# Distribution of Diagonal Elements of a General Central Complex Wishart Matrix

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***Abstract*—We consider a general central complex Wishart matrix derived from non-circular complex Gaussian vectors. We present a technique for evaluation of the probability density, cumulative distribution, and moment generating functions of the joint distribution of the diagonal elements of the Wishart matrix. We give examples of application of the obtained results.**

***Index Terms*—Complex Wishart matrix, multivariate distributions, non-circular Gaussian processes.**

## I. Introduction

STATISTICAL properties of Wishart matrices [1] are widely applied to performance evaluation of multiple-input multiple-output (MIMO) systems. Many works on the complex Wishart distribution consider only matrices derived from circular complex-valued Gaussian vectors. The circular Gaussian processes fit well to modeling fading effects in homogeneous diffuse scattering fields where under a large number of randomly distributed point scatters, the in-phase (I) and quadrature (Q) components of the fading signal are normally distributed, independent, and have equal powers. In this case, the propagation environment is characterized by the Rayleigh or Rician fading models. Many results on the Wishart distribution are applicable only to these fading scenarios.

The real fading environment is, however, non-homogeneous [2]. Recently introduced $\eta-\mu$ [3] and $\eta-\lambda-\mu$ [4] fading models take into account the non-homogeneous structure of the propagation medium and assume that the I and Q components of the fading signal are correlated and (or) have different powers. A large number of practical examples given in [3]– [4] prove that these models show better fits to experimental data in non-line-of-sight propagation mediums than some widely used fading distributions, such as the Rayleigh or Nakagami-$m$ approximations. The underlying Gaussian processes in both $\eta-\mu$ and $\eta-\lambda-\mu$ fading models are non-circular. The full statistical characterization of the vector non-circular Gaussian variable $\mathbf{Z}$ requires not only the expectation $\mathbf{M} = E\{\mathbf{Z}\}$ and correlation matrix $\boldsymbol{\Sigma} = E\left\{\mathbf{Z}\mathbf{Z}^H\right\}$ (as in the case of the circular Gaussian vectors), but also the relation matrix $\mathbf{C} = E\left\{\mathbf{Z}\mathbf{Z}^T\right\}$ [5], where $E$ means the expectation, $H$ means transposition and complex conjugation, and $T$ means transposition.

Diagonal elements of the complex Wishart matrix characterize the effective signal-to-noise ratio (SNR) in MIMO systems under various transmission policies, and many works have been devoted to the statistical analysis of the diagonal elements of the real and complex Wishart matrices (see, for example, [6] and the references therein). If the underlying Gaussian vectors are circular, the diagonal elements of the Wishart matrix are chi-square distributed [6]. This is not, however, the case if the underlying Gaussian vectors are non-circular. Recently results on the multivariate $\eta-\mu$ distribution were presented [7]. But in this case, the underlying Gaussian processes represent just a special type of non-circular processes where the real and imaginary components are either correlated or have different powers. In this letter, we consider the complex Wishart matrix derived from the general zero-mean non-circular Gaussian vectors. We present a full statistical characterization of the diagonal elements of this matrix in terms of the probability density, cumulative distribution, and moment generating functions (PDF, CDF, and MGF).

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This work was supported by the Academy of Finland.

## II. Distribution of Diagonal Elements of the General Central Complex Wishart Matrix

### A. General Central Wishart Matrix

We consider a $K \times N$ matrix of identically distributed complex Gaussian variables with mutually independent columns $\mathbf{Z}_i = [Z_{1,i}, \ldots, Z_{K,i}]^T$, where $Z_{m,i} = X_{m,i} + jY_{m,i}$ . We assume that $\mathbf{Z}_i$ are non-circular and zero-mean Gaussian, and they are characterized by the correlation matrix $\boldsymbol{\Sigma}$ and the relation matrix $\mathbf{C}$ expressed in terms of $E\left\{X^2\right\} = \sigma_X^2$ , $E\left\{Y^2\right\} = \sigma_Y^2$ , and the correlation coefficients $\lambda_{X_{m,n}} = \frac{E\{X_{m,i}X_{n,i}\}}{\sigma_X^2}$ , $\lambda_{Y_{m,n}} = \frac{E\{Y_{m,i}Y_{n,i}\}}{\sigma_Y^2}$, and $\lambda_{m,n} = \frac{E\{X_{m,i}Y_{n,i}\}}{\sigma_X\sigma_Y}$:

