Cumulative Distribution Function of Bivariate Gamma Distribution with Arbitrary Parameters and Applications

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***Abstract*—In this letter, we consider bivariate gamma distributions with arbitrary parameters and obtain closed-form expressions for the cumulative distribution function for scenarios where the difference between the shape parameters of the marginal distributions is an integer.**

***Index Terms*—Bivariate gamma distribution, correlated fading, diversity.**

I. Introduction

In communication engineering, bivariate fading distributions are of interest, for example, in the analysis of multi-antenna wireless communication systems operating over correlated branches. Such scenarios are typical in communication terminals where the available resources do not provide independent fading [1].

The Nakagami-m distribution is a generalized fading model representing miscellaneous fading scenarios with different fading severities by varying the parameter m from $m = 1/2$ (meaning a very strong fading severity) to $m \to \infty$ (corresponding to the absence of fading) [1]-[2]. In Nakagami-m environment, the gamma distribution models channel power gains affecting the received signal-to-noise ratio (SNR). Due to a good approximating ability, the two-parameter gamma distribution is widely used as a substitute to more sophisticated fading models [3]-[5].

For scenarios with equal shape parameters m of the marginal distributions, the probability density function (PDF) of two correlated gamma variates was presented in the original work by M. Nakagami [2, eq. (125)], and for integer values of m, an expression for the cumulative distribution function (CDF) was recently derived in [6, eq. (13)]. Practical fading scenarios, however, are often characterized by non-integer values of m. Additionally, in multi-antenna systems, the fading statistics for diversity branches can be different [7]-[8]. Examples include macrodiversity, angle diversity, polarization diversity, and rake receivers where the distribution of signal power is non-uniform at different delays [7]. Full statistical studies of bivariate gamma distributions with arbitrary fading parameters were given in [8]-[9]. The latter work assumes also arbitrary cross-correlation between the in-phase and quadrature components of underlying Gaussian signals. An alternative PDF expression was obtained in [10] in terms of the Humbert hypergeometric function Φ_3. The CDF formulas are given, however, only in [8]-[9], and they are expressed via multiple infinite series with the summands containing special functions, which may be inconvenient in the practical use.

In this letter, using the approaches of [8] and [10], we derive closed-form CDF expressions for scenarios where the difference between the shape parameters of the marginal distributions is an integer. The results of this work can be used for analyzing multi-antenna systems.

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II. Closed-form CDF expressions

A. Preliminaries

The PDF of two correlated gamma variates can be represented as [10, eq. (12)]

$$p_{X_1,X_2}(x_1,x_2) = \frac{x_1^{m_1-1}x_2^{m_2-1}\alpha_1^{m_1}}{\Gamma(m_1)\Gamma(m_2)\theta_2{}^{m_2}}\exp\left(-\alpha_1 x_1 - \alpha_2 x_2\right) \times \Phi_3\left(m_2 - m_1; m_2; \rho\alpha_2 x_2, \rho\alpha_1\alpha_2 x_1 x_2\right) \quad (1)$$

where m_1, $m_2 \geq m_1$, and θ_1, θ_2 are the respective shape and scale parameters of the marginal distributions. The correlation coefficient $\rho = (E\{X_1 X_{2,a}\} - E\{X_1\}E\{X_2,a\})/(m_1\theta_1\theta_2)$ (with E denoting the expectation) characterizes the correlation between X_1 and one component of X_2, $X_{2,a}$, with the shape parameter m_1, while the other component $X_{2,b} = X_2 - X_{2,a}$ is assumed to be independent of X_1. Thus, the correlation coefficient ρ can be expressed via the correlation coefficient ρ_{C} between X_1 and X_2 as $\rho = \rho_{\mathrm{C}}\sqrt{\frac{m_2}{m_1}}$. In (1), $\alpha_1 = 1/[\theta_1(1-\rho)]$, $\alpha_2 = 1/[\theta_2(1-\rho)]$, $\Gamma(.)$ is the gamma function, and Φ_3 is the Humbert hypergeometric function, which is often defined as [11, vol. 3, eq. (7.2.4.7)]

$$\Phi_3(b; c; w, z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(b)_k w^k z^l}{(c)_{k+l}k!l!} \quad (2)$$

where$(.)_k$ denotes the Pochhammer symbol [11, vol. 3, II.2], and $c \neq 0, -1, -2, \ldots,$.

