and a function $f(\cdot)$ such that the integral $\int_a^b A_i(x)B_j(x)f(x)\,\mathrm{d}x$ exists, the multi-dimensional integral of the product of the determinants can be evaluated as

$$\int \cdots \int_{\mathcal{D}} \det\left(\boldsymbol{A}(\boldsymbol{x})\right) \det\left(\boldsymbol{B}(\boldsymbol{x})\right) \prod_{i=1}^{n} f(x_i)\mathrm{d}x_i =$$
$$\det\left(\int_a^b A_i(x)B_j(x)f(x)\,\mathrm{d}x\right), \quad (74)$$

where $\mathcal{D} = \{a \leq x_n \leq \ldots \leq x_1 \leq b\}$.

APPENDIX C
EXAMPLES OF EXACT VOLUME COMPUTATION

We give explicit expressions of $\mu(B(r))$ obtained by (29) and (30). The volumes have different polynomial representations between different consecutive integer values of r^2, i.e.

on the intervals $[0, 1]$, $[1, 2]$, ..., $[p - 1, p]$. The expressions given here are valid for any $r^2 \in [0, p]$. It can be verified for $r \leq 1$ that the expressions simplify (e.g. to a monomial for $p = q$) and match the results in [13].

1) $n = 4$ *and* $p = q = 2$:

$$\mu\left(B\left(r\right)\right) = -\frac{7}{2} + 8r^2 - 6r^4 + 2r^6 - \frac{(r^2-1)^3(7-2r^2+r^4)}{2|r^2-1|}$$

2) $n = 5$ *and* $p = q = 2$:

$$\mu\left(B\left(r\right)\right) = \frac{17}{2} - \frac{144r^2}{5} + 36r^4 - 20r^6 + \frac{9r^8}{2}$$
$$- \frac{(r^2-1)^4(85-33r^2+6r^4+2r^6)}{10|r^2-1|}$$

3) $n = 5$, $p = 2$ *and* $q = 3$:

$$\mu\left(B\left(r\right)\right) = -\frac{59}{10} + \frac{96r^2}{5} - 24r^4 + 16r^6 - \frac{9r^8}{2}$$
$$+ \left(\frac{59}{10} - \frac{3r^2}{2} + \frac{9r^4}{5} - \frac{r^6}{5}\right)|r^2-1|^3$$

4) $n = 6$ *and* $p = q = 2$:

$$\mu\left(B\left(r\right)\right) = -\frac{31}{2} + \frac{480r^2}{7} - 120r^4 + 104r^6 - 45r^8 + 8r^{10}$$
$$- \frac{(r^2-1)^5(217-92r^2+10r^4+4r^6+r^8)}{14|r^2-1|}$$

5) $n = 6$, $p = 2$ *and* $q = 3$:

$$\mu\left(B\left(r\right)\right) = \frac{263}{14} - \frac{576r^2}{7} + 144r^4 - 128r^6 + 60r^8 - 12r^{10}$$
$$- \frac{(r^2-1)^5(-263+100r^2-38r^4-12r^6+3r^8)}{14|r^2-1|}$$

6) $n = 6$ *and* $p = q = 3$:

$$\mu\left(B\left(r\right)\right) = -\frac{6547}{28} + \frac{19683r^2}{28} - \frac{6561r^4}{7} + 729r^6 - \frac{729r^8}{2}$$
$$+ \frac{243r^{10}}{2} - 27r^{12} + \frac{27r^{14}}{7} - \frac{9r^{16}}{28} + \frac{r^{18}}{42}$$
$$+ \frac{6(r^2-1)^7 - 9|r^2-1|^6 - \frac{18}{7}|r^2-1|^8 - \frac{1}{28}|r^2-1|^{10}}{|r^2-1|}$$
$$+ \frac{6(r^2-2)^7 + 9|r^2-2|^6 + \frac{18}{7}|r^2-2|^8 + \frac{1}{28}|r^2-2|^{10}}{|r^2-2|}$$

REFERENCES

[1] J. H. Conway, R. H. Hardin, and N. J. A. Sloane, "Packing lines, planes, etc.: Packings in Grassmannian space," *Exper. Math.*, vol. 5, pp. 139–159, 1996.

[2] A. Edelman, T. Arias, and S. T. Smith, "The geometry of algorithms with orthogonality constraints," *SIAM J. Matrix Anal. Appl.*, vol. 20, no. 2, pp. 303–353, 1998.

[3] K. Ye and L.-H. Lim, "Schubert varieties and distances between subspaces of different dimensions," *SIAM J. Matrix Anal. Appl.*, 2016, to appear, arXiv preprint arXiv:1407.0900.

[4] D. Agrawal, T. Richardson, and R. Urbanke, "Multiple-antenna signal constellations for fading channels," *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2618–2626, Sep. 2001.

[5] L. Zheng and D. Tse, "Communication on the Grassmann manifold: a geometric approach to the noncoherent multiple-antenna channel," *IEEE Trans. Inf. Theory*, vol. 48, no. 2, pp. 359 –383, Feb. 2002.

[6] A. Ashikhmin and A. Calderbank, "Grassmannian packings from operator Reed-Muller codes," *IEEE Trans. Inf. Theory*, vol. 56, no. 11, pp. 5689–5714, Nov. 2010.

[7] D. J. Love, R. W. Heath Jr., and T. Strohmer, "Grassmannian beamforming for multiple-input multiple-output wireless systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2735–2747, Oct. 2003.

[8] K. Mukkavilli, A. Sabharwal, E. Erkip, and B. Aazhang, "On beamforming with finite rate feedback in multiple-antenna systems," *IEEE Trans. on Inf. Theory*, vol. 49, no. 10, pp. 2562–2579, Oct. 2003.

