Multivariate $\eta - \mu$ Fading Distribution with Constant Correlation Model

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***Abstract*—In this letter, we present formulas for the probability density, cumulative distribution, and moment generating functions of the multivariate $\eta - \mu$ fading distribution with the constant correlation model. We give examples of application of the derived results.**

***Index Terms*—Correlated fading, diversity methods, multivariate distributions.**

I. Introduction

APPLICATION of various diversity reception techniques aiming at mitigating fading effects is common in modern communications. If antenna spacing is sufficiently small, fading over multiple paths is correlated. In this case, various performance metrics of the communication system can be obtained on the basis of multivariate fading distributions. Such distributions with different types of correlation have been analyzed for Rayleigh [1], Nakagami-m [2]– [5], gamma-gamma [6], Weibull [7], and $\alpha - \mu$ fading models [8].

The generalized $\eta - \mu$ distribution was recently introduced by M.-D. Yacoub [9]. It fits well to experimental data in non-line-of-sight scenarios and includes some widely used fading models, such as Nakagami-m and Nakagami-q distributions, as particular cases. The $\eta - \mu$ model treats the fading signal as the composition of independent multipath clusters with Gaussian fading components within each cluster. The model involves two groups of scenarios (formats): 1) cases of unequal powers of in-phase (I) and quadrature (Q) components of the fading signal, 2) scenarios where the I and Q components are correlated.

A broad review on the literature on $\eta - \mu$ fading revealed only three studies, [10]– [12], that consider correlated multiple branches. In [10], the bivariate $\eta - \mu$ fading distribution is introduced. In [11], the moment generating function (MGF) of the distribution is derived for an arbitrary correlation model but only for the maximal-ratio combining (MRC) diversity method. In [12], MRC schemes operating over equally correlated $\eta - \mu$ fading channels are analyzed. The results derived in [12] are also applicable only to the MRC schemes. In this letter, we obtain formulas for the probability density, cumulative distribution functions (PDF, CDF), and MGF of the multivariate distribution for the constant correlation model. Unlike [10], the results derived in this work are valid for the arbitrary number of the diversity branches, and in contrast to [11]– [12], we present a full statistical characterization of the multivariate $\eta - \mu$ distribution with the considered correlation model.

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II. Multivariate $\eta - \mu$ Fading Distribution

A. Marginal $\eta - \mu$ Distribution

The PDF $f_{\gamma_{\eta-\mu}}(x)$ of the $\eta - \mu$ power variable γ is [9]:

$$f_{\gamma_{\eta-\mu}}(x) = \frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}} h^{\mu} x^{\mu-\frac{1}{2}}}{\Gamma(\mu) H^{\mu-\frac{1}{2}} \Omega^{\mu+\frac{1}{2}}} \exp\left(-\frac{2\mu x h}{\Omega}\right) I_{\mu-\frac{1}{2}}\left(\frac{2\mu H x}{\Omega}\right) \tag{1}$$

where 2μ is the number of the multipath clusters, $\mu = \frac{E^2\{\gamma\}}{2\mathrm{var}\{\gamma\}}\left[1+\left(\frac{H}{h}\right)^2\right]$(with $E\{.\}$ and $\mathrm{var}\{.\}$ denoting the expectation and variance, respectively), $\Omega = E\{\gamma\}$, $\Gamma(.)$ is the gamma function, and $I_{\alpha}(.)$ is the modified Bessel function of the first kind of the order α. In Format 1, $0 < \eta < \infty$ is the power ratio of the in-phase and quadrature scattered waves in each multipath cluster; $H = (\eta^{-1} - \eta)/4$ and $h = (2 + \eta^{-1} + \eta)/4$. In Format 2, $-1 < \eta < 1$ is the correlation coefficient between the in-phase and quadrature scattered waves in each multipath cluster; $H = \eta/(1 - \eta^2)$ and $h = 1/(1 - \eta^2)$.

