Grassmannian Packings from Orbits of Projective Group Representations

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***Abstract*—We discuss group orbits to construct codes in the complex Grassmann manifold. Finite subgroups of the unitary group act naturally on the Grassmann manifold. Given an irreducible representation of the group of the appropriate degree, its center has no effect in orbit construction. Thus, to generate Grassmann orbit codes, projective unitary representations of finite groups are of specific interest. Following this principle, we derive basic properties and describe explicit constructions of group orbits leading to some optimum packings in 2 and 4 dimensions.**

I. INTRODUCTION

Grassmannian codes are a generalization of spherical codes with applications in the area of multiple-antenna transmission [1]–[3]. When a code is designed in order to maximize the minimum distance it is sometimes referred as *packing*. The lowest-dimensional complex Grassmann manifold is isometric to a real sphere, and Grassmannian line packing in $\mathbb{C}^2$ is equivalent to sphere packing [4]. Solutions of sphere packing problems are often vertices of polyhedra with a high degree of symmetry [5]. Spherical codes thus often have a natural interpretation as collections of orbits of a symmetry group [5], [6]. When codes consist of a single orbit, they are called *group codes* [7], *group orbits* [8] or *orbit codes* [9].

For higher-dimensional Grassmannians, there exists a non-bijective isometric spherical embedding equipped with the so-called chordal distance [8]. Therefore, the Grassmannian inherits spherical bounds, which are known to be achievable in some special cases [10], [11]. For Grassmann codes, few works have been addressing group orbit constructions. Real Grassmannian packings were first addressed in [8], where it was argued that in the 3-dimensional case, Grassmann codes corresponds to antipodal spherical codes, and the corresponding group structures are discussed. The optimum codes in [10]–[12] are recognized to be orbits of a large Clifford-type group. In [13], group orbits are used to construct Grassmann simplices, i.e. codes with one single distance in its distance distribution. More recently, the concept of group orbits have been used in [14] to recover codes in the Grassmann variety over a finite field.

The construction of orbit codes follows two steps: First we need to choose a finite group having a unitary representation of the appropriate degree. Secondly, we need to choose an appropriate initial point which leads to a code with a given cardinality and minimal distance. Given a linear representation of a group, the center has no effect in orbit construction, as the center does not change the Grassmannian equivalence class of an orbit element. Thus, to generate Grassmann orbit codes, groups having projective unitary representations are of specific interest. In this paper, we consider some finite groups having appropriate representations and find appropriate initial points heuristically. We consider a number of groups with projective 2D representations, and their orbits, finding that the octahedral group is the largest symmetry group of many low-cardinality optimum packings in 2D. Generalizing this to 4D, we consider orbits of the Clifford group. Selecting appropriate initial points we recover some codes from [15], and give new constructions with up to 2150 elements and squared chordal distance of 0.2.

II. PRELIMINARITIES

A. Grassmann Manifold

The complex Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$, with $p \leq n$, is the set of p-dimensional subspaces in the n-dimensional complex vector space $\mathbb{C}^n$. It can be expressed as a homogeneous space of the unitary group $\mathcal{U}_p$: $\mathcal{G}^{\mathbb{C}}_{n,p} \cong \frac{\mathcal{V}^{\mathbb{C}}_{n,p}}{\mathcal{U}_p}$ where $\mathcal{V}^{\mathbb{C}}_{n,p}$ is the complex Stiefel manifold, the space of orthonormal non-square matrices:

$$\mathcal{V}^{\mathbb{C}}_{n,p} = \left\{ Y \in \mathbb{C}^{n\times p} \quad | \quad Y^H Y = I_p \right\}. \tag{1}$$

A point in the Grassmann manifold is thus an equivalence class of $n \times p$ unitary matrices whose columns span the same space:

$$\mathcal{G}^{\mathbb{C}}_{n,p} = \{[Y] \quad | \quad Y \in \mathcal{V}^{\mathbb{C}}_{n,p}\}. \tag{2}$$

Here $Y \in \mathcal{V}^{\mathbb{C}}_{n,p}$ is a representative of

$$[Y] = \{Y U_p \quad | \quad U_p \in \mathcal{U}_p\}. \tag{3}$$

We define the identity element of $\mathcal{G}^{\mathbb{C}}_{n,p}$ by $[I_{n,p}]$ where $I_{n,p} = \begin{pmatrix} I_p \\ 0 \end{pmatrix}$. We simply write $[I]$ when there is no ambiguity.

