Beamforming Codebooks for Two Transmit Antenna Systems based on Optimum Grassmannian Packings

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***Abstract*—Precoding codebook design for limited feedback MIMO systems is known to reduce to a discretization problem on a Grassmann manifold. The case of two-antenna beamforming is special in that it is equivalent to quantizing the real sphere. The isometry between the Grassmannian $\mathrm{G}_{2,1}^{\mathbb{C}}$ and the real sphere S^2 shows that discretization problems in the Grassmannian $\mathrm{G}_{2,1}^{\mathbb{C}}$ are directly solved by corresponding spherical codes. Notably, the Grassmannian line packing problem in $\mathbb{C}^2$, namely maximizing the minimum distance, is equivalent to the Tammes problem on the real sphere, so that optimum spherical packings give optimum Grassmannian packings. Moreover, a simple isomorphism between $\mathrm{G}_{2,1}^{\mathbb{C}}$ and S^2 enables to analytically derive simple codebooks in closed-form having low implementation complexity. Using the simple geometry of some of these codebooks, we derive closed-form expressions of the probability density function of the relative SNR loss due to limited feedback. We also investigate codebooks based on other spherical arrangements, such as solutions maximizing the harmonic mean of the mutual distances among the codewords, which is known as the Thomson problem. We find that in some special cases, Grassmannian codebooks based on these other spherical arrangements outperform codebooks from Grassmannian packing.**

***Index Terms*—Grassmannian packings, quantization, rate distortion theory, multiple-input multiple-output (MIMO) communications, precoding, limited feedback.**

I. Introduction

Multiple-Input Multiple-Output (MIMO) wireless communications systems using linear precoding have been shown to achieve large capacity gains over traditional single-input-single-output (SISO) systems [1], [2]. Linear precoding has been investigated intensely for single stream diversity transmissions [3]–[5]. Full gains from precoding are achieved when the transmitter possesses perfect channel state information (CSI). Often, especially in frequency-division duplex systems, full CSI is not available at the transmitter. One solution is to use closed-loop codebook-based precoding in which the receiver selects a precoding vector from a predefined set of vectors and feeds back the index to the transmitter.

The problem of designing beamforming codebooks reduces to discretizing the complex Grassmann manifold [5], [6] and can be formulated as a quantization problem associated with an average distortion metric that has to be extremized [7]–[10]. In [5], [6] the beamforming codebook design problem was linked to a suboptimal approach, the Grassmannian line packing problem, i.e. maximizing the minimum distance of the codebook. While being a mathematical problem of independent interest [11], packings gives well performing beamforming codebooks, and maximizing the minimum distance has thus been retained as an appropriate design criterion. Since extensive tables of packing in the real Grassmann manifold exist but for the complex Grassmannian few results are available, complex Grassmannian packing has been the focus of many recent works.

Because analytical constructions are possible only in very special cases, precoding codebooks are mostly generated by computer searches by either directly minimizing the distortion of the codebook using vector quantization algorithms such as Lloyd-type algorithms [7], [9], [12]; or maximizing its minimum distance with brute-force search [5], modified Lloyd algorithm [13], alternating projection algorithm [14], and expansion-compression algorithm [15].

In this paper we show that beamforming codebook design with two transmit antennas reduces to a quantization problem on the real sphere. We introduce a simple isometry between the corresponding Grassmannian and the 2-sphere that enables to derive simple codebooks in closed-form from spherical arrangements – an arrangement problem in the complex projective line can be directly solved by solutions of the transposed problem in the real sphere. In particular, we show that the problem of packing complex lines in $\mathbb{C}^2$ is equivalent to the Tammes problem on the real sphere. Tammes problem is a limiting case of the generalized Thomson problem, solutions of which may also be used for beamforming.

The connection with spherical codes has not been explicitly recognized in earlier work, and investigation of this special case provides insights to Grassmannian precoding codebook design. First, for the packing problem, comparing to codebooks found by computer search, e.g. in [5], [13], our approach allows to derive better packings and/or leverage optimality from sphere packing literature. Second, for the purpose of beamforming, the analytic handle provided by the underlying spherical geometry allows to design more transparent, implementation-friendly codebooks by imposing additional constraints, such as generation from a finite alphabet. Suitable rotations found by geometric inspection are used to simplify the expression of codebooks making the designed codebook more beneficial for hardware implementation than arbitrary

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codebooks.

Further, we analyze codebook performance and calculate the probability density function (pdf) of the relative signal-to-noise ratio (SNR) loss, as well as the relative average SNR loss of some of the designed optimum packings. The pdf and average of the relative SNR loss encompass the general properties and performance of the codebook but are in general hard to derive exactly. Using the simple geometry of these codebooks, we derive closed-form expressions. Analytical expressions related to codebook performance have been previously derived e.g. in [7], [16] using approximations from high resolution quantization theory, and in [17] for random vector quantization (RVQ) codebooks.

We finally show examples of cases when Grassmannian line packing is not the optimal approach for designing beamforming codebook by comparing performance of codebooks from different spherical arrangements.

The rest of the paper is organized as follows. In Section II, the system model is presented. Useful definitions on the Grassmannian are provided in Section III and the discretization problem on this manifold is stated. Section IV reviews the link between beamforming codebook design and Grassmannian discretization. Then, the sections V, VI and VII focus on the case of two transmit antennas. Section V shows, for this specific case, that Grassmannian codebooks are isometric to spherical arrangements. Several well-known spherical arrangements are presented and a framework is provided to construct optimum codebooks based on the literature on spherical codes. In Section VI, we provided closed form codebooks based on the spherical codes described previously, and we briefly discuss the benefits for implementation and the constraint of imposing equal transmit power to the antennas. Section VII provides closed form performance analysis. In addition, the performance of the beamforming codebooks obtained from the different spherical arrangements presented in this paper are compared by simulation.

II. System model

We consider a multi-input single-output (MISO) system with n transmit antennas applying transmit beamforming. The problem of codebook design for single stream transmission has been shown to be independent of the number of receive antennas [5]. Thus, for simplicity, and without loss of generality, we only consider single antenna receivers. We assume flat, independently and identically distributed (i.i.d) block fading channels so that $\mathbf{h} = [h_1, \ldots, h_n]^T$ is a vector with complex Gaussian distributed entries: $h_k \sim \mathcal{CN}(0, \sigma_h^2)$, $\forall k \in [\![1, n]\!]$. The received signal reads

$$y = \mathbf{h}^T \mathbf{w} s + z, \tag{1}$$

where the transmitted symbol s is mapped to $\mathbb{C}^n$ via the unitary beamforming vector $\mathbf{w}$; and z is an additive white complex Gaussian noise with power N_0. Without loss of generality we assume the transmitted symbol normalized to unity, $\mathbb{E}\left[|s|^2\right] = 1$.

The channel coefficients are assumed to be perfectly known at the receiver and unknown at the transmitter. The transmitter has only access to a limited amount of side information through an error-free, zero delay, low-rate feedback channel. For this purpose, the receiver feeds back the index of a codeword from a pre-designed codebook shared with the transmitter, $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_N\}$. The receiver is designed to maximize the instantaneous SNR, $\gamma = \frac{|\mathbf{h}^T\mathbf{w}|^2}{N_0}$, by choosing the precoding vector maximizing the channel gain:

$$\mathbf{w}_* = \arg\max_{\mathbf{w}\in\mathcal{W}} |\mathbf{h}^T \mathbf{w}|^2. \tag{2}$$

With perfect side information, the optimum instantaneous SNR, $\gamma_{opt} = \frac{\|\mathbf{h}\|^2}{N_0}$, can be achieved with $\mathbf{w}_{opt} = \frac{\mathbf{h}^*}{\|\mathbf{h}\|}$.

III. Grassmann Manifold

A. Definition as a metric space

The complex Grassmann manifold $\mathrm{G}_{n,1}^{\mathbb{C}}$ is the set of one-dimensional subspaces in the n-dimensional complex vector space $\mathbb{C}^n$. An element in $\mathrm{G}_{n,1}^{\mathbb{C}}$ is thus a complex line through the origin which may be specified by a unitary vector $\mathbf{w}$ spanning this subspace. The non-uniqueness of $\mathbf{w}$ leads to an equivalent representation of the Grassmann manifold as a quotient space, in which an element $[\mathbf{w}]$ of the Grassmann manifold $\mathrm{G}_{n,1}^{\mathbb{C}}$ is defined as the equivalence class of unitary vectors that span the same complex line:

$$[\mathbf{w}] = \left\{\mathbf{w}e^{i\phi} : \ e^{i\phi} \in \mathcal{U}_1\right\}. \tag{3}$$

Here $\mathbf{w} \in \Omega^n$, Ω^n being the set of unit vectors in $\mathbb{C}^n$, and $\mathcal{U}_1$ is the group of 1×1 unitary transformations. In the following, $\mathbf{w}$ will be called a generator of the equivalence class $[\mathbf{w}]$.

