Flag Orbit Codes and Their Expansion to Stiefel Codes

Renaud-Alexandre Pitaval and Olav Tirkkonen
Aalto University, Department of Communications and Networking, Espoo, Finland

***Abstract*—We discuss group orbits codes in homogeneous spaces for the unitary group, known as flag manifolds. The distances used to describe the codes arise from embedding the flag manifolds into Euclidean hyperspheres, providing a generalization of the spherical embedding of Grassmann manifolds equipped with the so-called chordal distance. Flag orbits are constructed by acting with a unitary representation of a finite group. In the construction, the center of the finite group has no effect, and thus it is sufficient to consider its inner automorphism group. Accordingly, some explicit constructions from projective unitary representations of finite groups in 2 and 4 dimensions are described. We conclude with examples of codes on the Stiefel manifold constructed as orbits of the linear representation of the projective groups, and thus expansion of the flag codes considered.**

I. Introduction

Flag manifolds are quotient spaces of unitary groups [1], i.e. an element in a flag manifold can be interpreted as an equivalence class of unitary matrices. Flag codes are generalizations of spherical codes with applications in the area of multiple-antenna transmission. This includes Grassmann codes [2], [3] which are special examples of flag codes. Other types of flag codes have also been used in the literature, although not called as such, see [4] and references therein.

The construction of orbit codes follows two steps [5]: First we need to choose a finite group with representation of the appropriate degree. Secondly, we need to choose an appropriate initial point which leads to a code with a given cardinality and minimal distance. Orbits have been used to construct codes on specific flag manifolds, including spheres [5], [6] and Grassmann manifolds [7]–[10]

Grassmann manifolds, equipped with a so-called chordal distance, have isometric spherical embeddings [7]. In this paper, we generalize this embedding and the corresponding Grassmann chordal distance to more general flag manifolds. Consequently, flag codes with the considered distance inherit spherical coding bounds.

Given a linear representation of a group, its center does not change the equivalence class of an orbit element and thus has no effect for orbit construction in flag manifolds. Thus, to generate flag orbit codes, groups having projective unitary representations are of specific interest.

In this paper, the projective finite groups and initial points employed in [10] to construct Grassmann orbit codes are reconsidered to construct other type of flag orbit codes in 2 and 4D. The examples focus on codes where elements are equivalence classes of square unitary matrices modulo column permutations and columnwise rotations. Finally, we investigate Stiefel orbit codes arising from the linear representation of the projective group considered. By doing so, one obtain expansions of the Grassmann orbit codes of [10] by finite unitary groups acting from the right.

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II. Flag Manifolds

A. Unitary Group and Stiefel Manifold

The complex Stiefel manifold $\mathcal{V}^{\mathbb{C}}_{n,p}$ is defined as the space of orthonormal rectangular matrices (with $p \leq n$),

$$\mathcal{V}^{\mathbb{C}}_{n,p} = \left\{ Y \in \mathbb{C}^{n\times p} \quad | \quad Y^H Y = I_p \right\}, \tag{1}$$

isomorphic to the quotient space $\mathcal{U}_n / \mathcal{U}_{p-n}$. The special cases $p = n$ yields the unitary group $\mathcal{U}_n = \mathcal{V}^{\mathbb{C}}_{n,n}$. We define the identity element of $\mathcal{V}^{\mathbb{C}}_{n,p}$ by $I_{n,p} = \begin{pmatrix} I_p \\ 0 \end{pmatrix}$, where I_p is the identity in $\mathcal{U}_p$.

B. Grassmann Manifold

The complex Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$ is the set of all p-dimensional subspaces of $\mathbb{C}^n$. $\mathcal{G}^{\mathbb{C}}_{n,p}$ can be expressed as the quotient space of the Stiefel manifold by the unitary group,

$$\mathcal{G}^{\mathbb{C}}_{n,p} \cong \mathcal{V}^{\mathbb{C}}_{n,p} / \mathcal{U}_p. \tag{2}$$

Accordingly, a point $[Y] \in \mathcal{G}^{\mathbb{C}}_{n,p}$ is an equivalence class of $n \times p$ orthonormal matrices whose columns span the same subspace, i.e. given $Y \in \mathcal{V}^{\mathbb{C}}_{n,p}$, $[Y] = \{YU \quad | \quad U \in \mathcal{U}_p\}$. We define the identity element of $\mathcal{G}^{\mathbb{C}}_{n,p}$ by $[I_{n,p}]$.