$$\boldsymbol{\Sigma}_{m,n} = \lambda_{X_{m,n}}\sigma_X^2 + \lambda_{Y_{m,n}}\sigma_Y^2 + j\sigma_X\sigma_Y\left(\lambda_{n,m} - \lambda_{m,n}\right),$$

$$\begin{aligned}\mathbf{C}_{m,n} = \lambda_{X_{m,n}}\sigma_X^2 - \lambda_{Y_{m,n}}\sigma_Y^2 + j\sigma_X\sigma_Y \\ \times\left(\lambda_{n,m} + \lambda_{m,n}\right).\end{aligned} \tag{1}$$

Since $Z_{m,j}$ are identically distributed, we put $\lambda_{m,m} = \lambda$. Then the Wishart matrix is $\mathbf{W} = \mathbf{Z}\mathbf{Z}^H$, and its diagonal elements are:

$$\begin{aligned}\operatorname{diag}(\mathbf{W}) &= \left[\sum_{i=1}^{N}|Z_{1,i}|^2, \ldots, \sum_{i=1}^{N}|Z_{K,i}|^2\right]^T \\ &= [w_1, \ldots, w_K]^T = \mathbf{w}.\end{aligned} \tag{2}$$

Obviously, the second-order statistics of $\mathbf{w}$ are totally defined by the second-order statistics of the underlying Gaussain processes. The covariance $\operatorname{cov}\{w_m w_n\}$ is [8]:

$$\begin{aligned}\operatorname{cov}\{w_m w_n\} = 2N\Big[\lambda_{X_{m,n}}^2\sigma_X^4 + \lambda_{Y_{m,n}}^2\sigma_Y^4 \\ +\sigma_X^2\sigma_Y^2\left(\lambda_{n,m}^2 + \lambda_{m,n}^2\right)\Big].\end{aligned} \tag{3}$$

We introduce a normalized covariance matrix $\mathbf{R}$ with the elements $\mathbf{R}_{m,n} = \rho_{m,n} = \frac{\mathrm{cov}\{w_m w_n\}}{\mathrm{var}\{w\}}$, where var $\{.\}$ denotes the variance.

### *B. MGF, PDF, and CDF*

It is seen from (2) that $w_l$ is the sum of two correlated gamma variables $v_l$ and $t_l$ with the shape parameter $p = N/2$ and respective scale parameters $\alpha_X = 2{\sigma_X}^2$ and $\alpha_Y = 2{\sigma_Y}^2$. Using the Karhunen-Loeve expansion of the underlying Gaussian processes, $w_l$ can be represented as the sum of independent gamma variables [9]. The following lemma specifies the parameters of these gamma variables.

*Lemma:* The variable $w_l$ in (2) can be decomposed into the sum of two independent gamma variables $g_{1,l}$ and $g_{2,l}$ with the same shape parameter $N/2$ and the respective scale parameters $\theta_{1(2)} = \frac{2}{c_1+c_2-(+)\sqrt{(c_2-c_1)^2+4a^2}}$, where $a = \frac{\lambda}{2\sigma_X\sigma_Y(1-\lambda^2)}$, $c_1 = \left[2{\sigma_X}^2\left(1-\lambda^2\right)\right]^{-1}$, and $c_2 = \left[2{\sigma_Y}^2\left(1-\lambda^2\right)\right]^{-1}$.

*Proof:* See Appendix.