In this letter, we apply the Nuttall Q function $Q_{m,v}(\alpha,\beta)$ introduced in [12, eq. (86)] as

$$Q_{m,v}(\alpha,\beta) = \int_{\beta}^{\infty} x^m \exp\left(-\frac{\alpha^2 + x^2}{2}\right) I_v(\alpha x)dx \quad (3)$$

with $I_v(.)$ denoting the modified Bessel function of the first kind of the order v [13], and we also use the generalized

Marcum Q function $Q_m(\alpha,\beta)=\frac{1}{\alpha^{m-1}}Q_{m,m-1}(\alpha,\beta)$ [1, eq. (4.60)]. If $(m-v)$ in (3) is an odd integer, then [1, eq. (4.110)]

$$Q_{v+2k+1,v}(\alpha,\beta)=\sum_{l=1}^{k+1}c_l(k,v)\alpha^{v+2(l-1)}Q_{v+l}(\alpha,\beta)$$
$$+\exp\left(-\frac{\alpha^2+\beta^2}{2}\right)\sum_{l=1}^{k}P_{k,l}(v,\beta)\beta^{v+l+1}I_{v+l-1}(\alpha\beta) \quad (4)$$

where $c_l(k,v)=2^{k-l+1}\frac{k!}{(l-1)!}\binom{k+v}{k-l+1}$, and $P_{l,k}(v,\beta)=\sum_{j=0}^{k-l}2^{k-l-j}\frac{(k-1-j)!2^{k-l-j}}{(l-1)!}\binom{k+v}{k-l-j}\beta^{2j}$ with $\binom{\cdot}{\cdot}$ denoting the binomial coefficient.

B. Results

Lemma 1: Incomplete integrals of Bessel I functions, $\mathcal{I}_i(v,p,w,z)\triangleq\int_0^z t^{i+\frac{v-1}{2}}\exp(-pt)\,I_{v-1}\left(2\sqrt{wt}\right)dt$, where i is a non-negative integer, can be expressed as

$$\mathcal{I}_i(v,p,w,z)=\exp\left(\frac{w}{p}\right)\left[\frac{\Gamma(i+v)w^{\frac{v-1}{2}}i!}{\Gamma(v)p^{i+v}(v)_i}L_i^{v-1}\left(-\frac{w}{p}\right)\right.$$
$$\left.-\frac{Q_{2i+v,v-1}\left(\sqrt{2w/p},\sqrt{2pz}\right)}{p^{i+\frac{v+1}{2}}2^{i+\frac{v-1}{2}}}\right] \quad (5)$$

where $L_i^m(z)=\frac{(m+1)_i}{i!}\sum_{k=0}^{i}\frac{(-i)_k}{(m+1)_k k!}z^k$ is the generalized Laguerre polynomial [14], and the Nuttall Q function in (5) can be evaluated via (4).

Proof: Eq. (5) follows directly from a Laplace transform (LT) formula [11, vol. 4, eq. (3.15.2.5)] including the Kummer hypergeometric function ${}_1F_1(.)$, a transformation formula for ${}_1F_1(.)$ [11, vol. 3, eq. (7.11.15)], and (3). ■

Lemma 2: Incomplete integrals of Nuttall Q functions, $\mathcal{H}_l(v,w,q,z)\triangleq\int_0^z t^{\frac{v-1}{2}}Q_{2l+v,v-1}\left(\sqrt{wt},\sqrt{q}\right)dt$, where l is a positive integer, can be evaluated as

$$\mathcal{H}_l(v,w,q,z)=z^v w^{\frac{v-1}{2}}\sum_{k=1}^{l+1}c_k(l,v-1)\frac{(wz)^{k-1}}{v-1+k}$$
$$\times\left\{Q_{v-1+k}\left(\sqrt{wz},\sqrt{q}\right)-\left(\frac{q}{wz}\right)^{v-1+k}\right.$$
$$\left.\times\left[1-Q_{v+k}\left(\sqrt{q},\sqrt{wz}\right)\right]\right\}+\frac{2}{w^{\frac{v+1}{2}}}\sum_{k=1}^{l}q^{v+k-1}$$
$$\times P_{l,k}(v-1,\sqrt{q})\left[1-Q_{v-1+k}\left(\sqrt{q},\sqrt{wz}\right)\right]. \quad (6)$$

Proof: See Appendix A.