For each $[Y] \in \mathcal{G}^{\mathbb{C}}_{n,p}$, we associate the orthogonal projection from $\mathbb{C}^n$ to $[Y]$: $\Pi_Y = YY^H$. This projection is unique for every element of $\mathcal{G}^{\mathbb{C}}_{n,p}$ and independent of the equivalence class representative. Let $[Y]$, $[Z] \in \mathcal{G}^{\mathbb{C}}_{n,p}$ be two subspaces of $\mathbb{C}^n$, where Y, $Z \in \mathcal{V}^{\mathbb{C}}_{n,p}$ are representative of their respective equivalence classes. The chordal distance is defined as [8]

$$d_c(Y,Z) = \frac{1}{\sqrt{2}} \|YY^H - ZZ^H\|_F. \tag{4}$$

The representation of the elements of the Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$ by their projection matrices associated with the chordal distance gives an isometric embedding in a sphere of radius $\sqrt{\frac{p(n-p)}{2n}}$ in $\mathbb{R}^D$ with $D = n^2 - 1$ [8]. We have thus the following Rankin bounds [16]: For a packing of N points in $\mathcal{G}_{n,p}^{\mathbb{C}}$ equipped with the chordal distance, the minimum distance among the elements of the packing is bounded by:

1) The simplex bound:

$$\delta^2 \leq \frac{p(n-p)}{n} \cdot \frac{N}{N-1} \tag{5}$$

which is achievable only if $N \leq D+1 = n^2$.

2) The orthoplex bound: for $N > n^2$

$$\delta^2 \leq \frac{p(n-p)}{n} \tag{6}$$

which is achievable only if $N \leq 2D = 2(n^2-1)$.

B. Basic Definitions fom Group Theory

Given a group G defined by an abstract *presentation*, we define some basic group-theoretic terms below.

Order: The order of a group is its cardinality, i.e., the number of elements in G.

Subgroup and generating set: $H \leq G$ (resp. $H < G$) means that H is a (resp. proper) subgroup of G, i.e. $H \subset G$ (resp. $H \subsetneq G$) and H is a group. Given a subset $S \subset G$, $H = \langle S \rangle$ denotes the subgroup generated by S, i.e. every element of H can be expressed as a finite combination of elements of S.

Center: The center of a group G, denoted $Z(G)$, is the subgroup of G consisting of elements that commute with every element of G:

$$Z(G) = \{z \in G \mid \forall g \in G, zg = gz\} \tag{7}$$

A group is said to be *centerless* if $Z(G)$ is trivial, i.e. consists only of the identity element.

Inner automorphism group: We define the inner automorphism group of a group G by the quotient of the group by its center

$$\text{Inn}(G) = G/Z(G). \tag{8}$$

Stabilizer: Given a group G acting from the left on a set/space $\mathcal{Y}$, the subgroup

$$\text{Stab}_G(Y) = \{g \in G \quad | \quad gY = Y\} \tag{9}$$

is called the stabilizer of $Y \in \mathcal{Y}$ in G.

Orbit: The subset of $\mathcal{Y}$

$$GY = \{gY \quad | \quad g \in G\} \tag{10}$$

is the orbit of Y under the action of G.

*p-group:*A finite group is a p-group if and only if its order is a power of p, where p is a prime number. Every element in a p-group has order a power of p.

Extraspecial group: Given a prime p, a p-group P is said to be extraspecial if its center $Z(P)$ is cyclic and if $\text{Inn}(P)$ is elementary abelian [17, Ch. 8]. For each order, there are exactly two extra special groups up to isomorphism.

C. Basics of Representation Theory

A linear *representation* of a group G is a homomorphism $\rho : G \to GL(V)$. When there is no ambiguity we simply write G for $\rho(G)$. The dimension of V is called the *degree* of ρ. If ρ is injective it is said to be *faithful*. Two representations ρ_1 and ρ_2 are said to be *equivalent* if there exist an invertible matrix M such that $\rho_1(g) = M\rho_2(g)M^{-1}$ for all $g \in G$.