A metric space structure can be added with the *chordal distance* between two Grassmannian lines $[\mathbf{w}]$, $[\mathbf{v}] \in \mathrm{G}_{n,1}^{\mathbb{C}}$ [11]:

$$d_c([\mathbf{w}], [\mathbf{v}]) = \frac{1}{\sqrt{2}} \|\mathbf{w}\mathbf{w}^\dagger - \mathbf{v}\mathbf{v}^\dagger\|_F, \tag{4}$$

where $\|.\|_F$ is the Frobenius norm. Alternative formulations of the chordal distance can be expressed in terms of the principal angle between the subspaces, $\theta = \arccos(|\mathbf{w}^\dagger\mathbf{v}|) \in [0, \frac{\pi}{2}]$, or as a function of their absolute correlation $|\mathbf{w}^\dagger\mathbf{v}|$:

$$d_c([\mathbf{w}], [\mathbf{v}]) = \sqrt{1 - |\mathbf{w}^\dagger\mathbf{v}|^2} = \sin\theta. \tag{5}$$

B. Quantization on the Grassmann Manifold

In this section, we briefly present the approach of [10] and [18] on quantizing the Grassmann manifold. Given a codebook, i.e. a discretization of the manifold, a quantization map may be defined, which attaches to each point of the manifold a corresponding codeword, subject to a metric. With the metric on the Grassmannian defined above, we may define the quantization map $\mathcal{Q}_{[\mathcal{W}]}$ associated to the codebook $[\mathcal{W}] = \{[\mathbf{w}_i]\}_{i=1}^N \subset \mathrm{G}_{n,1}^{\mathbb{C}}$, as

$$\begin{aligned} \mathcal{Q}_{[\mathcal{W}]} : \quad \mathrm{G}_{n,1}^{\mathbb{C}} &\ \to \ [\mathcal{W}] \\ [\mathbf{v}] &\ \mapsto \ \arg\min_{[\mathbf{w}_i]\in[\mathcal{W}]} d_c^2([\mathbf{v}], [\mathbf{w}_i]). \end{aligned} \tag{6}$$

Given a random variable V distributed[1] on $\mathrm{G}_{n,1}^{\mathbb{C}}$, the classical approach of quantization theory on Euclidean vector

[1] The Haar measure can be used as a probability measure [18].

space [19] may be transposed to the metric space $(\mathrm{G}^{\mathbb{C}}_{n,1}, d_c)$. A suitable average distortion measure of the quantization $\mathcal{Q}_{[\mathcal{W}]}$ is

$$\mathcal{D}([\mathcal{W}]) = \mathbb{E}\left[d_c^2\left(V, \mathcal{Q}_{[\mathcal{W}]}(V)\right)\right]. \tag{7}$$

In order to achieve the minimum average distortion (7) for a codebook of a given size N, the codebook design criterion is:

$$[\mathcal{W}]_* = \arg \min_{[\mathcal{W}]\subset \mathrm{G}^{\mathbb{C}}_{n,1}} \mathcal{D}([\mathcal{W}]). \tag{8}$$

This codebook design criterion is difficult to solve directly. In case V is uniformly distributed, a suboptimal approach has been of interest. Love *et al.* [5] have shown that the distortion metric above can be bounded[2] by a decreasing function of the minimum distance of the codebook $\delta^2([\mathcal{W}]) = \min_{i\neq j} d_c^2([\mathbf{w}_i], [\mathbf{w}_j])$:

$$\mathcal{D}([\mathcal{W}]) \leq 1 - N\left(\frac{\delta^2([\mathcal{W}])}{4}\right)^{n-1}\left(1 - \frac{\delta^2([\mathcal{W}])}{4}\right). \tag{9}$$

Therefore, maximizing the minimum distance of the codebook minimizes this upper bound. The corresponding codebook design criterion may be restated as

$$[\mathcal{W}]_\ddagger = \arg \max_{[\mathcal{W}]\subset \mathrm{G}^{\mathbb{C}}_{n,1}} \delta^2([\mathcal{W}]). \tag{10}$$

This problem is known as the *Grassmannian line packing problem*. Even if it is clear that it is a suboptimal approach in the sense that $\mathcal{D}([\mathcal{W}]_*) \leq \mathcal{D}([\mathcal{W}]_\ddagger)$, the design criterion (10) has been recognized to capture the notion of uniformity and is an appropriate design criterion to obtain codebooks with small distortion, see discussion and simulations in [10].

The quality of a codebook can be gauged against the following lower bound on the distortion measure [10]:

$$\mathcal{D}([\mathcal{W}]) \geq \frac{n-1}{n} N^{\frac{-1}{n-1}}. \tag{11}$$

This bound was premeditated as an approximation in [21].

IV. Grassmannian beamforming

In order to study the performance of a beamforming codebook $\mathcal{W}$, we define the relative instantaneous SNR loss: $\gamma_{\text{loss}} = \frac{\gamma_{\text{opt}}-\gamma}{\gamma_{\text{opt}}}$. Rewriting the instantaneous SNR as

$$\begin{aligned}\gamma = \frac{|\mathbf{h}^T\mathbf{w}_*|^2}{N_0} &= \frac{\|\mathbf{h}\|^2}{N_0}|\mathbf{w}_{opt}^\dagger \mathbf{w}_*|^2 \\ &= \gamma_{\text{opt}}\left(1 - d_c^2([\mathbf{w}_{opt}], [\mathbf{w}_*])\right),\end{aligned} \tag{12}$$

reveals that the relative SNR loss is the squared chordal distance between the lines generated by the optimum and selected beamforming vectors, $\gamma_{\text{loss}} = d_c^2([\mathbf{w}_{opt}], [\mathbf{w}_*])$. The link between Grassmann manifold discretization and beamforming codebook design comes from the irrelevance of the overall phase of the beamforming vector in the instantaneous SNR. Indeed, due to the absolute value in the SNR expression, it is clear that two unitary beamforming vectors belonging to the same complex line will perform similarly, and the optimum instantaneous SNR can be reached with any vector $\mathbf{w} \in [\mathbf{w}_{opt}]$. Accordingly, the encoding function (2) can be rewritten as

$$\mathbf{w}_* = \arg\min_{\mathbf{w}\in\mathcal{W}} \gamma_{\text{loss}} = \arg\min_{\mathbf{w}\in\mathcal{W}} d_c^2([\mathbf{w}_{opt}], [\mathbf{w}]) .$$

and the line generated by $\mathbf{w}_*$ can be regarded as the quantization of the line generated by the optimum vector:

$$[\mathbf{w}_*] = \mathcal{Q}_{[\mathcal{W}]}([\mathbf{w}_{opt}]) ,$$

where $[\mathcal{W}] = \{[\mathbf{w}_i]\}_{i=1}^N$, $[\mathbf{w}_i] \in \mathrm{G}^{\mathbb{C}}_{n,1}$ is the Grassmannian codebook generated by $\mathcal{W}$.

In [5], [9], it was suggested that minimizing the relative average SNR loss, $\Gamma_{\text{loss}} = \mathbb{E}[\gamma_{\text{loss}}]$, could be used as a beamforming codebook design criterion. The average loss is

$$\Gamma_{\text{loss}} = \mathcal{D}([\mathcal{W}]) = \frac{\Gamma_{\text{opt}} - \Gamma}{\Gamma_{\text{opt}}}, \tag{13}$$

where $\Gamma_{\text{opt}} = \mathbb{E}[\gamma_{\text{opt}}]$, $\Gamma = \mathbb{E}[\gamma]$. The last equality in (13) comes from the independence of the random variables γ_{opt} and γ_{loss}, which is a consequence of the assumption that $\mathbf{h}$ is i.i.d Gaussian [12]. Therefore, designing a beamforming codebook maximizing the average SNR reduces to a quantization problem of the Grassmann manifold as described in Section III.

It is worth noticing that the distortion measure (13) is equivalent to the SNR gain previously defined by Narula *et al.* [12]:

$$\Gamma_{\text{g}} = \frac{\Gamma}{\Gamma_0} = n\left(1 - \Gamma_{\text{loss}}\right), \tag{14}$$

where $\Gamma_0 = \mathbb{E}[\gamma_0] = \frac{\mathbb{E}\left[|\sum h_i|^2\right]}{2N_0}$. The counterpart of (11) for the SNR gain is

$$\Gamma_{\text{g}} \leq n - (n-1)N^{\frac{-1}{n-1}}. \tag{15}$$

This bound was premeditated for the specific case of two transmit antenna SNR gain in [12]. The concept of SNR gain was proposed in [12] based on an upper bound of the ergodic capacity,

$$\begin{aligned}\mathcal{C} = \mathbb{E}[\log(1+\gamma)] &\leq \log(1 + \mathbb{E}[\gamma]) = \log(1 + \Gamma_{\text{g}}\Gamma_0) &(16)\\ &\leq \log\left(1 + \left(1 - \Gamma_{\text{loss}}\right)\Gamma_{\text{opt}}\right), &(17)\end{aligned}$$

the first inequality coming from the Jensen's inequality and the concavity of the logarithm function. Thus, minimizing the average SNR loss or maximizing the SNR gain maximizes an upper bound on the capacity. Similarly, gains from precoding in the symbol and bit-error rates of constellation symbols transmitted over i.i.d. Rayleigh channels are approximated by the SNR gain.

V. Grassmannian Codebooks on $\mathrm{G}^{\mathbb{C}}_{2,1}$

By showing an isometry between $\mathrm{G}^{\mathbb{C}}_{2,1}$ and the real sphere S^2, we leverage results from the spherical code literatures to build Grassmannian codebooks.