Lemma 3:

1. Incomplete integrals of Marcum Q functions, $\mathcal{Q}_m(p,w,q,z)\triangleq\int_0^z t^{m-1}\exp(-pt)Q_m\left(w\sqrt{t},q\right)dt$, where $m>0$, can be expressed as

$$\mathcal{Q}_m(p,w,q,z)=\sum_{k=0}^{\infty}\left(\frac{w^2}{2}\right)^k\Gamma\left(k+m,\frac{q^2}{2}\right)$$
$$\times\frac{\gamma\left[k+m,\left(p+\frac{w^2}{2}\right)z\right]}{\Gamma(k+m)\left(p+\frac{w^2}{2}\right)^{k+m}k!} \quad (7)$$

where $\gamma(a,x)$ and $\Gamma(a,x)=\Gamma(a)-\gamma(a,x)$ are the respective lower and upper incomplete gamma functions.

If the series in (7) is approximated by the sum of $(N_{\max}+1)$ terms, the remainder $\mathcal{R}_{N_{\max}}$ can be bounded as

$$\mathcal{R}_{N_{\max}}<\left(\frac{w^2}{2}\right)^{N_{\max}+1}\frac{\gamma(m+N_{\max}+1,pz)}{p^{m+N_{\max}+1}}$$
$$\times\frac{Q_{m+N_{\max}+1}\left(w\sqrt{z},q\right)}{(N_{\max}+1)!}. \quad (8)$$

2. If m is a positive integer, $\mathcal{Q}_m(p,w,q,z)$ can be evaluated as

$$\mathcal{Q}_m(p,w,q,z)=\frac{(m-1)!}{\delta^m}\exp\left(-\frac{q^2}{2}\right)\times$$
$$\left[\exp\left(\lambda\frac{q^2}{2}\right)\frac{1-((1-\lambda)q^2/2)^m}{(1-\lambda)^m(1-(1-\lambda)q^2/2)}\right.$$
$$\left.-\exp\left(-\delta\right)\sum_{k=0}^{m-1}\frac{(\delta z)^k}{k!}\mathcal{F}_{m,k}(z)\right] \quad (9)$$

where $\delta=p+w^2/2$, $\lambda=w^2/(w^2+2p)$, and $\mathcal{F}_{m,k}(z)=(1-\lambda)^{m-k}\left[\exp\left(\frac{q^2}{2}+\delta\lambda z\right)\left(\frac{q^2}{2}\right)^k\tilde{\Phi}_3\left(1,k+1,\frac{q^2}{2},\frac{\delta\lambda q^2 z}{2}\right)\right]$ $+\sum_{l=1}^{m-k}(1-\lambda)^{k+l-m-1}\left(\frac{q^2}{2}\right)^{k+l-1}\tilde{\Phi}_3\left(l,k+l,\lambda\frac{q^2}{2},\frac{\delta\lambda q^2 z}{2}\right)$ with $\tilde{\Phi}_3(b,g,w,z)=\frac{\Phi_3(b,g,w,z)}{\Gamma(g)}$.

Proof: See Appendix A.

Proposition 1: The CDF of bivariate gamma distribution (1) with integer values of (m_2-m_1) can be expressed as

$$F_{X_1,X_2}(z_1,z_2)=\frac{1}{\Gamma(m_1)}\left[\gamma\left(m_1,\frac{z_1}{\theta_1}\right)-\right.$$
$$\left.\left(\frac{z_1}{\theta_1}\right)^{m_1}\mathcal{Q}_{m_1}\left(\frac{z_1}{\theta_1},\sqrt{2\rho\alpha_1 z_1},\sqrt{2\alpha_2 z_2},1\right)\right] \quad (10)$$

if $m_1=m_2$.