C. Generalized Flag Manifolds

A *flag* in $\mathbb{C}^n$ is a sequence of nested subspaces $V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n$ with $1 \leq r \leq n$ [11], [12]. To every flag $(V_1, \cdots, V_r)$ we can associate a subspace decomposition $(W_1, \cdots, W_r)$ satisfying $V_i = \bigoplus_{j=1}^{i} W_j$. Given $(p_1, \ldots, p_r) \in \mathbb{N}^r$, we define[1] the flag manifold $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r}$ as the set of all $W_1 \oplus \cdots \oplus W_r$ in $\mathbb{C}^n$ such that $\dim W_i = p_i$.

Let $p = \sum_{i=1}^{r} p_i$. We represent a point on $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r}$ by a n-by-p Stiefel matrix W, i.e. $W^H W = I_p$. Under this representation, two matrices W_1, W_2 are equivalent in the sense that they represent the same flag if matrices $(U_1, \ldots, U_r) \in \mathcal{U}_{p_1} \times \cdots \times \mathcal{U}_{p_r}$ exist such that $W_1 = W_2 \,\mathrm{diag}(U_1, \ldots, U_r)$. The flag manifold can thus be expressed as the quotient space

$$\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r} \quad \cong \quad \frac{\mathcal{V}^{\mathbb{C}}_{n,p}}{\mathcal{U}_{p_1} \times \cdots \times \mathcal{U}_{p_r}}$$

[1] This is isomorphic to the definition where $\mathcal{F}l^{\mathbb{C}}_{n;d_1,\ldots,d_r}$ is the set of all flags $(V_1, \cdots, V_r)$ in $\mathbb{C}^n$ with $V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n$ and $\dim V_i = d_i$.

$$\cong \quad \frac{\mathcal{U}_n}{\mathcal{U}_{p_1} \times \cdots \times \mathcal{U}_{p_r} \times \mathcal{U}_{n-p}}, \tag{3}$$

where $p = \sum_{i=1}^{r} p_i$. We note that $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r} \cong \mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r,n-p}$. For $r = 1$ and $p_1 = p$, one recovers the Grassmann manifold $\mathcal{F}l^{\mathbb{C}}_{n;p} \cong \mathcal{G}^{\mathbb{C}}_{n,p}$.

D. Simple Flag Manifolds and Permutation-Invariance

Motivated by application in MIMO coding, the case when $p_1 = \ldots = p_r = 1$ is of special interest [4]. We use the notation

$$\mathcal{F}^{\mathbb{C}}_{n,p} \triangleq \mathcal{F}l^{\mathbb{C}}_{n;\underbrace{1,\ldots,1}_{p}} \cong \frac{\mathcal{V}^{\mathbb{C}}_{n,p}}{(\mathcal{U}_1)^p} \cong \frac{\mathcal{U}_n}{(\mathcal{U}_1)^p \times \mathcal{U}_{n-p}}\ , \tag{4}$$

so that for the case $p = n$, we have $\mathcal{F}^{\mathbb{C}}_{n,n} \cong \mathcal{U}_n/(\mathcal{U}_1)^n$.

We also consider an additional equivalence relation on $\mathcal{F}^{\mathbb{C}}_{n,p}$ by considering that two representatives $W_1, W_2 \in \mathcal{V}^{\mathbb{C}}_{n,p}$ are equivalent under columns permutation, i.e. if there exists a permuation matrix $P \in \mathbb{R}^{p\times p}$ such that $W_1 = W_2 P$, then $[W_1] = [W_2]$. The permutation corresponds to an orientation-invariance of the elements. We denote the corresponding space by

$$\vec{\mathcal{F}}^{\mathbb{C}}_{n,p} \triangleq \mathcal{F}^{\mathbb{C}}_{n,p}/S_p \tag{5}$$

where S_p is the symmetric group whose elements are all the permutations of the p symbols.