[2] A more general bound is given in [10], where the authors noted that the bound is decreasing only if $\delta(\mathcal{W})$ is below a certain value, which is in fact always true according to the Rankin bound of [11] and its generalization to the complex Grassmannian [20].

A. Isometric isomorphism: $\mathrm{G}_{2,1}^{\mathbb{C}} \cong S^2$

The Grassmann manifold $\mathrm{G}_{n,1}^{\mathbb{C}}$ is by definition the *complex projective space* $\mathbb{C}\mathrm{P}^{n-1}$ [22, p.15]. From the fibration of the unit $(2n-1)$-sphere as a circle bundle over $\mathbb{C}\mathrm{P}^{n-1}$ [23, p.135], we have[3]

$$\mathrm{G}_{n,1}^{\mathbb{C}} = \mathbb{C}\mathrm{P}^{n-1} \cong \frac{S^{2n-1}}{S^1}. \tag{18}$$

For the specific case $n = 2$, this quotient representation reduces to

$$\mathrm{G}_{2,1}^{\mathbb{C}} = \mathbb{C}\mathrm{P}^{1} \cong \frac{S^{3}}{S^1} = S^2, \tag{19}$$

where the last equality is related to the first Hopf map [24, Ex. 17.23]. Therefore, $\mathrm{G}_{2,1}^{\mathbb{C}}$, which can be identified as the *complex projective line*, is isomorphic to the unit sphere S^2. For the explicit form of the isomorphism we parameterize the unit vector $\mathbf{w}$, a generator of the equivalent class $[\mathbf{w}] \in \mathrm{G}_{2,1}^{\mathbb{C}}$, as follows

$$\mathbf{w}(\theta,\phi) = \begin{pmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{pmatrix}. \tag{20}$$

Since $[\mathbf{w}(\theta+\frac{\pi}{2},\phi)] = [\mathbf{w}(\frac{\pi}{2}-\theta,\phi+\pi)]$, by setting the range of θ and ϕ to $[0;\frac{\pi}{2}]$ and $[0;2\pi]$ respectively, we fully describe the Grassmannian. Interpreting (θ,ϕ) directly as spherical coordinates, these would describe a hemisphere. A simple morphism from a hemisphere to the whole sphere can be obtained by doubling the angle θ. The irrelevance of ϕ for $\theta = 0$ and $\frac{\pi}{2}$ in (20) leads us to the following result.

Lemma 1. *Let (ϑ,ϕ) be spherical coordinates parameterizing the unit sphere and $\mathbf{w}(\theta,\phi)$ a complex 2D unit vector according to* (20). *The map*

$$\begin{array}{cccc} \Xi: & S^2 & \to & \mathrm{G}_{2,1}^{\mathbb{C}} \\ & (\vartheta,\phi) & \mapsto & [\mathbf{w}(\frac{\vartheta}{2},\phi)] \end{array} \tag{21}$$

is an isomorphism.

For simplicity, the domain of Ξ have been chosen to be a sphere of radius one. Note that a similar map from a sphere with any strictly positive radius will be also an isomorphism. We now show that this isomorphism can be strengthened to an isometry.

Proposition 1. *The Grassmann manifold $\mathrm{G}_{2,1}^{\mathbb{C}}$ equipped with the chordal distance is isometric to the real sphere of radius one half.*

Proof: Let $[\mathbf{w}_1] = [\mathbf{w}(\theta_1,\phi_1)]$ and $[\mathbf{w}_2] = [\mathbf{w}(\theta_2,\phi_2)] \in \mathrm{G}_{2,1}^{\mathbb{C}}$ be two lines in $\mathbb{C}^n$, and θ_{12} the principal angle between these two lines. We associate to these lines the points on a sphere of radius r with spherical coordinates $\mathbf{x}_1 = (r, 2\theta_1, \phi_1)$ and $\mathbf{x}_2 = (r, 2\theta_2, \phi_2)$, and the corresponding vectors in the Euclidean space $\mathbb{R}^3$. The angle ϑ_{12} between $\mathbf{x}_1$ and $\mathbf{x}_2$ is given by the inner product in $\mathbb{R}^3$ as $\mathbf{x}_1 \cdot \mathbf{x}_2 \triangleq r^2\cos(\vartheta_{12})$. It is a direct verification to show that $\mathbf{x}_1 \cdot \mathbf{x}_2 = 2|\mathbf{w}_1^\dagger \mathbf{w}_2|^2 - 1$, and finally that $\vartheta_{12} = 2\theta_{12}$. The Euclidean distance between $\mathbf{x}_1, \mathbf{x}_2 \in S^2(r)$ is the length of the chord joining these two points,

$$|\mathbf{x}_1 - \mathbf{x}_2| = r\,\mathrm{Crd}(\vartheta_{12}) = 2r\sin\frac{\vartheta_{12}}{2} = 2r\,d_c([\mathbf{w}_1],[\mathbf{w}_2]).$$

The isometry holds if $r = 1/2$. ■

It is worth noticing that this isometry is a specific case of the isometric embedding of [11], [20], where the embedding is a bijective map.

The isometry in Proposition 1 implies that a discretization or quantization problem on $\mathrm{G}_{2,1}^{\mathbb{C}}$ can analogically be addressed on the the real sphere S^2.

B. Grassmannian codebooks from spherical arrangements

The problem of distributing a certain number of points uniformly over the surface of a sphere has been thoroughly studied [25]. We now describe some of the well studied spherical arrangements. Different criteria on the mutual distances among the codewords have been extremized in the literature, with motivation often arising from chemistry, biology and physics [26], [27]. For convenience, solutions are often described as the vertices of a convex polyhedron.

If $\mathcal{X} = \{\mathbf{x}_1,\ldots,\mathbf{x}_N\}$ is a spherical codebook on the unit sphere, we may obtain the corresponding Grassmannian codebook with the help of (21): $\Xi[\mathcal{X}] = \{\Xi[\mathbf{x}_1],\ldots,\Xi[\mathbf{x}_N]\}$. In a more direct approach, any spherical code, for example taken from Sloane's tables available at [28], can be transformed to a Grassmannian codebook by applying the corresponding simple change of variables. Cartesian coordinates (x,y,z) are first converted to spherical coordinates (ϑ,ϕ) and the latitude is divided by two $(\theta = \frac{\vartheta}{2},\phi)$[4]:

$$\theta = \frac{1}{2}\arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right),\ \phi = \arctan\left(\frac{y}{x}\right). \tag{22}$$

A generator of the corresponding Grassmannian line is then obtain by using (θ,ϕ) in (20). As a result, the chordal distance between two Grassmannian lines is half the distance between the respective spherical codewords: $d_c(\Xi[\mathbf{x}_j],\Xi[\mathbf{x}_k]) = \frac{1}{2}|\mathbf{x}_j - \mathbf{x}_k|$.

1) Grassmannian line packing or Tammes problem: The problem of placing N points on a sphere so as to maximize the minimum distance, also referred to as *Tammes problem* or *spherical packing*, is a specific case of spherical arrangements [25]. It follows from Proposition 1 that Grassmannian line packing (10) in $\mathrm{G}_{2,1}^{\mathbb{C}}$ is the same problem as Tammes problem; we can thus construct codebooks and leverage existing results from the spherical code literature by using the isomorphism of Lemma 1.

This yields the following bounds on the squared minimum distance:

Corollary 1. *Given a codebook $[\mathcal{W}] \subset \mathrm{G}_{2,1}^{\mathbb{C}}$ of cardinality N with minimum chordal distance $\delta([\mathcal{W}])$, we have*

a. The simplex bound

$$\delta^2([\mathcal{W}]) \le \frac{1}{2}\cdot\frac{N}{N-1}$$

[3] This isomorphism can be also seen directly from the definition (3).

[4] The arctangent must be defined to take into account the correct quadrant of y/x (using for example the so-called function atan2).

The bound is achievable only for $N \leq 4$ by forming a regular simplex (digon, triangle and tetrahedron).

b. The orthoplex bound *for $N > 4$,*

$$\delta^2([\mathcal{W}]) \leq \frac{1}{2}$$

The bound is achievable for $N = 5$ and 6 by forming a subset of an octahedron.

c. The Fejes Tóth bound *for $N > 2$,*

$$\delta^2([\mathcal{W}]) \leq 1 - \frac{1}{4\sin^2 \omega_N}$$

where $\omega_N = \frac{\pi}{6} \cdot \frac{N}{N-2}$. This bound is achievable for $N =$ 3, 4, 6 and 12.

Proof: Follows directly from Proposition 1. Cases a. and b. are in [11], utilizing the Rankin bounds [29] on the minimum distance of packing in $\mathrm{G}_{2,1}^{\mathbb{C}}$. Case c. utilizes an additional bound, the Fejes Tóth bound [30]. ■

The Fejes Tóth bound is specific for the 2-sphere which in this case is tighter than the bound provided in [13]. Other bounds and improvements such as the Levenshtein and the Boyvalenkov-Danev-Bumova bounds are discussed in [30, Ch. 3].