Otherwise,

$$F_{X_1,X_2}(z_1,z_2)=\frac{\gamma\left(m_1,\frac{z_1}{\theta_1}\right)}{\Gamma(m_1)}-\frac{{\theta_1}^{-m_1}}{\Gamma(m_1)}\times$$
$$\left\{{z_1}^{m_1}\mathcal{Q}_{m_1}\left(\frac{z_1}{\theta_1},\sqrt{2\rho\alpha_1 z_1},\sqrt{2\alpha_2 z_2},1\right)+\right.$$
$$\frac{\exp\left(-\frac{z_2}{\theta_2}\right)}{\rho^{m_1}}\sum_{i=0}^{m_2-m_1-1}\frac{\theta_2^{-i}}{i!}\sum_{l=0}^{i}\binom{i}{l}{z_2}^{i-l}$$
$$\times\left(-\frac{1}{\rho\alpha_2}\right)^l\left[\frac{z_1^{m_1}}{m_1+l}l!L_l^{m_1}(-\alpha_1 z_1)-\right.$$
$$\left.\left.\frac{{z_1}^{\frac{m_1+1}{2}}}{2^l(2\alpha_1)^{\frac{m_1-1}{2}}}\mathcal{H}_l\left(m_1,2\alpha_1 z_1,2\rho\alpha_2 z_2,1\right)\right]\right\}. \quad (11)$$

Proof: See Appendix B.

If m_1 is an integer, and $m_1=m_2$, the CDF can be assessed via [6, eq. (13)]. For arbitrary $m_1>0$ and $m_1=m_2$, (10) gives an approximate CDF formula with the accuracy specified by (8). If $(m_2-m_1)>0$ is an integer, (11) provides a CDF formula. In any case, one can avoid the evaluation of the double infinite series required so far for the CDF assessment [8, eq. (15)]. Additionally, taking into account (4) and a relation between the generalized Marcum Q function and Φ_3 [15], we see that (10) and (11) can be efficiently evaluated via standard software packages such as *Mathematica*.

III. APPLICATIONS

The derived results can be applied to assessing the outage probability (OP) $P_{\text{out}}(\gamma_0)$ in multiple-input multiple-output (MIMO) systems with N_{T} transmitting (Tx) and two receiving (Rx) antennas. Let $\mathbf{H} = [H_{i,j}]$ be the $2 \times N_{\text{T}}$ channel matrix. We assume that the spatial correlation only appears at the receiver side, as well as that $|H_{i,j}|^2$ are gamma-distributed with different shape parameters $\mu_{i,j}$ and scale parameters $\theta_{i,j}$, $|H_{i,j}|^2 \sim \mathcal{G}(\mu_{i,j}; \theta_{i,j})$. Then the diagonal elements of the Gramian matrix $\mathbf{G} = \mathbf{H}\mathbf{H}^H$, $d_i = \sum_{j=1}^{N_{\text{T}}} |H_{i,j}|^2$, $i = 1, 2$, are correlated random variables. Generally, d_i are approximately gamma-distributed, $d_i \simeq \mathcal{G}(m_i; \theta_i)$, since they are the sums of independent gamma variables [5]. In general, the fitting procedure [5, eqs. (10)-(11)] certainly does not result in integer values of $(m_2 - m_1)$. In this case, under scenarios described below, (10)-(11) can be used as approximate bounds on real OP values since the shape parameter is inversely proportional to the amount of fading [1, eq. (1.27)]. If $\theta_{i,j} = \theta_i$, d_i are correlated gamma variables, $d_i \sim \mathcal{G}\left(m_i = \sum_{j=1}^{N_{\text{T}}} \mu_{i,j}; \theta_i\right)$, and depending on the values of $(m_2 - m_1)$, (10)-(11) provide either real OP estimates or real bounds.