III. Spherical Embeddings and Chordal Distances

To investigate coding problems on the manifolds of interest, we need to define a notion of distance. Flag codes are a subclass of spherical codes as flag manifolds can be embedded into a hypersphere. We focus on distances that correspond to taking the natural Euclidean/chordal distance in the ambient space. A (N, δ^2)-code thus is a finite subset of N points in the manifold with minimum squared distance δ^2 among the elements.

A. Stiefel Chordal Distance

The Stiefel manifold $\mathcal{V}^{\mathbb{C}}_{n,p}$ has a canonical isometric spherical embedding into $\mathcal{S}^{2np-1}(\sqrt{p})$ with the distance

$$d_s(U, V) = \|U - V\|_F, \tag{6}$$

where $\|\cdot\|_F$ is the Frobenius norm.

B. Grassmann Chordal Distance

Let $[Y]$, $[Z] \in \mathcal{G}^{\mathbb{C}}_{n,p}$ be two subspaces of $\mathbb{C}^n$, where Y, $Z \in \mathcal{V}^{\mathbb{C}}_{n,p}$ are representatives of their respective equivalence classes. The chordal distance is defined as [7]

$$d_g(Y, Z) = \frac{1}{\sqrt{2}} \|YY^H - ZZ^H\|_F. \tag{7}$$

The representation of the elements of the Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$ by their projection matrices associated with the chordal distance gives an isometric embedding in a sphere of radius $\sqrt{\frac{p(n-p)}{2n}}$ in $\mathbb{R}^D$ with $D = n^2 - 1$ [7].

C. Spherical Embedding of Flag Manifolds

In [7], spherical embeddings of Grassmann manifolds are described. Indeed, all flag manifolds can be understood as submanifolds of the same sphere. Roughly speaking, for a given n, it is remarkable that a $(n^2 - 2)$-dimensional hypersphere can be decomposed so that except for a zero-measure set, it consists of a "fibration" of flag manifolds $\mathcal{F}^{\mathbb{C}}_{n,n}$ over a $(n-2)$-dimensional hypersphere with some singular submanifolds removed. The remaining of the sphere being singularities corresponding to other flags $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\cdots p_r}$ which includes the Grassmann manifolds $\mathcal{G}^{\mathbb{C}}_{n,p}$ as special cases.

To see this, consider the hypersphere constructed as a set of Hermitian matrices

$$\mathcal{S}^{n^2-1} \cong \{X \in \mathbb{C}^{n\times n} \mid X^H = X,\ \|X\|_F^2 = 1\}.$$

Define the hyperplane,

$$\mathcal{P} = \{X \in \mathbb{C}^{n\times n} \mid X^H = X,\ \mathrm{Tr}[X] = 0\}.$$

The intersection between $\mathcal{S}^{n^2-1}$ and $\mathcal{P}$ is an $(n^2-2)-$dimensional unit hypersphere $\mathcal{S}^{n^2-2} \cong \mathcal{S}^{n^2-1} \cap \mathcal{P}$.

Consider the set of diagonal real matrices in $\mathcal{S}^{n^2-1} \cap \mathcal{P}$:

$$\mathcal{D} = \{\mathrm{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n\times n} \mid \sum d_i^2 = 1,\ \sum d_i = 0\},$$

we have $\mathcal{D} \cong \mathcal{S}^{n-2}$.

Conjugation in the unitary group of a fixed D,

$$\begin{aligned} \Xi : \mathcal{U}_n \times \mathcal{S}^{n-2} &\rightarrow \mathcal{S}^{n^2-2} \\ (U, D) &\mapsto UDU^H, \end{aligned} \tag{8}$$