Optimum packings of N points on a sphere have been found for $N \leq 12$ and $N = 24$ [30], [31], with optimality proven geometrically. Accordingly, the optimum squared minimum distances of the Grassmannian packings for the corresponding configurations can be found in Table I:

TABLE I
SQUARED MINIMUM DISTANCES OF OPTIMUM GRASSMANNIAN PACKINGS

N	2	3	4	5	6	7	8
$\delta^2([\mathcal{W}])$	1	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{2}$	≈ 0.3949	$\frac{4-\sqrt{2}}{7}$

N	9	10	11	12	24
$\delta^2([\mathcal{W}])$	$\frac{1}{3}$	≈ 0.2978	$\frac{\sqrt{5}-1}{2\sqrt{5}}$	$\frac{\sqrt{5}-1}{2\sqrt{5}}$	≈ 0.1385

For N up to 130, the best known sphere packings are available at Sloane's webpage [32]. Fig. 1 shows the achieved minimum distance of the corresponding Grassmannian packings along with the bounds of Corollary 1, and numerical results from [13] using modified Lloyd search algorithm (numerical values available in [33]) and from [5] using brute-force computer search.

2) Generalized Thomson problem: We call the problem of maximizing the generalized p-mean of the mutual distances among the codewords the *generalized Thomson problem*:

$$M_p([\mathcal{W}]) = \left(\frac{2}{N(N-1)} \sum_{1 \leq j < k \leq N} d_c([\mathbf{w}_j], [\mathbf{w}_k])^p \right)^{1/p}. \tag{23}$$

It is the counterpart of a spherical arrangement problem which, due to its relevance to physics, is often formulated as the minimization problem of the Riesz s-energy $E_s(\mathcal{X}) = \binom{N}{2} \left(2M_{-s}(\Xi[\mathcal{X}])\right)^{-s}$ for $s > 0$. It is remarked in [25] that on S^2 this problem is only interesting for $p < 2$.

Some values of p have attracted special interest. The case $p = -1$ (sometimes also $p = -2$) is known as the (standard) *Thomson problem*. Solutions referred to as *Fekete points* have been found for $N =$ 2–4, 6, 12 [34]. Another distinguished problem is the problem of maximizing the product of the distances, known as *Whyte's problem*. This occurs when $p \to 0$ and can be restated equivalently as minimizing the logarithmic energy $E_0(\mathcal{X}) = \sum_{j<k} \log \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|}$. Solutions referred to as *logarithmic points* have been found for $N =$ 2–6, 12 [34]. The limiting case $p \to -\infty$ is the Tammes Problem discussed above.

These problems are not in general solved by identical arrangements. However due to the high symmetry of the optimum solutions of Tammes problem for 2–4, 6 and 12 points, these cases are conjectured to provide general solutions [25], [26], [34]. The principal approach to solve these problems on S^2 has been to use extensive computations, especially in high cardinality. Results may be found at [32], [35] for $p = -1$ and $-\infty$ respectively, and at [36] for p from 0 to -12.

3) Maximal volume spherical codes: In [37], a library of N-point arrangements on a sphere that maximize the volume of the convex hull is also available. These may also be used as a basis for constructing precoding codebooks.

VI. Closed-form construction with low implementation complexity

Most of the solutions of spherical arrangement problems described in the previous section are vertices of polyhedra with a high degree of symmetry which makes the derivation of closed-form Grassmannian codebooks possible. One benefit of having geometric insight on the codebooks, and the corresponding analytical handle on their design, is that suitable rotations can be found by geometric inspection. Such rotations can be used to simplify the representation of the codebook. This is beneficial from several perspectives. First, the codebook can be rotated so that it can be realized with a minimum number of different complex numbers without impairing performance. Typically, selection of the precoding codeword $\mathbf{w}_*$ in Eq. (2) is done at the receiver by exhaustive search over all codewords in the codebook. Codebooks with arbitrary complex entries result in many complex multiplications at the receiver. Reduced computing complexity, as well as reduced storage, is possible by constraining the data format of the entries to a finite alphabet set. Also, analytic control on the codebooks may be used to select how the codebooks distribute power across the antennas. Finally, analytic control of the codebooks, together with geometric intuition, allows investigating non-optimum codebooks, with possibly different symmetry properties than the optimum ones, in order to balance performance, storage and computing complexity.

A. Closed-form codebooks from spherical arrangements: examples

If a closed-from parametrization of a spherical codes is available, an equivalent closed-form Grassmannian codebook can constructed by direct computation of (22) and (20). For example, Cartesian coordinates

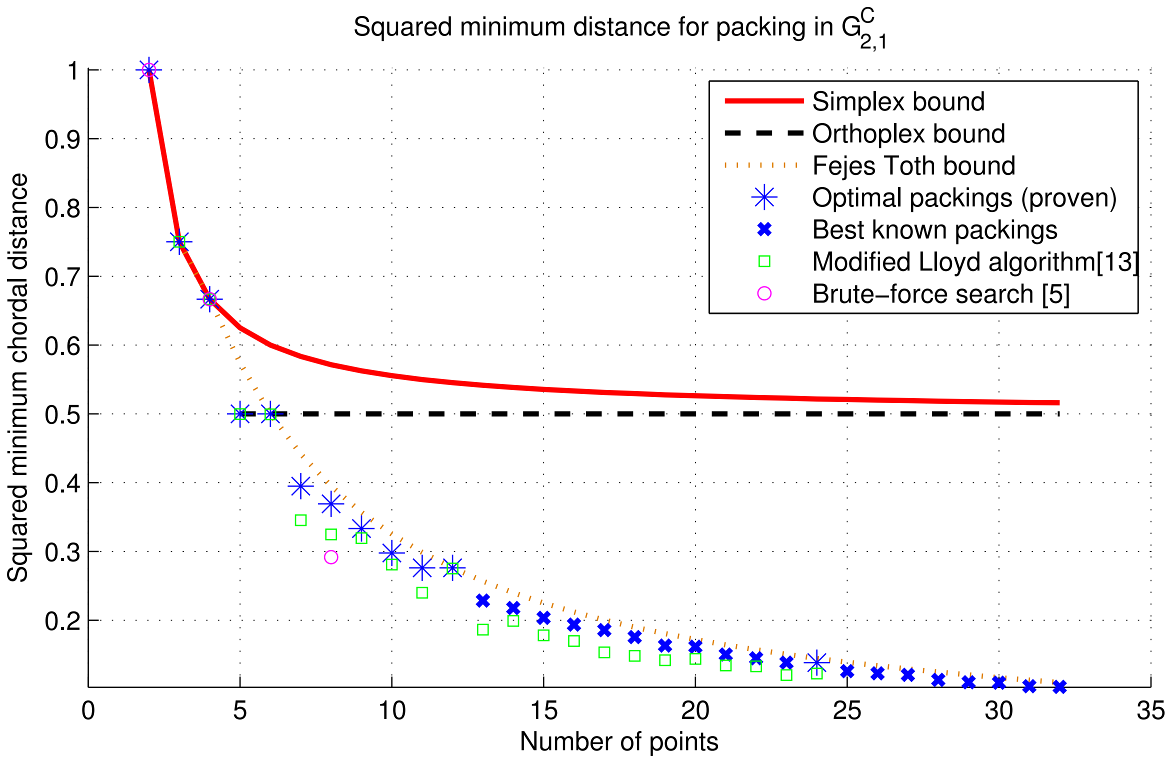

Fig. 1. Best known squared minimal chordal distance for packings in $G^{\mathbb{C}}_{2,1}$

of the vertices of a tetrahedron can be expressed as $\{(+1,+1,+1), (-1,-1,+1), (-1,+1,-1), (+1,-1,-1)\}$. Converting to spherical coordinates (22) leads to $\{(1/2 \arccos(1/\sqrt{3}), 0 \text{ or } \pi), (1/2 \arccos(-1/\sqrt{3}), \pi/2 \text{ or } 3\pi/2)\}$, then using (20) gives $\{(\alpha_+, \pm\alpha_-), (\alpha_-, \pm i\alpha_+)\}$ where the values of $\alpha_\pm$ are given in Table IV.

Examples of packings for $N = 2$–4, 6, 8, 12 and 16 are provided in Appendix IX-A. Out of these example codebooks, the ones for $N = 2$–4, 6, 8 and 12 are based on optimal packings whereas the one for $N = 16$ is putatively optimal. The polyhedra that the optimum Grassmannian packings are based on are

- $N = 2$: Digon
- $N = 3$: Triangle
- $N = 4$: Tetrahedron
- $N = 6$: Octahedron
- $N = 8$: Square antiprism
- $N = 12$: Regular icosahedron
- $N = 16$: Arrangement of 4 points on 4 latitudinal circles.

For $N = 2$, 3 and 4, two alternatives in term of power difference between the two antennas have been given. Of particular interest are the codebooks with $N = 2$, 4, 8 and 16 codewords, i.e. the 1-, 2-, 3- and 4-bit codebooks. These polyhedra are depicted in Fig. 2.

In some cases, suboptimal packings based on known polyhedra has interesting symmetry properties. Examples of such codebooks for $N = 8$, 12, 20 and 24 can be found in Appendix IX-B. The codebooks are provided with their minimum distance for comparison with optimum packings. These are based on the following polyhedra:

- $N = 8$: Cube / Stellated octahedron
- $N = 12$: Cuboctahedron
- $N = 20$: Dodecahedron
- $N = 24$: Rhombicuboctahedron.

For $N = 8$, the cube codebook has the specific property to be the concatenation of two tetrahedron codebooks, so called stellated octahedron.

