Simultaneous recovery of admittivity and body shape in electrical impedance tomography: An experimental evaluation

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Abstract. In this paper, the simultaneous retrieval of the exterior boundary shape and the interior admittivity distribution of an examined body in electrical impedance tomography is considered. The reconstruction method is built for the complete electrode model and it is based on the Fréchet derivative of the corresponding current-to-voltage map with respect to the body shape. The reconstruction problem is cast into the Bayesian framework, and maximum a posteriori estimates for the admittivity and the boundary geometry are computed. The feasibility of the approach is evaluated by experimental data from water tank measurements. The results demonstrate that the proposed method has potential for handling an unknown body shape in a practical setting.

Submitted to: *Inverse Problems*
1. Introduction

*Electrical impedance tomography* (EIT) is a noninvasive imaging technique which aims at reconstructing the internal admittivity distribution of a body from boundary measurements of current and voltage. EIT has been applied to, e.g., medical imaging, process tomography and nondestructive testing of materials [2, 5, 30]. Although the admittivity distribution is the quantity of primary interest in EIT, a practical measurement configuration often contains other unknowns as well: the information on the electrode positions, the contact impedances at the electrode-object interfaces and the shape of the examined body may be incomplete. These are major issues especially in medical applications of EIT. It is of utmost importance to take such uncertainties into account when developing reconstruction methods, since EIT is known to be highly intolerant to mismodelling of the measurement setting [1, 4, 21].

In this work, we generalize the iterative Newton-type output least squares methods that tolerate geometrical uncertainties from [8, 9] to a more practical setting, and evaluate the feasibility of the approach with experimental data from water tank measurements. The method introduced in [9] simultaneously reconstructs the admittivity distribution, the body shape and the electrode locations in a three-dimensional cylindrical geometry from simulated electrode measurements, assuming that the contact impedances are known. It is built in the framework of the *complete electrode model* (CEM) [6], which is the most accurate model for practical EIT. The theoretical basis of the algorithm is on the Fréchet derivative of the measurement map of the CEM with respect to the shape of the exterior object boundary; see, e.g., [17, 12, 13, 10] for more information on shape derivatives of boundary measurements.

The main difficulty in applying the algorithm of [9] to experimental (absolute) water tank data is dealing with the unknown contact impedances. Our leading idea is to compute the derivative of the CEM measurement map with respect to the contact impedances following [32] and to subsequently include their estimation as a part of the reconstruction algorithm. However, if the to-be-estimated contact impedances are relatively small — as they typically are in water tank experiments — the computation of the shape derivative of the measurement map becomes numerically unstable. This phenomenon can be justified from a theoretical view point: to put it short, the regularity of the CEM forward solution decreases as the contact impedances tend to zero, i.e., when one approaches the so-called shunt model [6], which assumes perfect contacts. As a consequence, we use artificially high contact impedances at the beginning of the iterative reconstruction algorithm, which provides a regulative effect, and start to fine-tune their values only after the reconstructions of the body shape and the admittivity have converged qualitatively. The functionality of this approach is demonstrated using several experimental data sets corresponding to different admittivity phantoms and water tank cross-sections.

It should be emphasized that we do not present the first algorithm designed for simultaneous reconstruction of admittivity and body shape from absolute EIT
measurements. A method based on allowing slightly anisotropic admittivities and on the use of sophisticated mathematical instruments such as quasiconformal maps and Teichmüller spaces was introduced in [18, 19] and applied to experimental data in [20]. However, the feasibility of the algorithm of [18, 19, 20] in purely three-dimensional, i.e., non-cylindrical, measurement settings remains unclear from the computational point of view, whereas the generalization of our reconstruction technique to more general configurations is straightforward. (Take note that all of our finite element computations are performed on three-dimensional meshes.) On the other hand, the so-called approximation error approach [16] was adapted to the compensation for errors resulting from an inaccurately known body shape in [26, 27]. For a more extensive discussion on handling of uncertainties in EIT, we refer to [27].

Apart from [8, 9], the previous publication that has the closest relation to our work is [3], where the conductivity and the exterior boundary shape are successfully reconstructed from experimental water tank data. However, since the algorithm of [3] is motivated by conformal mappings and is based on a (single) linearization, it is two-dimensional by nature and can arguably only handle small conductivity changes and boundary movements. As the object shape is perturbed in [3] by moving the boundary nodes of the finite element mesh, there even exists an explicit limit for the feasible boundary movements. It should also be emphasized that we work with absolute electrode measurements, whereas [3] considers (time) difference imaging, which is known to reduce the undesirable effects of a mismodelled object boundary [1, 4, 21]. (In fact, the point electrode model employed in [3] is properly justified only for difference measurements; see [11].)

This paper is organized as follows. In Section 2, we introduce the CEM, list related differentiability results and consider the effect of small contact impedances on the regularity of the CEM forward solution and its shape derivative. Section 3 describes our algorithm and Section 4 presents the corresponding reconstructions from experimental water tank data.

2. Electrical impedance tomography

In this section we review the theoretical background of our reconstruction method. We start by introducing the elliptic boundary value problem defining the CEM and then refer to some differentiability results for the associated measurement map. To conclude, we study the regularity of the CEM forward solution as the contact impedances approach zero, which is important for understanding the computational behavior of the reconstruction algorithm.