defines an equivalence class of unitary matrices such that $U_1 \equiv U_2$ if and only if $\Xi[(U_1, D)] = \Xi[(U_2, D)]$. For almost all elements of $\mathcal{D}$, $d_1 \neq \cdots \neq d_n$, we have $\Xi[(\mathcal{U}_n, D)] \cong \mathcal{F}^{\mathbb{C}}_{n,n}$. For special cases where some values of $\mathcal{D}$ are equal we get a related lower-dimensional flag manifold. For example, setting $d_1 = \cdots = d_p \neq d_{p+1} = \cdots = d_n$, we have $\Xi[(\mathcal{U}_n, D)] \cong \mathcal{G}^{\mathbb{C}}_{n,p}$. Reciprocally, by eigenvalue decomposition, every $X \in \mathcal{S}^{n^2-2}$ can be decomposed into an element in $\mathcal{D}$ and an equivalence class of unitary matrices, the decomposition being unique with ordered-eigenvalues.

D. Generalized Chordal Distance for Flag Manifolds

Given two flags $[W_1], [W_2] \in \mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r}$, represented by $W_1, W_2 \in \mathcal{V}^{\mathbb{C}}_{n,p}$, they can be decomposed as $W_k = (W_{k,1}, W_{k,2}, \ldots, W_{k,r})$, such that $W_{k,i} \in \mathcal{V}^{\mathbb{C}}_{n,p_i}$. The notion of Grassmannian chordal distance can be generalized to flag manifolds by

$$d_f([W_1], [W_2]) \quad = \quad \sqrt{\sum_{i=1}^{r} d_g^2([W_{1,i}], [W_{2,i}])}\ . \tag{9}$$

This distance naturally arises from the canonical embedding $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r} \hookrightarrow \mathcal{G}^{\mathbb{C}}_{n,p_1} \times \cdots \times \mathcal{G}^{\mathbb{C}}_{n,p_r}$ and taking the chordal distance on this space.

Proposition 1: The flag manifold $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\ldots,p_r}$ equipped with the chordal distance defined in (9) gives an isometric embedding into an r-product of (n^2-2)-spheres, which can be embedded into a sphere in $\mathbb{R}^{r(n^2-1)}$ with radius $\sqrt{\frac{pn-\sum p_i^2}{2n}}$:

$$\begin{aligned}(\mathcal{F}l^{\mathbb{C}}_{n;p_1,\dots,p_r}, d_f) \quad &\hookrightarrow \quad \prod_{i=1}^{r} \mathcal{S}^{n^2-2}\left(\sqrt{\frac{p_i(n-p_i)}{2n}}\right) \\ &\hookrightarrow \quad \mathcal{S}^{r(n^2-1)-1}\left(\sqrt{\frac{pn-\sum_{i=1}^{r} p_i^2}{2n}}\right).\end{aligned}$$

E. Permuation-invariance and MUness

The embedding above applies to the permutation-invariant flag manifold $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ as well. Two matrices that differ only by permutation maximize the distance d_f to $\sqrt{p}$. A point on $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ corresponds to a p-simplex inscribed in the embedding sphere. Accordingly, given two elements $[W_1], [W_2] \in \vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$, we define

$$d_p([W_1],[W_2]) = \min_{P\in S_p} d_f([W_1],[W_2P]). \tag{10}$$

With this distance function, to a (N, δ^2)-codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ corresponds a $(p!N, \delta^2)$-code in $\mathcal{F}^{\mathbb{C}}_{n,p}$. Coding bounds for $\mathcal{F}^{\mathbb{C}}_{n,p}$ would be rather loose for $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$, in the same manner as spherical bounds are loose for antipodal spherical codes.

Inspired by the literature in quantum information science on mutually unbiased bases (MUB) [13], we consider an alternative distance on $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ defined by

$$d_{\mathrm{mu}}([W_1],[W_2]) = \sqrt{p - \sum_{i,j=1}^{p} |w_{1,i}^H w_{2,j}|^4}. \tag{11}$$

where $w_{l,j}$ is the j-th columns of W_l. For the case $p = n$, this corresponds to the "MUness", a measure of mutually unbiasedness. Due to lack of space, we refer to [13] for details. We just note this metric corresponds to embedding $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ into the real Grassmann $\mathcal{G}^{\mathbb{R}}_{n^2-1,p-1}$ and taking the corresponding chordal distance, which is itself associated with a spherical embedding of dimension $\binom{n^2}{2} - 2$ and squared radius $\frac{(p-1)(n^2-p)}{n^2-1}$ [7]. Comparing to d_p, the advantage of d_{mu} is that metric calculations do not suffer from a combinatorial explosion.