Finally, Appendix IX-C provides some closed form solutions of Thomson problem on $G^{\mathbb{C}}_{2,1}$ for the case $p = -1$ with $N = 2$–7 ,12 and 32. As mentioned previously, Tammes solutions for $N = 2$–4, 6 and 12 are also Fekete points. Accordingly, those codebooks from Appendix IX-A are thus also solutions of the Thomson problem. The solutions to the Thomson problem that differ from the packings are based on the following polyhedra:

- $N = 5$: Trigonal bipyramid
- $N = 7$: Pentagonal bipyramid
- $N = 32$: Pentacis dodecahedron (the dual football)[5].

B. Equal gain transmit beamforming

Here, we discuss the problem of finding codebooks that set the average transmit power among the antennas equal. This feature is desirable in some cases, e.g., when transmit antennas are used in a power balanced manner.

[5]For $N = 32$ the codewords may be seen as the vertices of a pentakis dodecahedron or a rhombic triacontahedron. These two polyhedra have the same vertices but different edges.

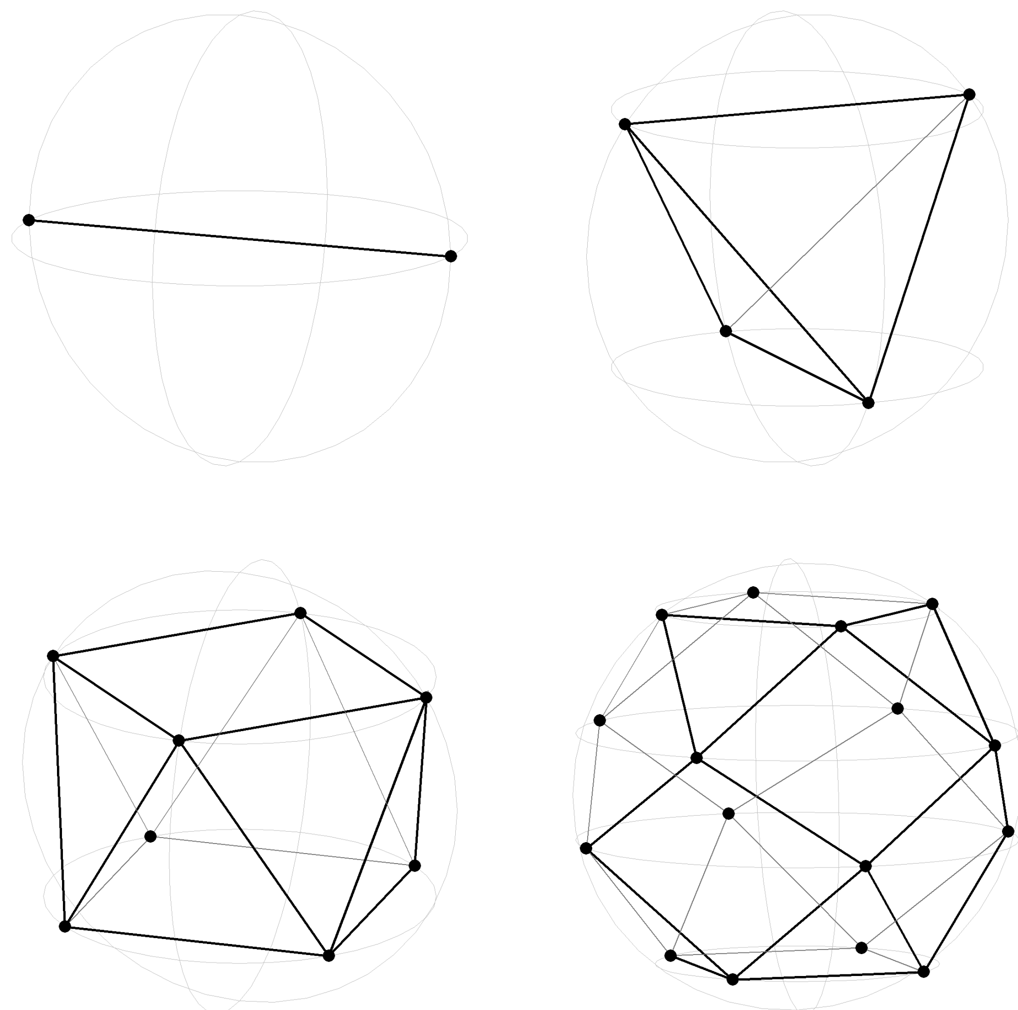

Fig. 2. Digon, tetrahedron, square antiprism and 4-bit spherical arrangement.

For the Grassmann manifold $\mathrm{G}_{2,1}^{\mathbb{C}}$, this restriction means that the moduli of the elements of the generator $\mathbf{w}(\theta, \phi)$ should be equal, requiring $\theta = \frac{\pi}{4}$. Then, points of the restricted $\mathrm{G}_{2,1}^{\mathbb{C}}$ lie on a circle, which is exactly the equator of the isomorphic sphere S^2. Consequently, the best packings obtained are the vertices of regular polygons. We can expressed the codebooks as follows:

$$\mathcal{EG}(N) = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{\frac{2i\pi k}{N}} \end{pmatrix} \mid k = 0 \dots N-1 \right\},$$

which is a specific case of the Fourier codebook in [38]. The corresponding minimum distance is $\delta(\mathcal{EG}(N)) = \sin\frac{\pi}{N}$ and equal gain solutions are optimum Grassmannian line packings only for $N = 2$ and 3.

C. Codebook storage and search complexity

The codebooks quoted in Appendices IX-A, IX-B and IX-C have been rotated in order to decrease search and storage complexity. To illustrate the implementation benefit of the closed-form representation, Tables II and III give a comparison in terms of the required number of multiplications and storage bits between random codebooks, and the 1-, 2-, 3- and 4-bit codebooks of Appendix IX-A depicted in Fig. 2. The required number of complex entries generating the codebooks has been decreased by rotating them so that several points are on the same latitude. Furthermore, if longitudinal separation between points on the same latitude are π, $\pi/2$, $\pi/4$ or a multiple of those, some complex multiplications can be reduced either to a sign change, a swap between the real and imaginary parts, additions, or a combination of such. Additionally, complexity of any codebook can be slightly decreased by scaling it so that the first entry of the first codeword is equal to one. Taking the Tetrahedron codebook as an example, this gives $\{(1, \pm c), (c, \pm i)\}$ with $c = \alpha_-/\alpha_+$, thus only one real value, c, needs to be stored, and only four real multiplications are needed in total. On the other hand, if the entries of the codebook are arbitrary complex numbers, each inner product between two vectors requires height real multiplications, and storage of four real values. In summary, with a random codebook of N codewords, the required number of multiplications is $4(2N-1)$, and the number of bits required for storage is $2(2N-1)K_b$, where K_b is the number of bits needed to represent a real number.

TABLE II
IMPLEMENTATION COMPLEXITY: NUMBER OF MULTIPLICATIONS

N	Proposed codebooks	Random codebooks
2	0	12
4	4	28
8	6	60
16	6	124

TABLE III
IMPLEMENTATION COMPLEXITY: STORAGE BITS

N	Proposed codebooks	Random codebooks
2	2	$6K_b$
4	$K_b + 21$	$14K_b$
8	$2K_b + 28$	$30K_b$
16	$4K_b + 42$	$62K_b$

VII. PERFORMANCE ANALYSIS

In this section we study the performance of the codebooks designed above, when used as precoding vectors in a MIMO system. System performance measures such as bit error rate, ergodic capacity and outage probability, can be obtained by integrating over the probability density function (pdf) of the instantaneous SNR γ. In the model we are investigating, the received SNR is a product of two independent random variables: $\gamma = \gamma_{\text{opt}}(1-\gamma_{\text{loss}})$. As γ_{opt} is independent of the codebook design, we are primarily interested in $\gamma_{\text{loss}} = d_c^2([\mathbf{w}_{opt}], [\mathbf{w}])$ which is the squared distance between the normalized channel gain and the chosen codeword. Since we have designed codebooks analytically, we may be able to compute the pdf of γ_{loss} as well as the average distortion measure (the SNR gain) in closed form.

A. Pdf of the relative SNR loss

Due to the isometry of Proposition 1, all the calculations can be done in terms of the equivalent spherical codes on the sphere $S^2(\frac{1}{2})$. We thus assume the codebook, $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ such that $\mathbf{x}_i \in S^2(\frac{1}{2}), \forall i$. We define the spherical cap $C_i(z)$ centered around $\mathbf{x}_i$ with height z by

$$C_i(z) = \{\mathbf{y} \in S^2(\tfrac{1}{2}) \ : \ |\mathbf{y} - \mathbf{x}_i|^2 \le z\}.$$

Since the radius of the sphere is one half, z is also the squared distance from $\mathbf{x}_i$ to the border of the cap. The border of the cap $C_i(z)$ is the set

$$\mathscr{C}_i(z) = \{\mathbf{y} \in S^2(\tfrac{1}{2}) \ : \ |\mathbf{y} - \mathbf{x}_i|^2 = z\}.$$

In order to calculate the pdf of γ_{loss}, we partition the surface of the sphere in N Voronoi cells[6]. The Voronoi cell around $\mathbf{x}_i$, denoted by V_i, is defined by

$$V_i = \{\mathbf{y} \in S^2(\tfrac{1}{2}) \ : \ |\mathbf{y} - \mathbf{x}_i| \le |\mathbf{y} - \mathbf{x}_j|, \forall j \in [\![1, N]\!]\}.$$

We understand the scalar z as a realization of the random variable γ_{loss}; it is convenient to regard z as the squared distance from a random point on $S^2(\frac{1}{2})$ to the closest point of the codebook. We will simply call this last random variable d_c^2 due to its analogy with the chordal distance on the Grassmannian. The probability that d_c^2 is less than or equal to a value z–the cumulative distribution function (CDF) of the squared distance in z–is expressed by

$$F_{d_c^2}(z) = \frac{\sum_{i=1}^{N} \mathcal{A}(C_i(z) \cap V_i)}{\mathcal{A}(S^2(\frac{1}{2}))},$$

where $\mathcal{A}(\cdot)$ is a function that computes area. If all the Voronoi cells are identical up to a unitary transformation, this simplifies to

$$F_{d_c^2}(z) = \frac{N}{\pi} \mathcal{A}(C_1(z) \cap V_1).$$

Here we took into account the fact that $\mathcal{A}(S^2(\frac{1}{2})) = \pi$.