We decompose the vector $\mathbf{w}$ in (2) into the sum $\mathbf{w} = \mathbf{g}_1 + \mathbf{g}_2$, where $\mathbf{g}_1 = [g_{1,1}, \ldots, g_{1,K}]^T$ and $\mathbf{g}_2 = [g_{2,1}, \ldots, g_{2,K}]^T$ are independent vector gamma variables. Then the correlation of the elements of $\mathbf{w}$ is due to the correlation of the elements of $\mathbf{g}_1$ and elements of $\mathbf{g}_2$, and the correlation coefficient $\rho_{i,j}$ is expressed in terms of the correlation coefficients $\rho_{m_{i,j}} \triangleq \frac{\mathrm{cov}\{g_{m,i}g_{m,j}\}}{\mathrm{var}\{g_{m,i}\}}$ between $g_{m,i}$ and $g_{m,j}$ , $m = 1, 2$:

$$\rho_{i,j} = \left(\rho_{1_{i,j}}\mathrm{var}\left\{g_{1,i}\right\} + \rho_{2_{i,j}}\mathrm{var}\left\{g_{2,i}\right\}\right)/\mathrm{var}\left\{w\right\}. \quad (4)$$

If $\rho_{1_{i,j}} = \rho_{2_{i,j}} = \rho_{i,j}$ , we obtain the given correlation model between the elements of $\mathbf{w}$.

The MGF $M_{\mathbf{w}}(\mathbf{s}) \triangleq E\{\exp(\mathbf{w}^T\mathbf{s})\} = M_{\mathbf{g}_1}(\mathbf{s})M_{\mathbf{g}_2}(\mathbf{s})$, where $M_{\mathbf{g}_1}(.)$ and $M_{\mathbf{g}_2}(.)$ are the respective MGFs of the multivariate gamma distributions of $\mathbf{g}_1$ and $\mathbf{g}_2$, and $\mathbf{s} = [s_1, \ldots, s_K]^T$. For the arbitrary correlation model [6, eq. (4)], we find that

$$M_{\mathbf{w}}(\mathbf{s}) = \left(|\mathbf{I} - \theta_1\mathbf{R}_1\mathbf{S}| \;\cdot|\mathbf{I} - \theta_2\mathbf{R}_1\mathbf{S}|\right)^{-N/2} \quad (5)$$

where $\mathbf{I}$ is the $K$-dimensional identity matrix, and $\mathbf{S} = \mathrm{diag}\{s_j\}$. The matrix $\mathbf{R}_1$ is the covariance matrix for the Gaussian vectors generating $\mathbf{g}_1$ and $\mathbf{g}_2$: $\mathbf{R}_{1_{i,j}} = \sqrt{\mathbf{R}_{i,j}}$ [13].

The PDF $f_{\mathbf{w}}(\underbrace{z_1, \ldots, z_K}_{\mathbf{z}})$ and CDF $F_{\mathbf{w}}(\mathbf{z})$ are [7]:

$$f_{\mathbf{w}}(\mathbf{z}) = \int_{\mathbf{0}}^{\mathbf{z}} f_1(\mathbf{t})f_2(\mathbf{z}-\mathbf{t})d\mathbf{t},$$
$$F_{\mathbf{w}}(\mathbf{z}) = \int_{\mathbf{0}}^{\mathbf{z}} f_1(\mathbf{t})F_2(\mathbf{z}-\mathbf{t})d\mathbf{t} \quad (6)$$

where $\mathbf{t} = [t_1, \ldots, t_K]^T$, $f_1(F_1)(.)$ and $f_2(F_2)(.)$ are the multivariate gamma PDFs (CDFs) of $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively. Expressions for the multivariate gamma PDFs have been presented for exponential correlation [10], constant correlation [11], and arbitrary correlation [12], [6] in the form of infinite series with the terms expressed via the products of the marginal gamma PDFs of the individual variables $z_j$. Hence, (6) shows that for any correlation model, $f_{\mathbf{w}}(F_{\mathbf{w}})(\mathbf{z})$ can been defined similarly, in the form of infinite series. The terms of the series are the scaled products of the following functions of the individual variables:

$$G_1(p_1, p_2, \vartheta_1, \vartheta_2, z_j) = \int_0^{z_j} f_{\gamma_1}(p_1, \vartheta_1, t)f_{\gamma_2}(p_2, \vartheta_2, z_j - t)dt$$
$$= \frac{\exp\left(-z_j/\vartheta_2\right) z_j^{p_1+p_2-1}}{\vartheta_1^{p1}\vartheta_2^{p2}\Gamma(p_1+p_2)}\,{}_1F_1(p_1; p_1+p_2; z_j\cdot\Delta\vartheta), \qquad \text{and}$$
$$G_2(p_1, p_2, \vartheta_1, \vartheta_2, z_j) = \int_0^{z_j} f_{\gamma_1}(p_1, \vartheta_1, t)F_{\gamma_2}(p_2, \vartheta_2, z_j - t)dt$$
$$= \frac{z_j^{p_1+p_2}\Phi_2(p1; p_2; p_1+p_2+1; -z_j/\vartheta_1; -z_j/\vartheta_2)}{\Gamma(p_1+p_2+1)\vartheta_1^{p1}\vartheta_2^{p_2}} \quad (7)$$

where $f_{\gamma_i}(F_{\gamma_i})(p_i, \vartheta_i, t)$ is the marginal gamma PDF (CDF) with the shape parameter $p_i$ and scale parameter $\vartheta_i$ [6], ${}_1F_1(.)$ is the Kummer hypergeometric function, $\Phi_2(.)$ is a confluent Lauricella function [14, vol. 3], $\Delta\vartheta = \left(\frac{1}{\vartheta_2} - \frac{1}{\vartheta_1}\right)$, and the scale factors are the pairwise products of the corresponding coefficients of the series representing $f_1(\mathbf{z})$ and $f_2(F_2)(\mathbf{z})$. The first integral in (7) is evaluated via [14, vol. 1, eq. (2.3.6.1)], and the second integral is evaluated via its Laplace transform and the use of [14, vol. 4, eq. (3.43.4)]. If $p_2$ is an integer, $F_{\gamma_2}(p_2, \vartheta_2, t)$ is a linear combination of the marginal gamma PDFs, and $G_2(p_1, p_2, \vartheta_1, \vartheta_2, z_j) = \frac{\gamma(p_1, z_j/\vartheta_1)}{\Gamma(p_1)} - \vartheta_2\sum_{i=0}^{p_2-1} G_1(p_1, i+1, \vartheta_1, \vartheta_2, z_j)$, where $\gamma(a, x)$ is the lower incomplete gamma function [14, vol. 2].

For example, for arbitrary correlation, we use the results of [6] ( that are directly extended to the case of non-integer values of $N/2$) and obtain that

$$f_{\mathbf{w}}(\mathbf{z}) = q^{2K}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n_1+\cdots+n_K=n,}\;\sum_{m_1+\cdots+m_K=m}$$
$$c\left(n_1, \cdots, n_K\right)c\left(m_1, \cdots, m_K\right)\prod_{j=1}^{K}\sum_{i1=0}^{n_j}\sum_{i2=0}^{m_j}(-n_j)_{i1}$$
$$\times\frac{(-m_j)_{i2}G_1\left(\frac{N}{2}+i1, \frac{N}{2}+i2, \theta_1, \theta_2, q^2z_j\right)}{i1!i2!} \quad (8)$$

where $(a)_i$ denotes the Pochhammer symbol [14, vol. 3], and $q$ is a weighting factor providing the convergence of the power series (see [6] for details). The parameters $c(n_1, \cdots n_K)$ are the coefficients of the series expansion of $|\mathbf{I} - (\mathbf{I} - \mathbf{Q}\mathbf{R}_1\mathbf{Q})|^{-N/2} = \sum_{n=0}^{\infty}\sum_{n_1+\cdots+n_K=n} c(n_1, ..., n_K)\prod_{j=1}^{K} u_j^{n_j}$ where $\mathbf{U} = \mathrm{diag}\{u_j\}$, and $\mathbf{Q} = q\mathbf{I}$. A Mathematica$^{\mathrm{TM}}$ program for the fast numerical evaluation of $c(n_1, \ldots, n_K)$ is given in [6].