IV. Flag Orbits Codes

We now consider a finite group $G \leq \mathcal{U}_n$ acting on $\mathcal{V}^{\mathbb{C}}_{n,p}$ and thus on quotient spaces of it. Given a group G and a initial point (or generator) Y in the manifold of interest, the orbit of Y under the action of G is the subset

$$GY = \{gY \quad | \quad g \in G\}. \tag{12}$$

The observation in [10] that group with projective representation are of specific interest to construct Grassmann orbit codes is generalized in this section to flag orbit codes.

A. Orbits from Projective Representation

The center of the unitary group $\mathcal{U}_n$ is $Z(\mathcal{U}_n) = \{e^{i\theta} I_n \quad | \quad \theta \in \mathbb{R}\} \cong \mathcal{U}_1$. The *projective unitary group* is the quotient of the unitary group by its center $\mathcal{PU}_n = \mathcal{U}_n/\mathcal{U}_1$. An element in $\mathcal{PU}_n$ is an equivalence class of unitary matrices under multiplication by a constant phase. If a group can be homomorphically mapped to $\mathcal{PU}_n$, it is said to have a projective unitary representation. Such can be naturally understood in terms of a linear representation of $\mathcal{U}_n$ acting on projection matrices by conjugation. For orbits under projective representation we have the following result.

Proposition 2: Given a group G having a faithful irreducible representation in $\mathcal{U}_n$, its inner automorphism group $\mathrm{Inn}(G)$ has a representation in $\mathcal{PU}_n$. Flag orbits of the action of G are orbits of the action of $\mathrm{Inn}(G)$: for any $[Y] \in \mathcal{F}l^{\mathbb{C}}_{n;p_1,\dots,p_r}$, we have $G[Y] = \mathrm{Inn}(G)[Y]$.

It follows that to construct flag orbit codes, we are primarily interested by groups having a representation in $\mathcal{PU}_n$.

B. Initial Points

The cardinality of the orbit code depends on the size of the stabilizer subgroup of the initial point in G.

Initial points with trivial stabilizer leads to orbit codes of the same cardinality as the group. This holds for almost every point in the manifold, except when the group has permutation elements then there is no point in $\vec{\mathcal{F}}^{\mathbb{C}}_{n,p}$ with trivial stabilizer.

Initial points leading to orbit codes of size less than the group size have by definition a stabilizer which is a non-trivial subgroup of G. Those are singularities and there is only a finite number of such codes. Such initial points in $\mathcal{F}l^{\mathbb{C}}_{n;p_1,\dots,p_r}$ are concatenations of r invariant subspaces of dimension $\{p_1,\dots,p_r\}$ of some non-trivial subgroup of G. In this case, appropriate initial points can be found from eigenspaces of the matrix representation of the group.

V. Examples of Flag Orbit Codes

In this section, examples of flag orbit codes in 2 and 4D are given. Projective representation of the groups considered and initial points can be found in [10]. Their cardinality and squared minimum flag distances are given in Table I, where δ_p and δ_{mu} being the minimum distance according to (10) and (11), respectively.