The CDF may be expressed by direct integration of the surface $C_i(z) \cap V_i$. For this purpose, we define the spherical coordinates (ϑ_i, ϕ_i) on $S^2(\frac{1}{2})$ such that $\mathbf{x}_i = (\vartheta_i = 0, \phi_i = 0)$:

$$\mathcal{A}(C_i(z) \cap V_i) = \iint_{C_i(z) \cap V_i} \frac{1}{4} \sin \vartheta_i \, \mathrm{d}\vartheta_i \, \mathrm{d}\phi_i.$$

Applying the change of variable $\mathfrak{z} = \sin^2 \frac{\vartheta_i}{2}$, where $\mathfrak{z}$ is the squared distance from $\mathbf{x}_i$ to (ϑ_i, ϕ_i) we get

$$\begin{aligned} \mathcal{A}(C_i(z) \cap V_i) &= \iint_{C_i(z) \cap V_i} \frac{1}{2} \, \mathrm{d}\mathfrak{z} \, \mathrm{d}\phi_i \\ &= \int_0^z \left(\frac{1}{2} \int_{\mathscr{C}_i(\mathfrak{z}) \cap V_i} \mathrm{d}\phi_i \right) \mathrm{d}\mathfrak{z}. \end{aligned}$$

The pdf of the squared distance $f_{d_c^2}$ is then obtained by straightforward differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}z} \left(\mathcal{A}(C_i(z) \cap V_i) \right) = \frac{1}{2} \int_{\mathscr{C}_i(z) \cap V_i} \mathrm{d}\phi_i,$$

and finally we have

$$f_{d_c^2}(z) = \frac{1}{2\pi} \sum_{i=1}^{N} \int_{\mathscr{C}_i(z) \cap V_i} \mathrm{d}\phi_i. \tag{24}$$

The integral in the last equality can be calculated by taking into account the fact that the discontinuities of $\mathscr{C}_i(z) \cap V_i$ belong to the borders of V_i. The borders of V_i are geodesics which may be expressed by the goniometric equation $\cos(\phi_i - d) = \lambda \cot(\vartheta_i)$, where d and λ are constants to be defined [39]. With a simple transformation, the angle ϕ_i may then be expressed as a function of the squared distance $z = \sin^2 \frac{\vartheta_i}{2}$, which gives $\phi_i(z) = \arccos \frac{\lambda(1-2z)}{2\sqrt{z-z^2}} + d$.

We note that for the specific case of z being less than half the way to the nearest neighbor: $z \le \frac{1-\sqrt{1-\delta^2(\mathcal{X})}}{2}$, $\mathcal{A}(C_i(z) \cap V_i) = \mathcal{A}(C_i(z)) = \pi z$ is independent of i, $F_{d_c^2}(z) = Nz$ and $f_{d_c^2}(z) = N$ is constant.

We have performed explicit calculations of $f_{d_c^2}$ for the codebooks of size 1, 2, 3 and 4 bits of the best known Grassmannian packings provided in Appendix IX-A and depicted in Fig. 2. For compactness, we used the notation $\Psi(\lambda; z) = \arccos \frac{\lambda(1-2z)}{2\sqrt{z-z^2}}$. As an example for the tetrahedron codebook, the four Voronoi cells are identical equilateral spherical triangles. The border of the Voronoi cell between two vertices is a geodesic that contains the two other vertices leading to the parameterization $\cos(\phi_i) = \sqrt{2} \cot(\vartheta_i)$ and then

[6] For example with "$N = 32$ Pentakis Dodecahedron" from Appendix IX-C, the partition give the familiar shape of a football, a spherical polyhedron analog to the truncated icosahedron whose dual polyhedron is the pentakis dodecahedron.

to $\phi_i(z) = \Psi(\sqrt{2}; z)$. Due to the high symmetry of the Voronoi cell, when z is more than half the way to the nearest neighbor, here α_-^2 with the notation of Table IV, the total angle length of latitudinal circle $C_i(z)$ outside the Voronoi cell V_i is $6\phi_i(z)$. The pdf of the SNR loss for the tetrahedron is then

$$f_{d_c^2}(z) = \begin{cases} 4 & \text{for } 0 \le z \le \alpha_-^2 \\ 4 - \frac{12}{\pi}\Psi(\sqrt{2}; z) & \text{for } \alpha_-^2 \le z \le \frac{1}{3} \end{cases}.$$

The pdfs of the square antiprism, the best known solution for $N = 16$ of Tammes problem, and the equal power solutions are given in Appendix IX-D. All these pdfs are drawn in Fig. 3. For equal power transmission, analytical expressions for the pdf of the SNR loss, calculated relative to perfect equal gain beamforming, are given in [16].

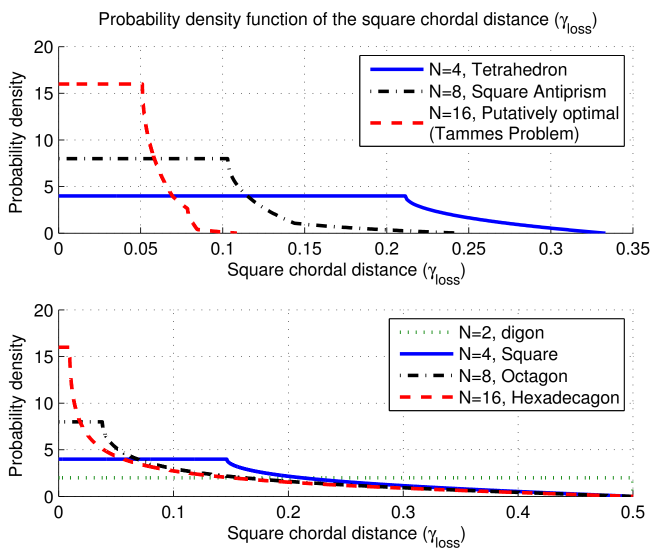

Fig. 3. Pdfs of the relative instantaneous SNR loss for different packings

B. Average distortion measure / relative SNR loss

The average distortion measure (13) of a beamforming codebook (i.e. the relative average SNR loss or equivalently the average SNR gain) may be calculated in closed form when the pdf of the relative SNR loss is known:

$$\Gamma_{\text{loss}} = \mathcal{D}([\mathcal{W}]) = \int_0^\infty z f_{d_c^2}(z)\,\mathrm{d}z.$$

For example, the corresponding average distortion for the tetrahedron codebook (2 bits) is $\Gamma_{\text{loss}} = \frac{1}{2} - \frac{\sqrt{6}}{4} + \frac{\sqrt{6}}{2\pi}\operatorname{arccot}\sqrt{2} \approx 2^{-2.97}$, and the corresponding average distortion for the square antiprism codebook (3 bits) is approximately $\Gamma_{\text{loss}} \approx 2^{-3.93}$ (its closed-form expression is unfortunately not compact). These are the minimum average distortion that may be achieved with Grassmannian packing. The tetrahedron is a highly symmetric polyhedron and the (conjectured) solution to all the spherical arrangement problems described in Section V, and additionally agrees with the best value found by Lloyd algorithm [40] up to numeric accuracy, thus we may conjecture that in this case the optimum packing codebook is the optimum average distortion codebook. However, for three bits codebook, computer search [40] reveals that Grassmannian packing

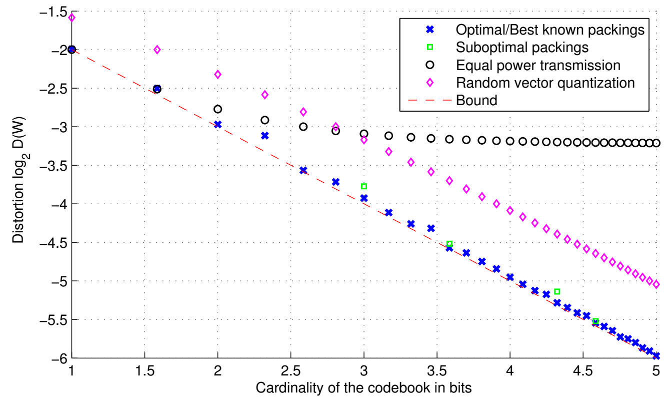

Fig. 4. Relative average SNR loss for best known Grassmannian line packings

does not give the best beamforming codebook in the sense of minimizing the SNR loss, we will see that it is actually the case for most of the cardinalities.