A CDF expression can be obtained from (6) as

$$F_{\mathbf{w}}(\mathbf{z}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n_1+\cdots+n_K=n,}\;\sum_{m_1+\cdots+m_K=m}$$
$$c\left(n_1, \cdots, n_K\right)c\left(m_1, \cdots, m_K\right)\prod_{j=1}^{K} P(n_j, m_j, z_j) \quad (9)$$

where $P(n_j, m_j, z_j) = \sum_{i1=0}^{n_j}\frac{(-n_j)_{i1}}{i1!}\sum_{i2=0}^{m_j-1}\frac{(-m_j+1)_{i2}}{i2!}$ $\times G_1\left(\frac{N}{2}+i1, \frac{N}{2}+i2+1, \theta_1, \theta_2, q^2z_j\right)$ if $m_j \neq 0$. Otherwise, $P(n_j, 0, z_j) = \frac{\sum_{i=0}^{n_j-1}\frac{(-n_j+1)_i G_1\left(\frac{N}{2}+i+1, \frac{N}{2}, \theta_1, \theta_2, q^2z_j\right)}{i!(q^2z_j)^i}}{\theta_1^{n_j-1}}$ if $n_j \neq 0$, and, Simpler formulas can be obtained for the case of $N_T = 2$ [15] and for the constant correlation model [7]. For the exponential correlation [10], i.e. for the case of $\rho_{i,j} = \rho^{|i-j|}$,

$$F_{\mathbf{w}}(\mathbf{z}) = \frac{(1-\rho)^N}{\left[\Gamma\left(\frac{N}{2}\right)\right]^2 (1+\rho)^{N(K-2)}} \sum_{k_1=0}^{\infty} \cdots \sum_{k_{2K-2}=0}^{\infty} \left[\frac{\rho}{(1+\rho)^{K-2}}\right]^{k_1+\ldots k_{2K-2}} \prod_{i=0}^{2K-2} \frac{1}{k_j!} \times \prod_{i=1}^{K} a_i G_2\left(\frac{N}{2}+n_i, \frac{N}{2}+n_{i+K}, \beta_{1,i}, \beta_{2,i}, z_i\right) \quad (10)$$

where $a_1 = a_K = 1$, $a_i = \prod_{j=0,1} \Gamma\left(\frac{N}{2} + k_{i-1+j\cdot(K-1)} + k_{i+j\cdot(K-1)}\right) / \prod_{j=0,1} \Gamma\left(\frac{N}{2} + k_{i+j\cdot(K-1)}\right)$ if $i = 2, \ldots, K-2$, $a_{K-1} = \prod_{j=0,1} \Gamma\left(k_{K-2+j\cdot(K-1)} + \frac{N}{2} + k_{K-1+j\cdot(K-1)}\right)$, $n_1 = k_1, n_K = k_{K-1}, n_{K+1} = k_K, n_{2K} = k_{2K-2}$, $n_i = k_{i-1} + k_i$, $\beta_{m,1} = \beta_{m,K} = \theta_m(1-\rho)$, and $\beta_{m,i} = \theta_m(1-\rho)/(1+\rho)$ for $i = 2, \ldots, K-1$, $m = 1, 2$.

The approximation of the sum of two gamma variables by a single gamma variable proposed in [16] reduces the considered problem to that solved in [6], [10]– [12]. In this case, $f_{\mathbf{w}}(\mathbf{z})$ is approximated by the multivariate gamma distribution $f_{\mathbf{g}}(\mathbf{z})$, with the shape parameter of the marginal distribution $p = \frac{E^2\{w_l\}}{E\{w_l^2\}-E^2\{w_l\}} = \left\{\left[(1+2/N)\left(\theta_1^2+\theta_2^2\right)+2\theta_1\theta_2\right](\theta_1+\theta_2)^2 - 1\right\}^{-1}$ and scale parameter $\theta = E\{w_l\}/p$ [16].

## III. Numerical Results

We apply the derived results to the evaluation of the outage probability $P_{\text{out}}(r) = Pr(\gamma \le r)$ in a MIMO system with $N_{\text{T}}$ transmitting (Tx) and $N_{\text{R}}$ receiving (Rx) antennas and with the Rx antenna selection where only one antenna with the maximal SNR is selected [6]. Let the $N_{\text{R}} \times N_{\text{T}}$ channel matrix be $\mathbf{W}$, and $E\{w_l\} = 1$, $l = 1, \ldots, N_{\text{R}}$. If the transmitted power is split equally between the Tx antennas, the SNR $\gamma$ at the output of the combiner is expressed as