B. Multivariate Distribution

We consider a vector $\mathbf{z} = \{z_1 \dots z_n\}$ of correlated and identically distributed $\eta - \mu$ power variables. It is proven in [13] that any $\eta - \mu$ power variable can be represented as the sum of two independent gamma variables with properly chosen parameters, i.e.

$$\mathbf{z} = \mathbf{x} + \mathbf{y} \tag{2}$$

where $\mathbf{x} = \{x_1 \dots x_n\}$ and $\mathbf{y} = \{y_1 \dots y_n\}$ are vector gamma variables with the components x_i and y_i subject to gamma distributions with the same shape parameter μ and scale parameters $\alpha_x = \Omega/\left[2\mu\left(h+H\right)\right]$ and $\alpha_y = \Omega/\left[2\mu\left(h-H\right)\right]$, respectively [13], [10]. Since $\mathbf{x}$ and $\mathbf{y}$ are independent, correlation of the components of $\mathbf{z}$ is due to correlation of the components of $\mathbf{x}$ and components of $\mathbf{y}$. Let $\rho_{z_i z_j} = \frac{\mathrm{cov}\{z_i z_j\}}{\mathrm{var}\{z\}}$ be the correlation coefficient between z_i and z_j, where $\mathrm{cov}\{.\}$ denotes covariance, and $\rho_{x_i x_j}$ ($\rho_{y_i y_j}$) be that between x_i and x_j (y_i and y_j). Then $\rho_{z_i z_j} = \left(\rho_{x_i x_j}\mathrm{var}\{x\} + \rho_{y_i y_j}\mathrm{var}\{y\}\right)/\mathrm{var}\{z\}$. Thus, by assigning different values of $\rho_{x_i x_j}$ and $\rho_{y_i y_j}$ one can obtain different values of $\rho_{z_i z_j}$. We see, particularly, that if $\rho_{x_i x_j} = \rho_{y_i y_j} = \rho_{ij}$, then $\rho_{z_i z_j}$ and ρ_{ij} obey the same model, such as the constant or exponential correlation model.

The constant correlation model includes scenarios with closely spaced antennas, and it may be used as a benchmark for the worst case scenario [7]. In this case, the normalized covariance matrix $\mathbf{\Sigma} = \mathrm{cov}\{\mathbf{z}^T\mathbf{z}\}/\mathrm{var}\{z\}$ is such that $\mathbf{\Sigma}_{ij} = \rho_{zz} = \rho$ if $i \neq j$, and $\mathbf{\Sigma}_{ii} = 1$.

1) PDF and CDF: Let $\mathbf{Z} = \{Z_1 \dots Z_n\}$. We obtain from (2) that the conditional on $\mathbf{x}$ CDF $F_{\mathbf{z}}(\mathbf{Z})|_{\mathbf{x}}$ is:

$$F_{\mathbf{z}}(\mathbf{Z})|_{\mathbf{x}} \triangleq \Pr(z_1 < Z_1, \dots, z_n < Z_n)|_{x_1,\dots,x_n} = \int_{\mathbf{0}}^{\mathbf{Z}-\mathbf{x}} f_{\mathbf{y}}(\mathbf{y})\, d\mathbf{y} \quad (3)$$

where $f_{\mathbf{y}}(\mathbf{y})$ is the joint gamma PDF of $y_1, \dots, y_n$, and vector notations on the right-hand side of (3) are used to simplify the presentation of the multiple integral. Then the conditional PDF $f_{\mathbf{z}}(\mathbf{Z})|_{\mathbf{x}}$ is:

$$f_{\mathbf{z}}(\mathbf{Z})|_{\mathbf{x}} \triangleq \frac{d^n F_{\mathbf{z}}(Z_1, \dots, Z_n)|_{\mathbf{x}}}{dZ_1 \dots dZ_n} = f_{\mathbf{y}}(\mathbf{Z} - \mathbf{x}). \quad (4)$$

Eq. (4) is obtained from (3) by sequential application of the Leibniz integral rule. The unconditional PDF $f_{\mathbf{z}}(\mathbf{z})$ can be obtained from (4) by averaging over $\mathbf{x}$ varying from $\mathbf{0}$ to ∞. But the PDF $f_{\mathbf{y}}(\mathbf{y}) = 0$ if any $y_i < 0$. Hence

$$f_{\mathbf{z}}(\mathbf{Z}) = \int_{\mathbf{0}}^{\mathbf{Z}} f_{\mathbf{x}}(\mathbf{x})\, f_{\mathbf{y}}(\mathbf{Z} - \mathbf{x})\, d\mathbf{x} \quad (5)$$

where $f_{\mathbf{x}}(\mathbf{x})$ is the joint gamma PDF of $x_1, \dots, x_n$.