The average distortion obtained of the best known packings are plotted in Fig. 4 with the respective bound (11) which for two antennas is $2^{-1-\log_2 N}$. It is notable that the distortion is very close to the lower bound. The Grassmannian line packing approach gives, if not optimal, nearly optimal performance in term of SNR, which is sufficiently accurate for engineering purpose. Fig. 4 includes also the performance of the non-optimum codebooks provided in Appendix IX-B. The distortion of two other codebook designs are given for comparision: equal power transmission easily assimilated as a Fourier codebook [38] and random vector quantization (RVQ). The SNR loss for an equal power transmission is $\Gamma_{\text{loss}} = \frac{1}{2} - \frac{N}{8}\sin\frac{\pi}{N}$ [41]. Equal gain transmit beamforming has thus an asymptotic performance of $\lim_{N\to\infty}\Gamma_{\text{loss}} = \frac{1}{2} - \frac{\pi}{8} \approx 2^{-3.22}$, as it can be seen from Fig. 4. In RVQ design, the codebook is drawn randomly with uniform distribution. The distortion of random codebooks, averaged over all possible codebook realizations, is $\Gamma_{\text{loss}} = N\mathrm{B}(N, 2)$ [17] where $\mathrm{B}(x, y)$ is the Beta function.

Nevertheless, for many cases, performance of packings may be slightly improved by other arrangements. The bound (11) would be reached if all the Voronoi cells would be identical spherical caps so that the pdf of the SNR loss would be rectangular. This is only possible for $N = 2$. The best performance is attained when the Voronoi cells are as close as possible to this ideal case. Arrangements which take into account the distances to all the neighbors and not only the nearest one have thus potential to perform better.

Optimum distortion codebooks are achievable by designing spherical codes minimizing the distortion measure $\mathcal{D}(\mathcal{X}) = \mathbb{E}\left[|X - q_{\mathcal{X}}(X)|^2\right] = 4\mathcal{D}(\Xi[\mathcal{X}])$, where X is a random variable on the unit sphere and $q_{\mathcal{X}}$ is a quantizer on the sphere such that $q_{\mathcal{X}}(X) = \Xi^{-1}[\mathcal{Q}_{\Xi[\mathcal{X}]}(\Xi[X])]$. In practice, Lloyd-type algorithm is the standard approach to solve this problem [7], [9], [12]. We have proceeded with numerical simulations to compare the performance of codebooks generated with Lloyd algorithm and the different spherical configurations listed in [28], [36], and described in Section V. The number of

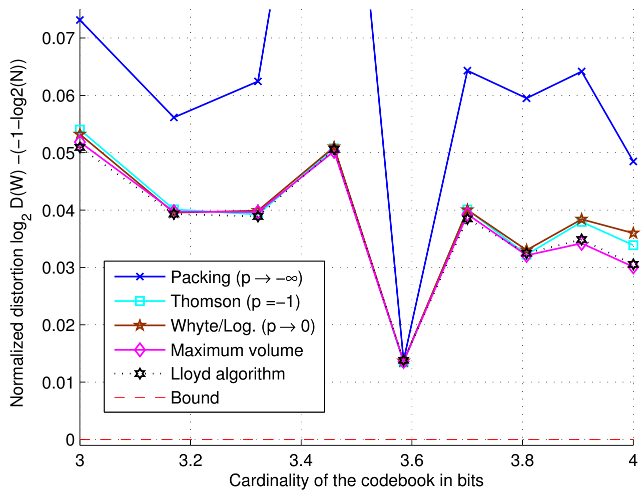

Fig. 5. Relative average SNR loss comparison for the different spherical arrangements

samples used in the Lloyd algorithm and for computing the average distortion is 2×10^7.

Fig. 5 gives a comparison of the average distortion for solutions of Tammes (Grassmannian packing), Thomson and Whyte problems, as well as for the maximum volume arrangements. The number of points is between $N = 8$ and 16. The best performance is always attained with the maximum volume configuration, closely followed by Thomson and Whyte solutions whereas packings perform slightly worse. The best results found by the Lloyd algorithm coincide with the performance of the maximum volume configurations, up to numeric accuracy. Related to deciding whether Thomson solutions or Whyte solutions are better for precoding, the results are inconclusive. With $N =$12, all the design problems give the same result. To understand the effect on performance measures, e.g. (11), the SNR gains for the best Lloyd algorithm codebooks are 2.720 dB for 3 bits and 2.869 dB for 4 bits. For the optimum packings, the SNR gains are 0.005 dB and 0.002 dB smaller, respectively – i.e. the difference is insignificant.

VIII. Conclusion

In this paper we have discussed the problem of designing closed-form beamforming codebooks for two transmit antennas. The problem reduces to a quantization problem on a real 2-sphere. Utilizing a simple isomorphism, we were able to derive simple closed form codebooks from spherical arrangements. Simple polyhedra yield some of the best codebooks. Example constructions were done for codebooks with up to 5 bits. The geometry of the 1- and 2-bit packings of [5] was revealed to be based on the digon and tetrahedron. For 3, and 4, and 5 bits, the codebooks are based on the square antiprism, an arrangement of 4×4 points on 4 latitudes, and the pentacis dodecahedron, respectively. Geometric intuition based on can be used for codebook design with low implementation complexity when additional constraints are posed on e.g. generation from a finite alphabet. For example, codebooks designed under the constraint of equal transmit power between the antennas correspond to equatorial points on the two-sphere. Using the simple geometry of some of these codebooks, we derived closed-form expressions of the corresponding relative SNR loss due to beamforming. In many cases the Grassmannian line packing is not the optimal approach for designing beamforming codebooks. Packing or Tammes problem is a specific case ($p \to -\infty$) of the more general Thomson problem, where the generalized p-mean of the mutual distances among the points is to be maximized. In addition to the packing problem with $p \to -\infty$, we have also considered the generalized Thompson problem with $p \to 0$ (Whyte problem) and $p = -1$ (standard Thomson problem), as well as the maximal volume spherical codes, for precoding codebook design. These methods yield slightly better performing codebooks than Grassmannian line packings. We observe that the distortion from maximal volume spherical codes closely matches the results from vector quantization algorithm minimizing the distortion measure.

For more transmit antennas, the link between Grassmannian codes and spherical codes is not reciprocal. In higher dimensions, there exists an isometric embedding from the Grassmannian to a hypersphere [11] which is not bijective, i.e. every Grassmannian code can be mapped to a spherical code but a spherical code can not always be mapped to a Grassmannian code keeping the distance properties. Accordingly, the discussed construction cannot be straight forwardly generalized to higher dimensions. The results in [42], [43] show that in some specific cases, algebraic constructions enable constructing isometries between Grassmannian and spherical codes even in higher dimensions.

IX. Appendix

TABLE IV
Notation used in the tables

$i = \sqrt{-1}$	$\alpha_\pm = \sqrt{\frac{1}{6}\left(3 \pm \sqrt{3}\right)}$
$\beta_\pm = \sqrt{\frac{1}{2} \pm \frac{1}{2\sqrt{1+2\sqrt{2}}}}$	$\gamma = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$
$\omega_\pm = \frac{\sqrt{2\pm\sqrt{2}}}{2}$	$\varrho_\pm = \sqrt{\frac{5\pm\sqrt{5}}{10}}$
$\varsigma = e^{i\frac{\pi}{5}} = \sqrt{\frac{5-\sqrt{5}}{8}}i + \frac{1+\sqrt{5}}{4}$	$\phi_\pm = \sqrt{\frac{15\pm\sqrt{75+30\sqrt{5}}}{30}}$
$a_\pm = \sqrt{\frac{1}{2} \pm \frac{1}{2\sqrt{5+2\sqrt{2}}}}$	$b_\pm = \sqrt{\frac{1}{2} \pm \frac{1+\sqrt{2}}{2\sqrt{5+2\sqrt{2}}}}$
$\varpi = e^{i\frac{\pi}{8}} = \omega_+ + i\omega_-$	$\varphi_\pm = \sqrt{\frac{15\pm\sqrt{75-30\sqrt{5}}}{30}}$
$\xi_1 \approx 0.78249$ is the only real postive root of $-1 - 2X^2 - 3X^4 + 4X^6 + 5X^8 + 6X^{10} + 23X^{12}$	
$\xi_2 = -3 + 2\sqrt{2} + 2(\sqrt{2}-2)\xi_1^2$	
$\zeta_{1\pm} = \sqrt{\frac{1\pm\xi_1}{2}}$	$\zeta_{2\pm} = \sqrt{\frac{\sqrt{2}\pm\sqrt{1+\xi_2}}{2\sqrt{2}}}$

A. Optimal or putatively optimal codebooks for Grassmannian line packing (Tammes problem) in $\mathrm{G}_{2,1}^{\mathbb{C}}$

$N=2$ Digon ($\delta^2=1$)

North and south poles:

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$$

$\mathcal{EG}(2)$:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{\pm 1}{\sqrt{2}}\end{bmatrix}$$

$N=3$ Triangle ($\delta^2=\frac{3}{4}$)

With a vertex at the north pole:

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}\frac{1}{2}\\ \frac{\pm\sqrt{3}i}{2}\end{bmatrix}$$

$\mathcal{EG}(3)$:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1\pm\sqrt{3}i}{2\sqrt{2}}\end{bmatrix}$$