CDF formulas (10)-(11) can be applied to the OP evaluation for MIMO systems exploiting Rx antenna selection where only one Rx antenna with the highest SNR is selected. If the transmitted power is split equally among the Tx antennas, the SNR γ at the combiner output is $\gamma = \frac{\bar{\gamma}_{\text{T}}}{N_{\text{T}}} \max\{d_1, d_2\}$, and $P_{\text{out}}(\gamma_0) = F_{d_1,d_2}\left(N_{\text{T}} \frac{\gamma_0}{\bar{\gamma}_{\text{T}}}, N_{\text{T}} \frac{\gamma_0}{\bar{\gamma}_{\text{T}}}\right)$. For this scenario, we present in Fig. 1 analytical estimates as well as simulation results for the OP versus the normalized threshold $\gamma_{\text{norm}} = \gamma/\theta_1$. We consider both the balanced (with $E\{d_1\} = E\{d_2\}$) and unbalanced branches. The latter scenario corresponds to the case of different shape and scale parameters of 2×2 channel matrix, where one gamma variable is used as a substitute to the sum of two gamma variables [5].

With the help of (10)-(11), an upper bound on the OP can be evaluated in transmit beamforming (TB) MIMO maximal ratio combining (MRC) systems where the SNR γ at the combiner output is the product of the transmitted SNR $\bar{\gamma}_{\text{T}}$ and the maximal eigenvalue λ_1 of $\mathbf{G}$: $\gamma = \bar{\gamma}_{\text{T}} \lambda_1$ [16]. Under these conditions, $P_{\text{out}}(\gamma_0) \leq F_{d_1,d_2}(\frac{\gamma_0}{\bar{\gamma}_{\text{T}}}, \frac{\gamma_0}{\bar{\gamma}_{\text{T}}})$ [17].

Furthermore, CDF expressions (10)-(11) can be applied to the OP evaluation for $2 \times N_{\text{T}}$ TB MIMO MRC systems with limited-rate feedback channels, see [19] for more details.

IV. CONCLUSION

In this letter, we derived formulas for the CDF of two correlated gamma variates for scenarios where the difference between the shape parameters of the marginal distributions is an integer. For integer values of the smallest shape parameter m_1, we presented exact formulas, and for arbitrary values of m_1, we obtained approximate CDF expressions and derived an upper bound on the approximation error.

Since the shape parameter m is inversely proportional to the amount of fading, the presented formulas provide CDF bounds for scenarios with non-integer values of $(m_2 - m_1)$. Thus, (10)-(11) can be applied to the CDF assessment in Nakagami-m fading with arbitrary values of m_1 and $m_2 \geq m_1$. Important application scenarios include cases of non-integer values of m_1 such as severe fading scenarios with $1/2 \leq m_1 < 1$, as well as fading scenarios where the gamma distribution is used as a substitute to real fading models.

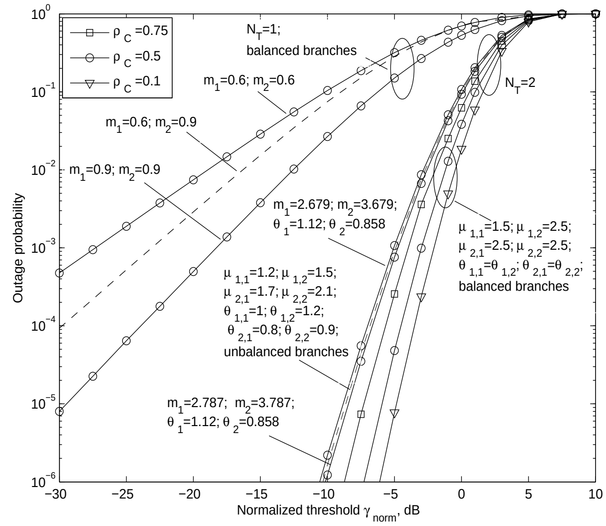

Fig. 1. Outage probability for MIMO systems with N_{T} Tx antennas, two Rx antennas, and Rx antenna selection. Single points report simulation results for integer values of $(m_2 - m_1)$, and dashed lines show simulation results for scenarios where analytical estimates do not exist.