An expression for the multivariate Nakagami-m PDF $f_{\mathbf{x}}(\mathbf{x})$ with the constant correlation model is given in [5, eq. (21)]:

$$f_{\mathbf{x}}(\mathbf{x}) = \frac{\left(1-\sqrt{\rho}\right)^m}{\Gamma(m)\left[1+(n-1)\sqrt{\rho}\right]^m} \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{\Gamma(m + k_1 + \ldots + k_n)}{\prod_{i=1}^n k_i!} \left(\frac{\sqrt{\rho}}{1+(n-1)\sqrt{\rho}}\right)^{k_1+\ldots+k_n} \times \prod_{i=1}^{n} f_{\mu+k_i}\left(x_i, \Xi\left(1-\sqrt{\rho}\right)\right) \quad (6)$$

where m is the Nakagami-m parameter, $\Xi = E\{x\}/m$, and $f_l(x,\theta) = \frac{x^{l-1}}{\Gamma(l)\cdot\theta^l}\exp\left(-\frac{x}{\theta}\right)$ is the marginal gamma PDF. It is seen that (6) can be expressed in terms of the multivariate hypergeometric series $\Psi_2^{(n)}(a;\mathbf{b};\mathbf{x}) = \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{(a)_{k_1+\ldots+k_n}}{\prod_{i=1}^n (b_i)_{k_i}\cdot k_i!} \prod_{i=1}^n x_i^{k_i}$, where $(\cdot)_i$ is the Pochhammer index [14, vol. 3, Section 6.6]. The function $\Psi_2^{(n)}(a;\mathbf{b};\mathbf{x})$ is absolutely convergent [15]. This fact justifies the change of integration and summation in (5). Evaluating the integral in (5) via an integration formula [14, vol. 1, eq. (2.3.6.1)] we obtain that

$$f_{\mathbf{z}}(\mathbf{Z}) = \frac{\left(1-\sqrt{\rho}\right)^{2\mu}(c_x c_y)^{n\cdot\mu}}{[\Gamma(\mu)]^2} \prod_{i=0}^{n} Z_i^{2\mu-1}\exp(-c_y \cdot Z_i) \times \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{\left(\sqrt{\rho}\right)^{k_1+\ldots+k_{2n}}}{\left[1+(n-1)\sqrt{\rho}\right]^{2\mu+k_1+\ldots+k_{2n}}} \times \frac{\Gamma\left(\mu + \sum_{i=1}^n k_i\right)\Gamma\left(\mu + \sum_{i=n+1}^{2n} k_i\right)}{\prod_{i=1}^{2n} k_i!} \times c_x^{k_1+\ldots+k_n} \cdot c_y^{k_{n+1}+\ldots+k_{2n}} \cdot \prod_{i=1}^{n} \frac{Z_i^{k_i+k_{n+i}}}{\Gamma(2\mu + k_i + k_{n+i})} \times {}_1F_1\left[\mu + k_i; 2\mu + k_i + k_{n+i}; (c_y - c_x) Z_i\right] \quad (7)$$

where $c_x = \left[\alpha_x\left(1-\sqrt{\rho}\right)\right]^{-1}$, $c_y = \left[\alpha_y\left(1-\sqrt{\rho}\right)\right]^{-1}$, and ${}_1F_1(\cdot)$ is a hypergeometric function [14, vol. 3, Section 6.6]. The multiple series (7) is a result of evaluation of the multiple convolution integral (5). Each factor in the integrand is the joint gamma PDF $f_{\mathbf{x}}(\mathbf{x})$ that is bounded for $0 \leq x_i < \infty$, and $f_{\mathbf{x}}(\mathbf{x}) \to 0$ as any $x_i \to \infty$ (see (6)). Thus $f_{\mathbf{z}}(\mathbf{Z})$ is bounded for $0 \leq Z_i < \infty$, and this fact proves convergence of (7) that is a representation of $f_{\mathbf{z}}(\mathbf{Z})$ defined by (5).

Using a series representation of ${}_1F_1(a;b;t) = \sum_{i=0}^{\infty} \frac{(a)_i}{(b)_i i!} t^i$, we obtain that the CDF $F_{\mathbf{z}}(Z_1, \dots, Z_n) \triangleq \int_{\mathbf{0}}^{\mathbf{Z}} f_{\mathbf{z}}(\mathbf{z})\, d\mathbf{z}$ can be expressed as