A representation is called *reducible* if there exist an invariant subspace $0 \subsetneq V \subsetneq \mathbb{C}^n$ such that for all $g \in G$, $\rho(G)V \subset V$, otherwise it is called *irreducible*.

Schur's lemma states that given a group G with irreducible representation ρ, the only elements of $GL(V)$ that commute with all $g \in \rho(G)$ are the scalar matrices. A corollary of Schur's lemma is that any element in the center of a irreducible matrix group is a scalar matrix.

III. Grassmannian Orbits Codes

We now consider a finite group $G \leq \mathcal{U}_n$ acting on $\mathcal{V}_{n,p}^{\mathbb{C}}$ and $\mathcal{G}_{n,p}^{\mathbb{C}}$. We first described basic properties of orbit codes, most of them have their counterpart for Grassmann variety in [14] and we refer to [14] for proofs.

A. Basic Properties

1) Given $X, Y \in \mathcal{V}_{n,p}^{\mathbb{C}}$ and for any $g \in \mathcal{U}_n$, the chordal distance is left-invariant under unitary transform: $d_c(gX, gY) = d_c(X, Y)$.
2) If $X, Y \in \mathcal{V}_{n,p}^{\mathbb{C}}$ generate the same Grassmannian plane, i.e. $X \in [Y] \in \mathcal{G}_{n,p}^{\mathbb{C}}$ and $d_c(X, Y) = 0$, we have $gX \in [gY]$ and $d_c(gX, gY) = 0$.
3) Unitary lef-action on $[Y] \in \mathcal{G}_{n,p}^{\mathbb{C}}$ implies conjugation action on the corresponding projector Π_Y. For any $g \in \mathcal{U}_n$, we have $\Pi_{gY} = g\Pi_Y g^H$.
4) For any $[Y] \in \mathcal{G}_{n,p}^{\mathbb{C}}$, we have

$$\mathcal{G}_{n,p}^{\mathbb{C}} \cong \mathcal{U}_n/\text{Stab}([Y]). \tag{11}$$

Specifically by taking $Y = I_{n,p}$,

$$\mathcal{G}_{n,p}^{\mathbb{C}} \cong \mathcal{U}_n / \begin{pmatrix} \mathcal{U}_p & 0 \\ 0 & \mathcal{U}_{n-p} \end{pmatrix}. \tag{12}$$

5) Orbit-stabilizer theorem: Let $C = G[Y_0]$ be an orbit code. The cardinality of the code is (orbit-stabilizer theorem)

$$|C| = \frac{|G|}{|\text{Stab}_G([Y_0])|} \tag{13}$$

and the minimum distance of the code is

$$\delta_c(C) = \min_{g \in G \setminus \text{Stab}_G([Y_0])} d_c(Y_0, gY_0). \tag{14}$$

6) Every orbit code $C = G[Y_0]$ has an isometric orbit code $\hat{C} = \hat{G}[I]$ for some equivalent representation of the group G, $\hat{G} = UGU^H$ with $U \in \mathcal{U}_n$.
7) The minimum distance of any orbit of the identity $C = G[I]$ is given by

$$\delta_c^2(C) = p - \max_{g \in G \setminus \text{Stab}_G([Y_0])} \|g[1,p]\|_F^2 \tag{15}$$

where $g[1,p] = I_{n,p}^H g I_{n,p}$ is the upper left-square p-by-p submatrix of g.

B. Orbits from Projective Representation

Following Schur's lemma, the center of the unitary group $\mathcal{U}_n$ is $Z(\mathcal{U}_n) = \{e^{i\theta} I_n \quad | \quad \theta \in \mathbb{R}\} \cong \mathcal{U}_1$. The *projective unitary group* is the quotient of the unitary group by its center $\mathcal{PU}_n = \mathcal{U}_n/\mathcal{U}_1$. An element in $\mathcal{PU}_n$ is an equivalence class of unitary matrices under multiplication by a constant phase. We note that $\mathcal{PU}_n$ is isomorphic to the projective special unitary group $\mathcal{PSU}_n$. For orbits under projective representation we have the following result.