$$\gamma = \frac{\gamma_{\text{T}}}{N_{\text{T}}} \max_{1\le i\le N_{\text{R}}} \operatorname{diag}\left\{\mathbf{W}\mathbf{W}^H\right\} = \frac{\gamma_{\text{T}}}{N_{\text{T}}} \max\{w_1, \ldots, w_{N_{\text{R}}}\} \quad (11)$$

where $\gamma_{\text{T}}$ is the transmitted SNR. Hence, $P_{\text{out}}(r) = \Pr(w_1 \le r \cdot N_{\text{T}}/\gamma_{\text{T}}, \ldots, w_{N_{\text{R}}} \le r \cdot N_{\text{T}}/\gamma_{\text{T}}) \triangleq F_{\mathbf{w}}(\underbrace{r \cdot N_{\text{T}}/\gamma_{\text{T}}, \ldots, r \cdot N_{\text{T}}/\gamma_{\text{T}}}_{N_{\text{R}}})$. The technique given in Section II is applicable if the spatial correlation is observed at the receiver side only. In the case of spatial correlation at the transmitter side, $w_l$ in (2) is the sum of $J > 2$ correlated gamma variables, which can be transformed into the statistically equivalent sum of a number (larger than two) of independent gamma variables [9]. Application of the decomposition method in this case is apparently unreasonable since it will result in much more complicated expressions than those obtained in this work.

In Fig. 1, we show the CDF of the received SNR for a few values of the transmitted SNR and antenna configurations: $N_{\text{T}} = 2$, $N_{\text{R}} = 4$, and $N_{\text{R}} = 2$. For $N_{\text{R}} = 4$, the matrix $\mathbf{R} = \begin{pmatrix} 1 & 0.3 & 0.25 & 0.2 \\ 0.3 & 1 & 0.16 & 0.12 \\ 0.25 & 0.16 & 1 & 0.1 \\ 0.2 & 0.12 & 0.1 & 10 \end{pmatrix}$, and $\rho = 0.3$ for $N_{\text{R}} = 2$. Unfortunately, the evaluations on the basis of (9) are very time-consuming in a common PC due to a large number of terms required for the evaluation of the truncated series. To solve the problem, we use an approximate method presented in Section II and apply the efficient Mathematica$^{\text{TM}}$ program given in [6]. In Fig. 2, the outage probability $P_{\text{out}}(r)$ is plotted as a function of the transmitted SNR. We consider the cases of exponential correlation with $\rho = 0.1$ and $\rho = 0.81$. The value of the threshold $r = E\{w_l\}$. The presented estimates clearly show effects of increasing the number of the antennas, spatial correlation, and homogeneity of the fading environment on the outage probability. It is seen, particularly, that the outage probability gets smaller as the fading environment becomes more homogeneous.

The numerical evaluations require truncation of the infinite series. The maximal number of terms taken into account ($N_{\max}$) depends on the concrete scenario, and an example for $\lambda = 0.9, \sigma_X^2/\sigma_Y^2 = 0.1, N_{\text{T}} = 2$, and $N_{\text{R}} = 3$ is shown in Table I.

TABLE I
$N_{\max}$ NEEDED TO PROVIDE $10^{-6}$ ACCURACY IN (10).

| SNR, dB | $\rho$ | $N_{\max}$ |
|---|---|---|
| 0 | 0.1 | 5 |
| 10 | 0.1 | 2 |
| 0 | 0.81 | 42 |
| 10 | 0.81 | 31 |

## IV. Conclusion

In this paper, we analyze the statistical distribution of the diagonal elements of the general central complex Wishart matrix derived from the non-circular Gaussian vectors. We present a full statistical characterization of the diagonal elements in terms of the PDF, CDF, and MGF, and give examples of application of the obtained results for performance evaluation of MIMO systems. The results of this work are based on the bivariate gamma distribution [17] and on the multivariate central chi-square distribution [6], [10]– [12], and thus they are applicable only to the central Wishart matrix.