$$F_{\mathbf{z}}(\mathbf{Z}) = \frac{\left(1-\sqrt{\rho}\right)^{2\mu}(c_x/c_y)^{n\cdot\mu}}{[\Gamma(\mu)]^2} \times \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{\left(c_x\sqrt{\rho}\right)^{k_1+\ldots+k_n}\left(c_y\sqrt{\rho}\right)^{k_{n+1}+\ldots+k_{2n}}}{\left[1+(n-1)\sqrt{\rho}\right]^{2\mu+k_1+\ldots+k_{2n}}} \times \frac{\Gamma\left(\mu + \sum_{i=1}^n k_i\right)\Gamma\left(\mu + \sum_{i=n+1}^{2n} k_i\right)}{\prod_{i=1}^{2n} k_i!} \times \prod_{i=1}^{n} \frac{1}{\Gamma(2\mu + k_i + k_{i+n})\, c_y^{k_i+k_{n+i}}} \times \sum_{l_1=0}^{\infty} \dots \sum_{l_n=0}^{\infty} \frac{(\mu + k_i)_{l_i}(1 - c_x/c_y)^{l_i}}{(2\mu + k_i + k_{n+i})_{l_i}\, l_i!} \times \gamma(2\mu + k_i + k_{n+i} + l_i, c_y \cdot Z_i) \quad (8)$$

where $\gamma(a,t) = \int_0^t t^{a-1}\exp(-t)\, dt$ is an incomplete gamma function.

2) MGF: Due to independence of $\mathbf{x}$ and $\mathbf{y}$ in (2), the MGF of the multivariate distribution can be expressed as

$$M_{\mathbf{z}}(\mathbf{s}) \triangleq E\{\mathbf{exp}\left(\mathbf{z}\cdot\mathbf{s}^T\right)\} = E\{\mathbf{exp}\left(\mathbf{x}\cdot\mathbf{s}^T\right)\} \times E\{\mathbf{exp}\left(\mathbf{y}\cdot\mathbf{s}^T\right)\} = M_{\mathbf{x}}(\mathbf{s}) \cdot M_{\mathbf{y}}(\mathbf{s}) \quad (9)$$

where $M_{\mathbf{x}}(\mathbf{s})$ and $M_{\mathbf{y}}(\mathbf{s})$ are the MGFs of the multivariate gamma distributions with the PDF (6), and $\mathbf{s} = \{s_1 \dots s_n\}$. From [5, eq. (21)], we have that

$$M_{\mathbf{x}}(\mathbf{s}) = \frac{\left(1-\sqrt{\rho}\right)^{\mu}}{\Gamma(\mu)\left[1+(n-1)\sqrt{\rho}\right]^{\mu}} \sum_{k_1=0}^{\infty} \dots \sum_{k_n=0}^{\infty} \frac{\Gamma\left(\mu + \sum_{i=1}^n k_i\right)}{\prod_{i=1}^n k_i!} \left(\frac{\sqrt{\rho}}{1+(n-1)\sqrt{\rho}}\right)^{k_1+\ldots+k_n} \times \prod_{i=1}^{n} \frac{1}{\left[1 - s_i \cdot \alpha_x\left(1-\sqrt{\rho}\right)\right]^{\mu+k_i}}. \quad (10)$$

Eq. (10) represents also the MCF $M_{\mathbf{y}}(\mathbf{s})$ after changing α_x to α_y. Thus (9)-(10) define the MGF of the multivariate $\eta-\mu$ distribution with the constant correlation model.

Remark: The approach used in this paper can be applied to any correlation model. In this case, the multivariate PDF $f_{\mathbf{x}}(\mathbf{x})$ in (5) must correspond to the given correlation model.

III. Numerical Results

We apply the obtained theoretical results to the evaluation of the outage probability $P_{\mathrm{out}}(q)$ of a communication system employing the selection combining (SC) method and operating over the correlated $\eta-\mu$ branches. Estimates for Format 1 are shown in Fig. 1, and those for Format 2 are presented in Fig. 2. In both figures, $P_{\text{out}}(q)$ is plotted as a function of the received signal-to-noise ratio (SNR) that is involved in the evaluations via α_x and α_y (more precisely, $\Omega \propto \text{SNR}$). The outage probability under SC is expressed as

$$P_{\text{out}}(q) = \Pr\left(\max_{1\le i\le n} \text{SNR}_{\text{r}_i} \le q\right) = F_{\mathbf{z}}(\mathbf{q}) \qquad (11)$$

where SNR_{r_i} is the received SNR over the i th branch (i.e. the product of the transmitted SNR and channel power gain), and $\mathbf{q} = \{\underbrace{q,\ldots,q}_{n}\}$.