Proposition 1: Given a group G having a faithful irreducible representation in $\mathcal{U}_n$, its inner automorphism group $\mathrm{Inn}(G)$ has a representation in $\mathcal{PU}_n$. Grassmannian orbits of the action of G are orbits of the action of $\mathrm{Inn}(G)$: for any $[Y] \in \mathcal{G}^{\mathbb{C}}_{n,p}$, we have $G[Y] = \mathrm{Inn}(G)[Y]$.

Proof: This follows directly from Schur's lemma. An element in $Z(G)$ is a scalar matrix, thus if $g \in Z(G)$, $\Pi_{gY} = \Pi_Y$. The center of the group thus has no effect. ■

From Proposition 1, to construct orbit codes in the Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$, we are primarily interested in groups having a representation in $\mathcal{PU}_n$. If a group is centerless its linear representation in $\mathcal{U}_n$ is also a projective representation in $\mathcal{PU}_n$.

C. Initial Points

As a consequence of the orbit stabilizer theorem, given a group G of order N_g, an initial point has a stabilizer of order N_s which is a divisor of N_g, and the code obtained has cardinality $N = N_g/N_s$. There is two different cases to consider:

– Non trivial stabilizer ($N_s > 1$): Initial points leading to orbit codes of $N < N_g$ have a stabilizer which is a nontrivial subgroup of G. Such initial points are clearly invariant subspaces of their stabilizer subgroup. We have,

Proposition 2: Given $[Y] \in \mathcal{G}^{\mathbb{C}}_{n,p}$ and a projective group representation $\mathrm{Inn}(G) \leq \mathcal{PU}_n$, the orbit code $\mathrm{Inn}(G)[Y]$ has cardinality $N < |\mathrm{Inn}(G)|$ if and only if $[Y]$ is an invariant subspace of a non-trivial subroup $1 < S < \mathrm{Inn}(G)$.

Therefore there is only a finite number of such codes, and appropriate initial points can be heuristically found from the eigenspaces of the matrix representation of the groups.

– Trivial stabilizer $N_s = 1$: Initial points leading to orbit codes of the cardinality of the group $N = N_g$ have a stabilizer of order 1. This holds for almost every point in the Grassmannian, except the singularities described above. In this case, there is a continuum of parametrizable orbits and accordingly an infinity of codes with cardinality $N = N_g$. Using an appropriate parametrization, it may be possible to optimize the minimum distance of the code.

IV. Examples

The Grassmann manifold $\mathcal{G}^{\mathbb{C}}_{n,p}$ equipped with the chordal distance is isometrically embedded on a sphere in a Euclidean space of dimension $n^2 - 1$. Any finite group in $\mathcal{PU}_n$ acts on the basis of this Euclidean space and is a subgroup of the orthogonal group $\mathcal{SO}(n^2 - 1)$. Except for $n = 2$, where $\mathcal{SO}(3) \cong \mathcal{PU}(2)$, $\mathcal{SO}(n^2 - 1)$ is larger than $\mathcal{PU}_n$ and thus we cannot realize any rotation in the Euclidean space with this projective representation. In 2D, we give explicit constructions of group orbits recovering the optimum packings of [4]. For higher dimension, we can then look for groups that have a relatively simple action on the basis. It is natural to consider the Clifford group, employed in quantum theory, that permutes the basis of the considered space up to sign changes. We described several constructions arising from the Clifford group in 4D recovering some codes from [15]. The results are summarized in Table I and II. Some codes meet the bounds (5) or (6). Other justifications of optimality for $\mathcal{G}^{\mathbb{C}}_{2,1}$ can be found in [4].

A. Codes in $\mathcal{G}^{\mathbb{C}}_{2,1}$

In 2D, it is straightforward to find a number of small groups with projective representations [18].

1) Klein 4-group V_4: This is an abelian group of order 4. A projective representation of the Klein 4-group can be constructed from a linear representation of the dihedral group D_8.