[9] A. Barg and D. Nogin, "Bounds on packings of spheres in the Grassmann manifold," *IEEE Trans. Inf. Theory*, vol. 48, no. 9, pp. 2450–2454, Sep. 2002.

[10] O. Henkel, "Sphere packing bounds in the Grassmann and Stiefel manifolds," *IEEE Trans. Inf. Theory*, vol. 51, no. 10, pp. 3445–3456, Oct. 2005.

[11] J. Roh and B. Rao, "Transmit beamforming in multiple-antenna systems with finite rate feedback: a VQ-based approach," *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 1101–1112, Mar. 2006.

[12] P. Xia and G. Giannakis, "Design and analysis of transmit-beamforming based on limited-rate feedback," *IEEE Trans. on Signal Process.*, vol. 54, no. 5, pp. 1853–1863, May 2006.

[13] W. Dai, Y. Liu, and B. Rider, "Quantization bounds on Grassmann manifolds and applications to MIMO communications," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1108–1123, Mar. 2008.

[14] W. Dai, Y. Liu, B. Rider, and V. Lau, "On the information rate of MIMO systems with finite rate channel state feedback using beamforming and power on/off strategy," *IEEE Trans. Inf. Theory*, vol. 55, no. 11, pp. 5032–5047, Nov. 2009.

[15] W. Dai, B. Rider, and Y. Liu, "Volume growth and general rate quantization on Grassmann manifolds," in *Proc. IEEE Global Telecom. Conf.*, Nov. 2007, pp. 1441–1445.

[16] D. Zhu, B. Li, and P. Liang, "Normalized volume of hyperball in complex Grassmann manifold and its application in large-scale MU-MIMO communication systems," *arXiv preprint*, 2014, arXiv:1402.4543.

[17] N. Jindal, "A feedback reduction technique for MIMO broadcast channels," in *Proc. IEEE Int. Symp. Inf. Theory*, July 2006, pp. 2699–2703.

[18] B. S. Thomas, L. Lin, L.-H. Lim, and S. Mukherjee, "Learning subspaces of different dimensions," *arXiv preprint*, 2014, arXiv:1404.6841.

[19] J. H. Lee and W. Choi, "Multiuser diversity for secrecy communications using opportunistic jammer selection: Secure DoF and jammer scaling law," *IEEE Trans. Signal Process.*, vol. 62, no. 4, pp. 828–839, Feb. 2014.

[20] A. T. James, "Normal multivariate analysis and the orthogonal group," *Ann. Math. Statist.*, vol. 25, no. 1, pp. 40–75, 1954.

[21] M. Adler and P. V. Moerbeke, "Integrals over Grassmannians and random permutations," *Adv. Math.*, vol. 181, no. 1, p. 19049, Feb. 2004.

[22] K. Johansson, "On random matrices from the compact classical groups," *Ann. Math.*, vol. 145, no. 3, pp. 519–545, May 1997.

[23] D. Love and R. Heath, "Limited feedback unitary precoding for spatial multiplexing systems," *IEEE Trans. Inf. Theory*, vol. 51, no. 8, pp. 2967–2976, Aug. 2005.

[24] C. Bachoc, Y. Ben-Haim, and S. Litsyn, "Bounds for codes in products of spaces, Grassmann, and Stiefel manifolds," *IEEE Trans. Inf. Theory*, vol. 54, no. 3, pp. 1024 –1035, Mar. 2008.

[25] H.-C. Wang, "Two-point homogeneous spaces," *Ann. Math*, vol. 55, no. 1, pp. 177–191, Jan. 1952.

[26] R.-A. Pitaval and O. Tirkkonen, "Flag orbit codes and their expansion to Stiefel codes," in *Proc. IEEE Inf. Theory Workshop*, Sep. 2013, pp. 1–5.

[27] ——, "Volume of ball and Hamming-type bounds for Stiefel manifold with Euclidean distance," in *Proc. Asilomar Conf. on Sig., Syst. and Comp.*, Nov. 2012, pp. 483–487.

[28] G. Han and J. Rosenthal, "Unitary space-time constellation analysis: An upper bound for the diversity," *IEEE Trans. Inf. Theory*, vol. 52, no. 10, pp. 4713–4721, Oct. 2006.

[29] L. Wei, R.-A. Pitaval, J. Corander, and O. Tirkkonen, "From random matrix theory to coding theory: Volume of a metric ball in unitary group," *arXiv preprint*, 2015, arXiv:1508.01358.

[30] E. Basor, Y. Chen, and T. Ehrhardt, "Painlevé V and time-dependent Jacobi polynomials," *J. Phys. A: Math. Theor.*, vol. 43, 2010.

[31] C. Andréief, "Note sur une relation entre les intégrales définies des produits des fonctions," *Mém. de la Soc. Sci. Bordeaux 2*, 1883.

[32] M. Chiani, M. Z. Win, and A. Zanella, "On the capacity of spatially correlated MIMO Rayleigh-fading channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2363–2371, Oct. 2003.

[33] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products.* 7th Ed., San Diego: Academic Press, 2007.

[34] F. Liese and I. Vajda, "On divergences and informations in statistics and information theory," *IEEE Trans. Inf. Theory*, vol. 52, no. 10, pp. 4394–4412, Oct. 2006.

[35] T. Kailath, "The divergence and Bhattacharyya distance measures in signal selection," *IEEE Trans. Commun. Tech.*, vol. 15, no. 1, pp. 52–60, Feb. 1967.

[36] R.-A. Pitaval, O. Tirkkonen, and S. D. Blostein, "Density and bounds for Grassmannian codes with chordal distance," in *Proc. IEEE Int. Symp. Inf. Theory*, Aug. 2011, pp. 1–5.