## Appendix
## Proof of Lemma

We evaluate the PDF $f_w(z)$ of the sum of two correlated gamma variables $w = v + t$ with the joint PDF $f(v,t)$ defined by [17, eq. (125)-(126)] using [8, eq. (6-38)] :

$$f_w(z) = \int_0^z f(z-t, t)\, dt = \underbrace{\frac{\exp(-c_1 z)}{\Gamma(p)(\alpha_X \alpha_Y)^{(p+1)/2} \lambda^{p-1}(1-\lambda^2)}}_{A} \times \int_0^z (z-t)^{(p-1)/2} t^{(p-1)/2} \exp\left[-(c_2-c_1)t\right]$$

Fig. 1. CDF of received SNR. $\lambda = 0.8$, $\sigma_X^2/\sigma_Y^2 = 0.1$. Solid lines represent approximate estimates, and circles show simulation results for $N_{\mathrm{R}} = 4$. Single points report simulation results for $N_{\mathrm{R}} = 2$.

Fig. 2. Outage probability for the MIMO system with Rx antenna selection. $N_{\mathbf{R}} = 3$, $\lambda = 0.9$. Single points report simulation results.

$$\times I_{p-1}\left[2\frac{\lambda\sqrt{t(z-t)}}{(1-\lambda^2)\sqrt{\alpha_X\alpha_Y}}\right]dt = A\frac{\sqrt{\pi}\exp\left[-(c_2+c_1)\frac{z}{2}\right]}{\Gamma(p)}$$
$$\times a^{p-1}\sum_{k=0}^{\infty}\frac{a^{2k}}{k!}\left(\frac{z}{c_2-c_1}\right)^{p-1/2+k} I_{p+k-\frac{1}{2}}\left[(c_2-c_1)\frac{z}{2}\right]$$
$$= \frac{\sqrt{\pi}}{\Gamma(p)}\frac{a^{p-1}\exp\left[-(c_2+c_1)\frac{z}{2}\right]\left(\frac{z}{c_2-c_1}\right)^{p-\frac{1}{2}}}{(\alpha_X\alpha_Y)^{(p+1)/2}\lambda^{p-1}(1-\lambda^2)}$$
$$\times I_{p-\frac{1}{2}}\left[\frac{z}{2}\sqrt{(c_2-c_1)^2+4a^2}\right]\left(1+\frac{4a^2}{(c_2-c_1)^2}\right)^{-\frac{\left(p-\frac{1}{2}\right)}{2}} \quad (12)$$

where $I_\nu(.)$ is the modified Bessel function of the first kind of the order $\nu$ . To obtain (12), we took into account that the correlation coefficient between ${X_{i,j}}^2$ and ${Y_{i,j}}^2$ is $\lambda^2$ [13, eq. (28)]. We also use a series expansion of $I_{p-1}(.)$ in (12) [14, vol. 1, eq. (5.2.10.1)], an integration formula [14, vol. 1, eq. (2.3.6.2)], and summation formula [14, vol. 2, eq. (5.8.2.4)].

Evaluating the PDF $f_{\mathrm{sum}}(z)$ of the sum of two independent gamma variables with the same shape parameter $p$ and some scale parameters $\theta_1$ and $\theta_2$, we obtain on the basis of [8, eq. (6-39)] and [14, vol. 1, eq. (2.3.6.2)] that

$$f_{\mathrm{sum}}(z) = \frac{\exp\left(-\frac{z}{\theta_1}\right)}{[\Gamma(p)]^2(\theta_1\theta_2)^p}\int_0^z (z-y)^{p-1}y^{p-1}$$
$$\times\exp\left[-\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right)y\right]dy = \sqrt{\pi}\frac{\exp\left[-\frac{z}{2}\left(\frac{1}{\theta_1}+\frac{1}{\theta_2}\right)\right]}{\Gamma(p)(\theta_1\theta_2)^p}$$
$$\times\left[\frac{z}{\frac{1}{\theta_2}-\frac{1}{\theta_1}}\right]^{p-\frac{1}{2}} I_{p-\frac{1}{2}}\left[\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right)\frac{z}{2}\right]. \quad (13)$$

We note that (12) and (13) are equivalent if $\theta_1 = \frac{2}{c_1+c_2-\sqrt{(c_2-c_1)^2+4a^2}}$ and $\theta_2 = \frac{2}{c_1+c_2+\sqrt{(c_2-c_1)^2+4a^2}}$ . ■

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