In 2D, the dihedral group D_8 can be represented as

$$D_8 = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle. \tag{16}$$

Its center is $Z(D_8) = \langle -I \rangle \cong \mathbb{Z}_2$ and we get the projective representation of $V_4 = D_8/Z(D_8)$.

A point in $\mathcal{G}^{\mathbb{C}}_{2,1}$ has a stabilizer in V_4 of order of 2 or 1. The Grassmannian line generated by $I_{2,1}$ has a stabilizer of order 2 in V_4, and the code $V_4[I_{2,1}]$ forms a digon of distance 1. This is the optimum codebook for $N = 2$.

After optimization over a parametrizable family, we find that the line generated by

$$Y_{\text{tetra}} = \begin{pmatrix} \sqrt{\frac{1}{2} + \frac{1}{2\sqrt{3}}} \\ e^{\frac{i\pi}{4}} \sqrt{\frac{1}{2} - \frac{1}{2\sqrt{3}}} \end{pmatrix} \tag{17}$$

has a stabilizer of order 1 in V_4, and the code $V_4[Y_{\text{tetra}}]$ forms a tetrahedron, which is an optimum codebook for $N = 4$.

2) Symmetric group S_3: This group of order 6 has a standard representation of degree 2. As the group is centerless, this linear representation is a projective representation as well,

$$S_3 = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{-2\pi i}{3}} \end{pmatrix} \rangle. \tag{18}$$

The Grassmannian line generated by $I_{2,1}$ has a stabilizer of order 3 in S_3, and the code $S_3[I_{2,1}]$ forms a digon of distance 1, optimum for $N = 2$. The Grassmannian line generated by $Y_{\text{trian}} = (1\ 1)^T/\sqrt{2}$ has a stabilizer of order 2 in S_3, and its orbit $S_3[Y_{\text{trian}}]$ is a $(3, \frac{\sqrt{3}}{2})$-code, which is optimum for $N = 3$.

After optimization over a parametrizable family, we find that the line generated by

$$Y_{\text{oct}} = \begin{pmatrix} \frac{1}{\sqrt{3-\sqrt{3}}} \\ \frac{e^{\frac{i\pi}{6}}}{\sqrt{3+\sqrt{3}}} \end{pmatrix} \tag{19}$$

that has a stabilizer of order 1 in S_3 leads to an octahedron, which is optimum for $N = 6$.

TABLE I

ORBIT CODES IN $\mathcal{G}_{2,1}^{\mathbb{C}}$ OF CARDINALITY N AND MINIMUM SQUARED DISTANCE δ^2

Group	Order	Initial point	N	δ^2	Comment
$V_4 \cong \mathrm{Inn}(D_8)$	4	$I_{2,1}$	2	1	Digon (optimum)
		Y_{tetra}	4	$\frac{2}{3}$	Tetrahedron (optimum)
S_3	6	$I_{2,1}$	2	1	Digon (optimum)
		Y_{trian}	3	$\frac{3}{4}$	Triangle (optimum)
		Y_{oct}	6	$\frac{1}{2}$	Octahedron (optimum)
$D_8 \cong \mathrm{Inn}(D_{16})$	8	Y_{sq}	4	$\frac{1}{2}$	Square
		Y_{sqAnti}	8	$\frac{4-\sqrt{2}}{7}$	Square antiprism (optimum)
$T \cong A_4 \cong \mathrm{Inn}(2T)$	12	Y_{tetra2}	4	$\frac{2}{3}$	Tetrahedron (optimum)
		Y_{icosa}	12	$\frac{\sqrt{5}-1}{2\sqrt{5}}$	Icosahedron (optimum)
$O \cong S_4 \cong \mathrm{Inn}(2O)$	24	$I_{2,1}$	6	$\frac{1}{2}$	Octahedron (optimum)
		Y_{cube}	8	$\frac{1}{3}$	Cube
		Y_{snub}	24	≈ 0.1385	Snub cube (optimum)

3) Dihedral group D_8: This is an extraspecial group of order 2^3. A projective representation of D_8 can be obtained from the linear representation of the dihedral groupD_{16}.