$N=4$ Tetrahedron ($\delta^2=\frac{2}{3}$)

With a vertex at the north pole:

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{-1\pm\sqrt{3}i}{\sqrt{6}}\end{bmatrix}$$

With minimum power difference between the antennas:

$$\begin{bmatrix}\alpha_+\\ \pm\alpha_-\end{bmatrix} \begin{bmatrix}\alpha_-\\ \pm i\alpha_+\end{bmatrix}$$

$N=6$ Octahedron ($\delta^2=\frac{1}{2}$)

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{bmatrix}$$

$N=8$ Square antiprism ($\delta^2=\frac{4-\sqrt{2}}{7}$)

$$\begin{bmatrix}\beta_+\\ \pm i\beta_-\end{bmatrix} \begin{bmatrix}\beta_+\\ \pm\beta_-\end{bmatrix}$$

$$\begin{bmatrix}\beta_-\\ \pm\gamma\beta_+\end{bmatrix} \begin{bmatrix}\beta_-\\ \pm i\gamma\beta_+\end{bmatrix}$$

$N=12$ Regular icosahedron ($\delta^2=\frac{\sqrt{5}-1}{2\sqrt{5}}$)

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}\varrho_\pm\\ \mp\varsigma^{2n}\varrho_\mp\end{bmatrix}$$

$n=0,1,\ldots,4$

$N=16$ Putatively optimal with $\delta^2=1-\frac{1+{\xi_1}^2}{2}$:

$$\begin{bmatrix}\zeta_{1+}\\ \gamma^{2n}\zeta_{1-}\end{bmatrix} \begin{bmatrix}\zeta_{1-}\\ \gamma^{2n+1}\zeta_{1+}\end{bmatrix}$$

$$\begin{bmatrix}\zeta_{2+}\\ \gamma^{2n+1}\zeta_{2-}\end{bmatrix} \begin{bmatrix}\zeta_{2-}\\ \gamma^{2n}\zeta_{2+}\end{bmatrix}$$

$n=0,1,2,3$

B. Some simple (suboptimal) codebooks for $\mathrm{G}_{2,1}^{\mathbb{C}}$

$N=8$ Cube/Stellated octahedron

Combination of 2 tetrahedra ($\delta^2=\frac{1}{3}$)

$$\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{\sqrt{2}}{\sqrt{3}}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{3}}\\ \frac{-1\pm\sqrt{3}i}{\sqrt{6}}\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}\frac{\sqrt{2}}{\sqrt{3}}\\ \frac{-1}{\sqrt{3}}\end{bmatrix} \begin{bmatrix}\frac{\sqrt{2}}{\sqrt{3}}\\ \frac{\sqrt{3}\pm 3i}{6}\end{bmatrix}$$

$$\begin{bmatrix}\alpha_+\\ \pm\alpha_-\end{bmatrix} \begin{bmatrix}\alpha_-\\ \pm i\alpha_+\end{bmatrix} \begin{bmatrix}\alpha_+\\ \pm i\alpha_-\end{bmatrix} \begin{bmatrix}\alpha_-\\ \pm\alpha_+\end{bmatrix}$$

$N=12$ Cuboctahedron ($\delta^2=0.25$)

$$\begin{bmatrix}\omega_+\\ \pm\omega_-\end{bmatrix} \begin{bmatrix}\omega_+\\ \pm i\omega_-\end{bmatrix} \begin{bmatrix}\omega_-\\ \pm\omega_+\end{bmatrix}$$

$$\begin{bmatrix}\omega_-\\ \pm i\omega_+\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1\pm i}{2}\end{bmatrix} \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1\pm i}{2}\end{bmatrix}$$

$N=20$ Dodecahedron ($\delta^2=\frac{3-\sqrt{5}}{6}$)

$$\begin{bmatrix}\phi_\pm\\ \pm\phi_\mp\varsigma^{2n}\end{bmatrix} \begin{bmatrix}\varphi_\pm\\ \pm\varsigma^{2n}\varphi_\mp\end{bmatrix}$$

$n=0,1,\ldots,4$

$N=24$ Rhombicuboctahedron: ($\delta^2=\frac{1}{5+2\sqrt{2}}$)

$$\begin{bmatrix}a_\pm\\ \varpi^{\pm 1}a_\mp\end{bmatrix} \begin{bmatrix}a_\pm\\ \varpi^{\pm 3}a_\mp\end{bmatrix} \begin{bmatrix}a_\pm\\ \varpi^{\pm 5}a_\mp\end{bmatrix}$$

$$\begin{bmatrix}a_\pm\\ \varpi^{\pm 7}a_\mp\end{bmatrix} \begin{bmatrix}b_\pm\\ \gamma^{\pm 1}b_\mp\end{bmatrix} \begin{bmatrix}b_\pm\\ \gamma^{\pm 3}b_\mp\end{bmatrix}$$

C. Optimal or putatively optimal codebooks for Thomson problem ($p=-1$) in $\mathrm{G}_{2,1}^{\mathbb{C}}$

$N=2-4$, 6 and 12

Same solutions than for Tammes problem.

$N=5$ Trigonal bipyramid ($\delta^2=\frac{1}{2}$)

Combine "$N=2$ north and south poles" with $\mathcal{EG}(3)$.

$N=7$ Pentagonal bipyramid ($\delta^2=\frac{5-\sqrt{5}}{8}$)

Combine "$N=2$ north and south poles" with $\mathcal{EG}(5)$

$N=32$ Pentakis Dodecahedron

($\delta^2\approx 0.1026$, best known packing has $\delta^2\approx 0.1031$)

Combine "N=12 Regular icosahedron" and "N=20 Dodecahedron"

D. Pdfs

Equal power per antenna:

$$f_{d_c^2}(z)=\begin{cases} N & \text{for } 0\le z\le \sin^2(\frac{\pi}{2N})\\ N-\frac{2N}{\pi}\Psi(\tan\frac{\pi}{N};z) & \text{for } \sin^2(\frac{\pi}{2N})<z\le\frac{1}{2}\end{cases}$$

where $\Psi(\lambda;z)=\arccos\frac{\lambda(1-2z)}{2\sqrt{z-z^2}}$.

Square antiprism:

$$f_{d_c^2}(z)=\begin{cases} 8 & \text{for } 0\le z\le z_a\\ 8-\frac{32}{\pi}\Psi(2\omega_-;z) & \text{for } z_a\le z\le z_b\\ \frac{8}{\pi}\arctan\sqrt{1+2\sqrt{2}}-\frac{8}{\pi}\Psi(2\omega_-;z) & \text{for } z_b\le z\le\beta_-^2\end{cases}$$

where $z_a=\frac{7-\sqrt{7(3+\sqrt{2})}}{14}$ and $z_b=\frac{3\sqrt{4\sqrt{2}-5}-\sqrt{3}}{6\sqrt{4\sqrt{2}-5}}$.

16-point configuration:

$$f_{d_c^2}(z)=$$
$$\begin{cases} 16 & \text{for } 0\le z\le z_1\\ 16-\frac{64}{\pi}\Psi(\tan\vartheta_1;z) & \text{for } z_1<z\le z_2\\ 16-\frac{16}{\pi}(\phi_2-\phi_1+\phi_3)-\frac{40}{\pi}\Psi(\tan\vartheta_1;z) & \text{for } z_2<z\le z_3\\ 16-\frac{16}{\pi}(\phi_2-\phi_1+\phi_3)-\frac{40}{\pi}\Psi(\tan\vartheta_1;z) & \\ \qquad -\frac{16}{\pi}\Psi(\tan\vartheta_2;z) & \text{for } z_3<z\le z_4\\ \frac{16}{\pi}\phi_1-\frac{8}{\pi}\Psi(\tan\vartheta_1;z) & \text{for } z_4<z\le z_5\end{cases}$$

where

$$\vartheta_1=\tfrac{1}{2}\arccos\xi_1^2,\qquad \vartheta_2=\tfrac{1}{2}\arccos(-\xi_1)-\tfrac{1}{4}\arccos\xi_2,$$
$$\phi_1=\tfrac{1}{2}\operatorname{arccot}\xi_1,\qquad \phi_2=\tfrac{\pi}{2}+\tfrac{1}{2}\arctan\frac{\sqrt{1-\xi_2}}{\sqrt{2-2\xi_1^2}\sqrt{1+\xi_2}-\xi_1\sqrt{1-\xi_2}},$$
$$\phi_3=\tfrac{1}{2}\arctan\frac{\sqrt{2-2\xi_1^2}}{\xi_1\sqrt{2-2\xi_2}-\sqrt{1-\xi_1^2}\sqrt{1+\xi_2}},$$
$$h(x)=(1+x^2)^{-\frac{1}{2}},\qquad g(x)=\tfrac{1}{2}(1-xh(x)),$$
$$z_1=\sin^2\tfrac{\vartheta_1}{2},\quad z_2=g\left(\cot\vartheta_1\cos(\phi_2-\phi_1)\right),\quad z_3=\sin^2\tfrac{\vartheta_2}{2},$$
$$z_4=g\left(\cot\vartheta_1 h\left(\frac{\tan\vartheta_2}{\tan\vartheta_1\sin 2\phi_2}+\cot 2\phi_2\right)\right),\qquad z_5=\frac{1-\xi_1}{2}.$$

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