A. The Specific Cases of $\mathcal{F}^{\mathbb{C}}_{2,2}$ and $\vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$

The lowest dimensional flag manifolds ($n = 2$) are very specific cases. We have $\mathcal{F}^{\mathbb{C}}_{2,2} \cong \mathcal{F}^{\mathbb{C}}_{2,1} \cong \mathcal{G}^{\mathbb{C}}_{2,1}$, which further reduces to the real unit sphere $\mathcal{F}^{\mathbb{C}}_{2,2} \cong S^2$. It follows that designing codes in $\mathcal{F}^{\mathbb{C}}_{2,2}$ is equivalent to designing spherical codes [10], [14]. In addition, each colummn of a 2×2 unitary matrix generating a point in $\mathcal{F}^{\mathbb{C}}_{2,2}$ can be seen as two ordered antipodal points. It follows that $\vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$ is the set of spherical antipodal points, or equivalently the set of lines in 3D, also known as the real Grassmannian $\mathcal{G}^{\mathbb{R}}_{3,1} \cong \vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$ [7]. Codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$ can thus be constructed by leveraging results from known antipodal spherical codes [15]. Some optimal orbit codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$ can be obtained by pairing antipodals of orbit codes in $\mathcal{G}^{\mathbb{C}}_{2,1}$. Examples of optimal orbit codes are simplices of cardinality 3 and 4, orbits of the octahedral group O, forming a octahedron and a cube on the sphere. The maximum simplicial configuration, i.e. of cardinality 6, forms an icosahedron, an orbit of the tetrahedral group. As expected, the MUness

TABLE I

Some (N, δ^2)-flag orbit codes.

$\vec{\mathcal{F}}^{\mathbb{C}}_{2,2}$		
N	δ_p^2	δ_{mu}^2
3	1	1
4	0.66	0.88
6	0.55	0.8
15	0.19	0.35

$\vec{\mathcal{F}}^{\mathbb{C}}_{4,4}$		
N	δ_p^2	δ_{mu}^2
15	2	2
90	1	1
180	0.59	1
360	1.25	1.75
960	0.5	0.75
1440	0.22	0.40

distances from Table I, match the result [7], meeting the Rankin bound. A suboptimal packing of size 15 is also given as orbit of the icosahedral group $A5$, inner automorphism group of the binary icosahdral group $2I$. The obtained squared mutual unbiasedness distance is 0.35, for comparision the putatively optimum code has $\delta_{\text{mu}}^2 \approx 0.38$.

B. Codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{4,4}$ fom Clifford Group

In this space, code elements are 4×4 unitary matrices modulo column permutations and columnwise rotations. Here we described some codes obtained from the Clifford group known to lead to good codes in the Grassmann manifold.

Consider

$$H = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad P = e^{\frac{3i\pi}{4}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \tag{13}$$

and define the octahedral group , or first-order Clifford group, by

$$C_1 = \langle H, P \rangle / \mathbb{Z}_2 \subset \mathcal{PU}_2. \tag{14}$$

Clifford groups C_n of cardinality $|C_n| = 2^{n^2+2n} \prod_{j=1}^{n}(4^j - 1)$ with representation in $\mathcal{PU}_n$ are defined in [16] (an alternative definition can be found in [17]). The Clifford group in 4D is obtained by tensor multiplication of the elements of C_1, and an additional element $CNOT$ which is essentially a column-permutation. It follows that the $CNOT$ operator is irrelevant for constructing codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{4,4}$, so that we can only consider the subgroup of C_2 defined by

$$\widetilde{C}_2 = \langle H \otimes I_2, I_2 \otimes H, P \otimes I_2, I_2 \otimes P \rangle / \mathbb{Z}_4 \subset \mathcal{PU}_4 \tag{15}$$

From the eigenvectors of the group elements, we found some initial points with non-trivial stabilizers of different orders. The resulting codes with cardinality and minimum distance are presented in Table I. From the table, one can notice than the two considered distance functions behave quite similarly except for the code of size 180. The generator of the 15-points codes is the identity matrix, i.e. the code corresponds to taking the finite group directly as a code itself. This code is a collection of 3 maximal sets of MUBs. The other generating points are given in the Appendix. Recall that to a (N, δ_p^2)-codes in $\vec{\mathcal{F}}^{\mathbb{C}}_{4,4}$, shown in Table I, corresponds a $(24N, \delta_p^2)$-code in $\mathcal{F}^{\mathbb{C}}_{4,4}$.