The dihedral group D_{16} has order 16, and center $\mathbb{Z}_2$. Its subgroups are $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_8$, the Klein 4-group V_4, and $D_8 \cong D_{16}/\mathbb{Z}_2$, which is also its inner automorphism group. A representation in terms of real orthogonal matrices is given by:

$$D_{16} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \rangle. \tag{20}$$

The Grassmannian line generated by

$$Y_{\text{sq}} = \frac{1}{\sqrt{2\left(2+\sqrt{2}\right)}} \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} \tag{21}$$

has a stabilizer of order 2 in D_8, and the code $D_8[Y_{sq}]$ forms a square corresponding to the Mode 1 codebook [19].

After optimization over all the possible points with order-1 stabilizer, the following gives the largest minimum distance:

$$Y_{\text{sqAnti}} = \begin{pmatrix} \cos\frac{1}{4}\arccos\left(\frac{3}{7} - \frac{6\sqrt{2}}{7}\right) \\ \left(\frac{1}{2^{1/4}} + i\sqrt{1-\frac{1}{\sqrt{2}}}\right)\sin\frac{1}{4}\arccos\left(\frac{3}{7} - \frac{6\sqrt{2}}{7}\right) \end{pmatrix}. \tag{22}$$

The corresponding orbit forms a square antiprism, which is optimum $N = 8$.

4) Tetrahedral group T: The symmetry group of chiral tetrahedral symmetry is a group of order 12. It is isomorphic to A_4 the alternating group of degree 4, and also to the projective special linear group of degree two over the field of three elements $PSL(2,3)$. It has also the following subgroups: cyclic groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, and as normal subgroup the Klein 4-group. This group is centerless but does not have an irreducible representation in $\mathcal{U}_2$. A projective representation can however be obtained from the linear representation of the binary tetrahedral group $2T$ of order 24, isomorphic to $SL(2,3)$.

The binary tetrahedral group $2T$ is a group of order 24, isomorphic to the special linear group $SL(2,3)$. Its center is $\mathbb{Z}_2$; its other subgroups are the cyclic groups $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$ and the quaternion group Q. A faithful unitary linear representation of degree 2 is given by

$$2T = \langle \frac{1}{2} \begin{pmatrix} 1+i & 1+i \\ -1+i & 1-i \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1-i & 1+i \\ -1+i & 1+i \end{pmatrix} \rangle. \tag{23}$$

With this representation, we get a tetrahedron codebook as an orbit of the line generated by

$$Y_{\text{tetra2}} = \begin{pmatrix} \sqrt{\frac{1}{6}\left(3-\sqrt{3}\right)} \\ -e^{\frac{i\pi}{4}}\sqrt{\frac{1}{6}\left(3+\sqrt{3}\right)} \end{pmatrix} \tag{24}$$

which as order 3 in T.

The Grassmannian line generated by $I_{2,1}$ has a stabilizer of order 2 in T, and the code $T[I_{2,1}]$ forms an octahedron codebook, which is optimum for $N = 6$.

Among the Grassmannian lines with a stabilizer of order 1 in T, we find

$$Y_{\text{icosa}} = \begin{pmatrix} \frac{1}{2}\sqrt{2+\sqrt{2+\frac{2}{\sqrt{5}}}} \\ \frac{1}{2}\sqrt{2-\sqrt{2+\frac{2}{\sqrt{5}}}} \end{pmatrix} \tag{25}$$

that generate the icosahedron code $T[Y_{\text{icosa}}]$ which is optimum for $N = 12$.

5) Octahedral group O: The symmetry group of chiral octahedral symmetry is a group of order $2^2 3! = 24$. It is isomorphic to symmetric group S_4. This group is centerless and has subgroups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, the Klein 4-group, S_3, Dihedral group D_8, and the Tetrahedral group T.