In our numerical evaluations, we use two values of η: $\eta_1 = 0.1$ and $\eta_2 = 0.9$. In each case, $\mu = 1.2$ and $q = 1$. We evaluate $P_{\text{out}}(q)$ by truncating the infinite sums in (8) and use N_1 terms in each sum over indices k_i and N_2 terms in each sum over indices l_i. We observe that the number of terms providing a fixed accuracy depends strongly on the values of ρ_{zz}. The more ρ_{zz} is, the more the terms must be included into the truncated series for providing the fixed accuracy. This is a fact to be expected since the arguments of the functions ${}_1F_1(\cdot)$ in (7) and $\gamma(\cdot)$ in (8) increase as ρ_{zz} increases. For example, in the case of Format 2 with $\eta = 0.9$, a three-branch diversity receiver, SNR=10 dB, and $\rho_{zz} = 0.05$, we observe that the numbers of terms in (8) providing 10^{-6} accuracy are $N_1 = 3$ and $N_2 = 8$. If $\rho_{zz} = 0.8$, the same accuracy is provided by $N_1 = 23$ and $N_2 = 42$. Alternatively, we obtain estimates of $P_{\text{out}}(q)$ through computer simulations where the correlated $\eta-\mu$ variables are generated via correlated gamma variables and the decomposition formula (2) (see also [10]). Two types of estimates agree well, and they are practically not distinguished in Fig. 1–2.

If the diversity branches are independent then $P_{\text{out}}(q) = \left[F_{\gamma_{\eta-\mu}}(q)\right]^n$, where $F_{\gamma_{\eta-\mu}}(q)$ is the CDF of the $\eta-\mu$ power variable γ. In this case our numerical estimates are consistent with estimates obtained on the basis of [16].

In the case of Format 1, our numerical evaluations show that the outage probability gets smaller as the fading environment becomes more homogenous ($\eta \to 1$). In the case of Format 2, the outage probability increases as the correlation coefficient between the I and Q components of the fading signal increases ($\eta \to 1$).

IV. Conclusion

In this paper, we derive expressions for the PDF, CDF, and MGF of the multivariate $\eta-\mu$ distribution with the constant correlation model. As it is common for multivariate statistical distributions (see, for example, [2]– [6]), the formulas are obtained in the forms of infinite series. In numerical evaluations, the series must be truncated. The number of the terms in the truncated series depends strongly on the value of the correlation coefficient. The larger the value of the correlation coefficient, the larger the number of the terms that must be included into the truncated series to provide a given accuracy.

For the particular case of $n = 2$, the formulas obtained in this paper give the same results as the corresponding formulas in [10], but the forms of the representation differ. As it was pointed out in [5], the different forms of representation of the same result stem from different series expansions of the MGF of the bivariate distribution.

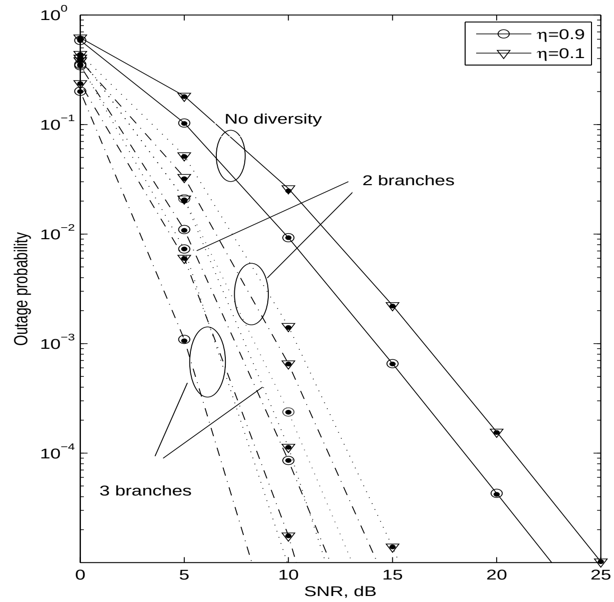

Fig. 1. Outage probability under selection combining - Format 1. Dotted lines represent correlated branches with $\rho = 0.4$, and dashed-dotted lines represent independent branches. Black points report simulation results.