VI. Stiefel Codes from Grassmann Orbit Codes

Here, we consider Stiefel orbit codes arising from the linear representation of the projective group considered in the previous section and in [10]. The codes are expansions of the Grassmann orbit codes of [10] as direct products of a Grassmannian (or flag) code and a unitary code. Indeed, the obtained codes are more than just a central extension of the Grassmann code. To see this consider a unitary group $G \subset \mathcal{U}_n$ and an initial point $V_0 \in \mathcal{V}^{\mathbb{C}}_{n,p}$. Assume that $[V_0] \in \mathcal{G}^{\mathbb{C}}_{n,p}$ has a non-trivial stabilizer $S = \{s_1, \ldots, s_n\}$ in G. By definition, for any $s_i \in S$ there exists $r_i \in \mathcal{U}_p$ such that $s_i V_0 = V_0 r_i$. Define $R \in \mathcal{U}_p$ to be the set of all these right unitary rotations. Then

Lemma 1: The set $R \in \mathcal{U}_p$ is a group.

Proof: Given $s_i \in S$ and $r_k \in R$ satisfying $s_i V_0 = V_0 r_k$, it is a direct verification that $s_i^H V_0 = V_0 r_k^H$. Thus the inverse of r_k is in R. Given also $s_j \in S$ and $r_l \in R$ satisfying $s_j V_0 = V_0 r_l$, we have $s_i s_j V_0 = V_0 r_l r_k$, and as $s_i s_j \in S$ a $r_m \in R$ exists such that $r_m = r_l r_k$. ■

The code obtained is an extension of the Grassmannian code by the group R and the center of G. Non-trivial stabilizers are only possible if some non-trivial group elements have eigenvalue 1. Otherwise, the size of the code is of the size of the linear group considered.

In the following, we give examples of Stiefel codes arising from the Grassmann codes of [10]. Their cardinality and minimum distance are summarized in Table II, where N_g and δ_g stand for the cardinality and minimum distance in the Grassmann manifold, whereas N_s and δ_s stand for the cardinality and minimum distance in the Stiefel manifold. Their Stiefel squared minimum distances are evaluated in percentage of the Hamming-type bound from [18].

A. Examples in $\mathcal{V}^{\mathbb{C}}_{2,1}$

The Stiefel manifold $\mathcal{V}^{\mathbb{C}}_{2,1}$ is isomorphic to the 3-sphere, and these two spaces can be easily mapped to each other. Codes described below are thus not new and are only interesting as tutorial examples.

From the Klein group V_4, inner automorphism group of the Dihedral group D_8, we obtained Grassmann digon and tetrahedron codes in [10]. The initial point $[I_{2,1}] \in \mathcal{G}^{\mathbb{C}}_{2,1}$ had an order 2 stabilizer in V_4, its Stiefel representative $I_{2,1} \in \mathcal{V}^{\mathbb{C}}_{2,1}$ has also an order 2 stabilizer in D_8 leading to a $(4, 2)$-code, worse than the optimum simplex configuration. The Grassmann digon can also be generated from the orbit of $(1\ i)^T/\sqrt{2}$ in V_4, this initial point has a trivial stabilizer in D_8 leading to an optimal $(8, 2)$-orthoplex code. The orbit of the tetrahedron codebook by D_8 leads to a $(8, 2-2/\sqrt{3})$- Stiefel code.

The orbit of the identity $I_{2,1}$ by the symmetric group S_3 leads to a $(6, 2)$-Stiefel code. The orbit of $(1\ 1)^T/\sqrt{2}$ by S_3, leading to an optimum triangle in the Grassmannian, leads also to an optimum triangle in the Stiefel manifold. This is a remarkable example of optimum joint Grassmann-Stiefel packing [19]. The Grassmann octahedron generated by S_3 is a $(6, 2-\sqrt{2})$- Stiefel code.

The initial point of the square code in [10] has a non-trivial stabilizer of order 2 in D_{16}. This gives a $(8, 2-\sqrt{2})$- Stiefel code. From the Grassmann square anteprism code, we obtain a Stiefel code of size 16 and squared minimum distance of ≈ 0.41, far from the best-known packing of ≈ 1.22.

Orbits from the binary tetrahedral group $2T$ give a $(24, 1)$-

TABLE II
SOME OF (N_s, δ_s^2)-STIEFEL ORBIT CODES.
EXPANSION OF (N_g, δ_g^2)-GRASSMANN ORBIT CODES.