A projective representation is obtained from the linear representation of the binary octahedral group $2O$ isomorphic to $S(2,3)$. It is a 48-order group with center $\mathbb{Z}_2$. Subgroups are $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, $\mathbb{Z}_8$, the quaternion group Q, dicyclic group 12, the generalized quaternion group, and the binary tetrahedral group $2T$. Its inner automorphism group is the octahedral group. A faithful unitary linear representation of degree 2 is given by

$$2O = \langle \frac{1}{2} \begin{pmatrix} -1-i & -1-i \\ 1-i & -1+i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \rangle. \tag{26}$$

The Grassmannian line generated by $I_{2,1}$ has a order 4 Stabilizer in O, and the code $O[I_{2,1}]$ gives an octahedron. The Grassmannian line generated by

$$Y_{\text{cube}} = \begin{pmatrix} \sqrt{\frac{1}{6}\left(3-\sqrt{3}\right)} \\ -e^{\frac{i\pi}{4}}\sqrt{\frac{1}{6}\left(3+\sqrt{3}\right)} \end{pmatrix} \tag{27}$$

has a stabilizer of order 3 in O, and the code $O[Y_{\text{cube}}]$ forms a cube.

There exists an initial point having a stabilizer of order 1 in O generating a snub cube, optimum for $N = 24$. The

TABLE II
CODES FROM CLIFFORD GROUP IN $\mathcal{G}_{4,2}^{\mathbb{C}}$ OF CARDINALITY N AND MINIMUM SQUARED DISTANCE δ^2

N	δ^2	Comments
30	1	$C_2[Y_{30}]$, orthoplex (optimum)
120	0.75	Subset of $C_2[Y_{320}]$
320	0.44	$C_2[Y_{320}]$
360	0.5	$C_2[Y_{360}]$
390	0.5	$C_2[Y_{30}] \cup C_2[Y_{360}]$
480	0.32	Subset of $C_2[Y_{1440}]$
710	0.44	$C_2[Y_{30}] \cup C_2[Y_{320}] \cup C_2[Y_{360}]$
1440	0.2	$C_2[Y_{1440}]$
2150	0.2	$C_2[Y_{30}] \cup C_2[Y_{320}] \cup C_2[Y_{360}] \cup C_2[Y_{1440}]$

corresponding generator has a complicated closed form, it is approximately

$$Y_{\text{snub}} \approx \begin{pmatrix} 0.962 \\ 0.240 + 0.131i \end{pmatrix}. \quad (28)$$

B. Codes in $\mathcal{G}_{4,2}^{\mathbb{C}}$

From the examples in $\mathcal{G}_{2,1}^{\mathbb{C}}$, we see that the largest investigated group, the octahedral group, includes most of the smaller groups discussed as a subgroup. A generalization of the octahedral group to higher dimensions is the Clifford group C_n of cardinality $|C_n| = 2^{n^2+2n}\prod_{j=1}^{n}(4^j - 1)$ with representation in $\mathcal{PU}_n$. Here, we follow the definition of [20], which slightly differs from the one in [12].

Consider

$$H = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad P = e^{\frac{3i\pi}{4}}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \quad (29)$$

We have normalized these so that they are special unitary: by using generators in $\mathcal{SU}_2$, we reduce the size of the generated linear representation and its center. We recover the linear representation of the binary octahedral group $2O = \langle H, P\rangle$, and the Clifford group C_1 is the exactly the following inner automorphism group

$$C_1 = O = \langle H, P\rangle/\mathbb{Z}_2 \subset \mathcal{PU}_2 \quad (30)$$

The Clifford group in 4D is obtained by tensor multiplication of the element of C_1 and an additional element $CNOT$:

$$C_2 = \langle H\otimes I_2, I_2\otimes H, P\otimes I_2, I_2\otimes P, CNOT\rangle/\mathbb{Z}_4 \subset \mathcal{PU}_4 \quad (31)$$

where

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (32)$$

The following initial points: $Y_{30} = I_{4,2}$,

$$Y_{320} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, Y_{360} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}, Y_{1140} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

have stabilizers of order 384, 36, 32, and 8 respectively. The characteristics of the corresponding orbit codes are given in Table II. The orbit of cardinality 30 is an optimum packing meeting the orthoplex bound with maximal cardinality. By combining orbits, we may obtain other codes with good distance, for example by combining the two orbits of cardinality 30 and 360, we obtain a code of the same minimal distance as [15].

V. CONCLUSION

We discussed Grassmann orbit codes arising from projective unitary group representations. We gave basic properties and described few examples in 2D and 4D. Future work includes systematic search for invariant subspaces of subgroups of the Clifford group and other classified groups.

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