APPENDIX A
PROOFS OF LEMMA 2 AND LEMMA 3

To obtain (6), we use (4), and the integral of $Q_m(.)$ is evaluated by parts taking into account that $\frac{\partial Q_\mu(ax,b)}{\partial x} = \frac{b^\mu \exp\left(-\frac{a^2x^2+b^2}{2}\right)}{a^{\mu-2}x^{\mu-1}} I_\mu(abx)$ [18, eqs. (18), (4)]. Then using (5) with $i = 0$, we obtain (6).

To derive (7), we apply the Maclaurin series expansion of $Q_{m_1}(\alpha, \beta)$ in the variable α^2 given in [20, eq. (29)] as

$$Q_{m_1}(\alpha, \beta) = \exp\left(-\frac{\alpha^2}{2}\right) \sum_{k=0}^{\infty} \frac{\alpha^{2k} \Gamma\left(k + m_1, \frac{\beta^2}{2}\right)}{2^k k! \Gamma(k + m_1)}. \quad (12)$$

If the series in (7) is truncated after $(N_{\max} + 1)$ terms, the remainder $\mathcal{R}_{N_{\max}}$ can be assessed by estimating the remainder in series (12), $r_{N_{\max}}$, which can be evaluated, for instance, by using the Lagrange form of the reminder [11, vol. 3, I.3.6]. We note that (12) can be derived by using the fact that $\frac{\partial^k}{\partial \alpha^{2k}} \left[\exp\left(\frac{\alpha^2}{2}\right) Q_\mu(\alpha, \beta)\right] = 2^{-k} \left[\exp\left(\frac{\alpha^2}{2}\right) Q_\mu(\alpha, \beta)\right]$ [18, eq. (15)]. Then putting $\alpha = w\sqrt{t}$ and applying the Lagrange form of the residual, we obtain that $r_{N_{\max}}(w^2 t) = \frac{\left(\frac{w^2 t}{2}\right)^{N_{\max}+1}}{(N_{\max}+1)!} Q_{m+N_{\max}+1}\left(\vartheta \cdot w\sqrt{t}, q\right)$, where $0 < \vartheta < 1$. Since $Q_{m+N_{\max}+1}\left(w\sqrt{t}, q\right)$ is a monotonically increasing function of t, $t \in [0, z]$, see $\frac{\partial Q_\mu(ax,b)}{\partial x}$ above, we find that $r_{N_{\max}}(w^2 t) < \frac{\left(\frac{w^2 t}{2}\right)^{N_{\max}+1}}{(N_{\max}+1)!} Q_{m+N_{\max}+1}\left(w\sqrt{z}, q\right)$, $t \in [0, z]$. Putting this inequality into the integral formula specifying $\mathcal{Q}_m(p, w, q, z)$, we obtain (8).

For integer values of m, using [21, eqs. (15)-(16)], we find that $\mathcal{Q}_m(p,w,q,z)$ can be evaluated via (9). ■

Appendix B
Proof of Proposition 1

In view of (1), the CDF $F_{X_1,X_2}(z_1,z_2) \triangleq \int_0^{z_1}\int_0^{z_2} p_{X_1,X_2}(x_1,x_2)dx_2dx_1$ can be expressed as

$$F_{X_1,X_2}(z_1,z_2) = \frac{\alpha_1^{m_1}}{\Gamma(m_1)\theta_2^{m_2}}\int_0^{z_1} x_1^{m_1-1}\exp\left(-\alpha_1 x_1\right) \times \int_0^{z_2} x_2^{m_2-1}\exp\left(-\alpha_2 x_2\right) \times \tilde{\Phi}_3\left(m_2-m_1; m_2; \rho\alpha_2 x_2, \rho\alpha_1\alpha_2 x_1 x_2\right)dx_2dx_1. \quad (13)$$