Dim	Group	Order	N_g	δ_g^2	N_s	δ_s^2	%HB
2×1	D_8	8	2	1	4	2	62
			2	1	8	2	86
			4	0.66	8	0.85	37
	S_3	6	2	1	6	2	75
			3	0.75	3	3	84
			6	0.5	6	0.59	22
	D_{16}	16	4	0.5	8	0.59	25
			8	0.37	16	0.41	26
	$2T$	24	4	0.66	24	1	81
	$2O$	48	6	0.5	48	0.59	73
4×1	$2C_2$	$2^6 6!$	60	0.5	480	0.59	32
			480	0.19	3840	0.23	21
4×2	$2C_2$	$2^6 6!$	30	1	5760	1.17	41
			320	0.44	15360	1.17	47
			360	0.5	23040	1	43
			1440	0.2	46080	0.4	19

packing, vertices of the 24-cell. This is a well-known polyhedron in 4D with well-understood symmetry, and known to lead to an optimal packing [20].

Orbits from the binary octahedral group $2O$ lead to a codebook of 48 points with squared minimum distance $2 - \sqrt{2} \approx 0.59$, a combination of the 24-cell and its dual which is also a 24-cell. For comparison, the best known packing of this size has a squared minimum distance of ≈ 0.62 [20].

B. Examples in $\mathcal{V}_{4,1}^{\mathbb{C}}$ and $\mathcal{V}_{4,2}^{\mathbb{C}}$

In [10], the projective representation in 4D of the Clifford group C_2 of order $2^4 6!$ was constructed as the inner automorphism group of a group with linear representation of order $2^6 6!$. We denote this central extension of C_2 by $4C_2$ since $C_2 = 4C_2/\mathbb{Z}_4$.

The orbit of $[I_{4,1}]$ by the Clifford group leads to a $(60, 0.5)$-Grassmann code. The element $I_{4,1}$ has also a non-trivial stabilizer in $4C_2$ leading to a $(480, 2-\sqrt{2})$- Stiefel code. The orbit of $[(1\ \ 1\ \ 1\ \ i)^T/2]$ by the Clifford group leads to a $(480, \frac{3}{16})$- Grassmann code, its generator has also a non-trivial stabilizer in $4C_2$ leading to a $(3840, 2-\frac{5}{2\sqrt{2}})$- Stiefel code.

The optimum Grassmann orthoplex orbit of $[I_{4,2}]$ by C_2 generates an extension to a $(5760, 4-2\sqrt{2})$- Stiefel code. The orbit of Y_{320} by $4C_2$ leads to a $(15360, 4-2\sqrt{2})$- Stiefel code. The orbit of Y_{360} by $4C_2$ leads to a $(23040, 1)$- Stiefel code. Finally, Y_{1440} has a trivial stabilizer in $4C_2$ resulting to a code of maximal carinality, i.e. $6!2^6 = 46080$ and square minimum distance 0.40.

VII. CONCLUSION

We discussed flag orbit codes arising from projective unitary group representations. We described a spherical embedding of flag manifolds and use their corresponding chordal distance. We gave few examples of codes where elements are 2×2 and 4×4 unitary matrices modulo column permutations and columnwise rotations. We also described 4×2 Stiefel orbit codes, as expansion of Grassmann orbit codes constructed previously from the same groups.

APPENDIX

In successive order: the 90-, 180, 360-, 960-, and 1440-point generators of Table I:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1-i}{2} & \frac{1-i}{2} \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$$

$$\begin{pmatrix} \frac{i}{2} & 0 & \frac{1-i}{2} & \frac{-1+i}{2\sqrt{2}} \\ \frac{-1-i}{2} & \frac{i}{2} & \frac{-i}{2} & 0 \\ \frac{1}{2} & \frac{1+i}{2} & 0 & \frac{-1-i}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-3i-\sqrt{3}}{6} & \frac{3i-\sqrt{3}}{6} \\ 0 & \frac{1}{\sqrt{3}} & \frac{3i-\sqrt{3}}{6} & \frac{-3i-\sqrt{3}}{6} \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-\sqrt{2}}{\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 & \frac{-1}{2} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{2\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{2} \end{pmatrix}$$

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