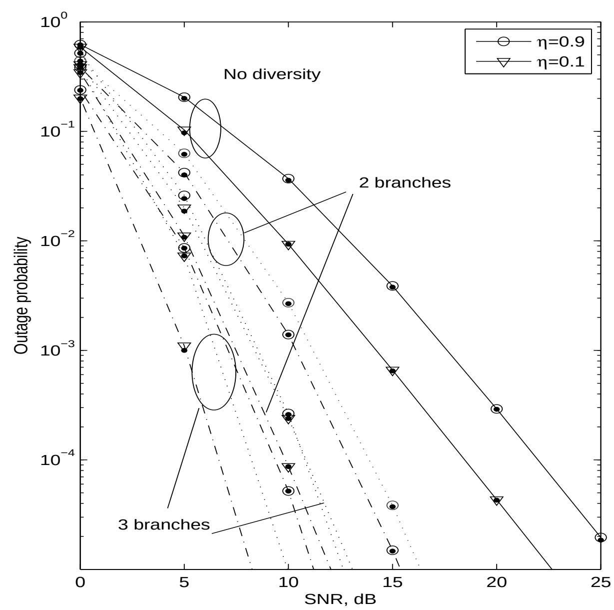

Fig. 2. Outage probability under selection combining - Format 2. Dotted lines represent correlated branches with $\rho = 0.4$, and dashed-dotted lines represent independent branches. Black points report simulation results.

References

[1] A. Abdi, "Comments on "On verifying the first-order Marcovian assumption for a Rayleigh fading channel model"," *IEEE Trans. Veh.*

Technol., vol. 48, no. 9, p. 1739, Sept. 1999.

[2] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "On the multivariate Nakagami-m distribution with exponential correlation," *IEEE Trans. Commun.*, vol. 51, no. 8, pp. 1240–1244, Aug. 2003.

[3] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "An efficient approach to multivariate Nakagami-m distribution using Green's matrix approximation," *IEEE Trans. Wireless Commun.*, vol. 2, no. 9, pp. 883–889, Sept. 2003.

[4] R. A. A. de Souza and M. D. Yacoub, "On multivariate Nakagami-m distribution with arbitrary correlation and fading parameters," in *Proc. IEEE Intern. Microwave and Optoelectronics Conf.*, Salvador, Brazil, Oct. 2007, pp. 812–816.

[5] J. Reig, "Multivariate Nakagami-m distribution with constant correlation model," *Intern. J. Electron. Commun.*, vol. 63, pp. 46–51, Jan. 2009.

[6] K. P. Peppas, G. C. Alexsandropoulus, C. K. Datsikas, and F. I. Lazarakis, "Multivariate gamma-gamma distribution with exponential correlation and its applications in radio frequency and optical wireless communictions," *IET Antennas and Propag.*, vol. 5, pp. 364–371, 2011.

[7] K. T. Hemachandra and N. C. Beaulieu,"New representations for the multivariate Weibull distribution with constant correlation," in *Proc. IEEE GLOBECOM* 2010, Miami, Florida, USA, Dec. 2010, pp. 1–5.

[8] R. A. A. de Souza and M. D. Yacoub, "On the multivariate $\alpha - \mu$ distribution with arbitrary correlation and fading parameters," in *Proc. IEEE Intern. Conf. Commun. (ICC' 08),* Beijing, China, May 2008, pp. 4456–4460.

[9] M. D. Yacoub, "The $\kappa - \mu$ distribution and the $\eta - \mu$ distribution," *IEEE Antennas and Propag. Mag.*, vol. 49, pp. 68–81, Feb. 2007.

[10] N. Y. Ermolova and O. Tirkkonen, "Bivariate $\eta - \mu$ fading distribution with application to analysis of diversity systems," *IEEE Trans. Wireless Commun.*, vol. 10, no. 10, pp. 3158–3162, Oct. 2011.

[11] V. Asghari, D. B. da Costa, and S. Aissa, "Symbol error probability of rectangular QAM in MRC systems with correlated η-μ fading channels," *IEEE Trans. Veh. Techn.*, vol. 59, no. 3, pp. 1497–1503, March 2010.

[12] R. Subadar and P. R. Sahu, "Performance of a L-MRC receiver over equally correlated $\eta - \mu$ fading channels," *IEEE Trans. Wireless Commun.*, vol. 10, no. 5, pp. 1351–1355, May 2011.

[13] K. P. Peppas, F. Lazarakis, T. Zervos, A. Alexandridis, and K. Dangakis, "Error performance of digital modulation schemes with MRC diversity reception over $\eta - \mu$ fading channels," *IEEE Trans. Wireless Commun.*, vol. 8, no. 10, pp. 4974–4980, Oct. 2009.

[14] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*, Gordon and Breach Science Publishers, 1986.

[15] H. Exton, *Multiple Hypergeometric Functions and Applications*, NY, Wiley, 1976.

[16] D. Morales-Jimenez and J. F. Paris, "Outage probability analysis for $\eta - \mu$ fading channels," *IEEE Commun. Lett.*, vol. 14, no. 6, pp. 521–523, June 2010.