If $\nu=(m_2-m_1)=0$, $\tilde{\Phi}_3\left(0;c;w,z\right) = z^{\frac{1-c}{2}}I_{c-1}\left(2\sqrt{z}\right)$ (see (2)), and the inner integral in (13) can be evaluated by using (5) resulting in (10). If $\nu \neq 0$, the inner integral in (13) can be evaluated by using LT formulas [11, vol. 4, eq. 3.43.8, eq. 1.1.3.1, and eq. 1.1.3.1]. The LT $\mathcal{L}$ of the inner integral in (13) can be expressed as

$$\mathcal{L}\left\{\int_0^{z_2} {x_2}^{m_2-1}\exp\left(-\alpha_2 x_2\right)\times \tilde{\Phi}_3\left(m_2-m_1; m_2; \rho\alpha_2 x_2, \rho\alpha_1\alpha_2 x_1 x_2\right)dx_2; \{z_2, p\}\right\} = \frac{\exp\left(-\frac{\rho\alpha_1\alpha_2 x_1}{p+\alpha_2}\right)}{(p+\alpha_2)^{m_1}p(p+1/\theta_2)^{m_2-m_1}}. \quad (14)$$

Then using LT formulas [11, vol. 4, eq. 3.15.2.8 and eq. 3.10.1.1], we find that the last line of (14) is the product of LTs $\mathcal{L}\left\{\underbrace{\left(\rho\alpha_1\alpha_2 x_1\right)^{\frac{1-m_1}{2}}\exp\left(-\alpha_2 z\right)z^{\frac{m_1-1}{2}}I_{m_1-1}(\sqrt{2\rho\alpha_1\alpha_2 x_1 z})}_{f_1(z)}; \{z,p\}\right\}$ and $\mathcal{L}\left\{\underbrace{\theta_2^{m_2-m_1}\gamma\left(m_2-m_1, \frac{z}{\theta_2}\right)}_{f_2(z)}; \{z,p\}\right\}$. Thus, the inner integral in (13) is the convolution of $f_1(z)$ and $f_2(z)$ [11, vol. 4, eq. (1.1.5.7]. If ν is a positive integer, $\gamma\left(\nu, \frac{z}{\theta_2}\right)$ can be expressed as [11, vol. 1, eq. 4.1.7.10]

$$\gamma\left(\nu, \frac{z}{\theta_2}\right) = \nu!\left[1-\exp\left(-\frac{z}{\theta_2}\right)\sum_{k=0}^{\nu-1}\frac{(z/\theta_2)^k}{k!}\right]. \quad (15)$$

Evaluating the convolution of $f_1(z)$ and $f_2(z)$ and using the binomial expansion in (15), we find that

$$F_{X_1,X_2}(z_1,z_2) = \frac{\gamma\left(m_1, \frac{z_1}{\theta_1}\right)}{\Gamma(m_1)} - \frac{{\theta_1}^{-m_1}}{\Gamma(m_1)}\Big\{{z_1}^{m_1}\times \int_0^1 x^{m_1-1}\exp\left(-x\frac{z_1}{\theta_1}\right)Q_{m_1}\left(\sqrt{2\rho\alpha_1 z_1 x}, \sqrt{2\alpha_2 z_2}\right)dx + \frac{\exp\left(-\frac{z_2}{\theta_2}\right)}{\rho^{m_1}}\sum_{i=0}^{m_2-m_1-1}\frac{\theta_2^{-i}}{i!}\sum_{l=0}^{i}\binom{i}{l}{z_2}^{i-l}\left(-\frac{1}{\rho\alpha_2}\right)^l \times\left[\frac{z_1^{m_1}}{m_1+l}l!L_l^{m_1}(-\alpha_1 z_1) - \frac{{z_1}^{\frac{m_1+1}{2}}}{2^l(2\alpha_1)^{\frac{m_1-1}{2}}}\int_0^1 t^{\frac{m_1-1}{2}} \times Q_{2l+m_1,m_1-1}\left(\sqrt{2\alpha_1 z_1 t}, \sqrt{2\rho\alpha_2 z_2}\right)dt\right]\Big\}. \quad (16)$$

Then applying lemma 2, we obtain (11). ■

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