2.1. Complete electrode model

In practical EIT, $M \geq 2$ contact electrodes $\{E_m\}_{m=1}^M$ are attached to the exterior surface of a body $\Omega$. A net current $I_m \in \mathbb{C}$ is conducted through each $E_m$ and the resulting
constant electrode potentials $U = [U_1, \ldots, U_M]^T \in \mathbb{C}^M$ are measured. It is reasonable to assume that there are no sinks or sources inside the object. Hence, any meaningful current pattern $I = [I_1, \ldots, I_M]^T$ satisfies the current conservation law, i.e., $\sum_m I_m = 0$. For brevity, we use the notation

$$C_\diamond M = \left\{ V \in \mathbb{C}^M \left| \sum_{m=1}^M V_m = 0 \right. \right\}$$

for the subspace of current patterns. The contact impedances at the electrode-object interfaces are modelled by $z = [z_1, \ldots, z_M]^T \in \mathbb{C}^M$.

We assume that $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded domain with a smooth boundary. Moreover, the electrodes $\{E_m\}_{m=1}^M$ are identified with the open, nonempty subsets of $\partial \Omega$ that they cover and assumed to be mutually well-separated, i.e., $\overline{E}_k \cap \overline{E}_l = \emptyset$ for $k \neq l$. We denote $E = \cup E_m$. The mathematical model that most accurately predicts real-life EIT measurements is the complete electrode model (CEM) [6], which is described by an elliptic mixed Neumann–Robin boundary value problem: the electromagnetic potential $u$ and the potentials on the electrodes $U$ satisfy

$$\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 \quad \text{in } \Omega, \\
\nu \cdot \sigma \nabla u &= 0 \quad \text{on } \partial \Omega \setminus \overline{E}, \\
u + z_m \nu \cdot \sigma \nabla u &= U_m \quad \text{on } E_m, \quad m = 1, \ldots, M, \\
\int_{E_m} \nu \cdot \sigma \nabla u \, dS &= I_m, \quad m = 1, \ldots, M, 
\end{align*}
$$

interpreted in the weak sense. Here, $\nu$ is the outward pointing unit normal of $\partial \Omega$ and the (real) symmetric admittivity distribution $\sigma : \Omega \to \mathbb{C}^{n \times n}$ that characterizes the electric properties of the medium is assumed to satisfy

$$\begin{align*}
\Re(\sigma \xi \cdot \overline{\xi}) &\geq \varsigma_- |\xi|^2, \\
|\sigma \xi \cdot \overline{\xi}| &\leq \varsigma_+ |\xi|^2, \\
&\varsigma_- < \varsigma_+ > 0,
\end{align*}
$$

for all $\xi \in \mathbb{C}^n$ almost everywhere in $\Omega$. Moreover, the contact impedances are positive in the real part and finite, that is,

$$0 < \varsigma_- := \min_{1 \leq m \leq M} \{\Re z_m\} \leq \max_{1 \leq m \leq M} \{|z_m|\} =: \varsigma_+ < \infty.$$

The first line of (1) is the standard conductivity equation that follows from the Maxwell’s equations under the quasi-static approximation [2], and the second formula expresses the fact that the current injection is fully confined to the electrodes. The third equation of (1) models the contact impedance effect: due to an electrochemical process, thin highly resistive layers are formed at the electrode-object interfaces, which causes potential jumps according to the Ohm’s law. It turns out that the introduction of the contact impedances can actually be seen as a regularization of the forward model (cf. [7, 33] and Section 2.3 below). Finally, the fourth equation of (1) indicates that the integral of the current density through $E_m$ equals the corresponding net current feed $I_m$.

Given an input current pattern $I \in C_\diamond M$, the admittivity $\sigma$ and the contact impedances $z$, the spatial electric potential $u \in H^1(\Omega)$ and the electrode potentials
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$U \in C^M$ are uniquely determined by (1) up to a common additive constant, i.e., up to the ground level of potential. This can be shown by considering the Hilbert space $V := (H^1(\Omega) \oplus C^M)/C$ with the norm

$$
\|(v, V)\|_V = \inf_{c \in C} \left\{ \|v-c\|_{H^1(\Omega)}^2 + \sum_{m=1}^{M} |V_m-c|^2 \right\}^{1/2}
$$

and the variational formulation of (1) given by [29]

$$
\int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx + \sum_{m=1}^{M} \frac{1}{z_m} \int_{E_m} (u - U_m)(V - V_m) \, dS = I \cdot V
$$

for all $(v, V) \in V$. By resorting to the Lax–Milgram theorem, it is not difficult to see that (4) is uniquely solvable and that the solution satisfies (1) [29, 14]. Moreover, the solution pair $(u, U) \in V$ depends continuously on the data,

$$
\|(u, U)\|_V \leq \frac{C}{\min\{\zeta^{-1}, \zeta^1\}} |I|,
$$

where $C = C(\Omega, E) > 0$ does not depend on $\sigma$ or $z$ (cf. [14, Section 2]).

We define the measurement, or current-to-voltage, map of the CEM as

$$
R : I \mapsto U, \quad C^M \rightarrow C^M/C.
$$

Due to an obvious symmetry of (4), $R$ can be represented as a symmetric complex $(M-1) \times (M-1)$ matrix (with respect to any chosen basis for $C^M_0 \sim C^M/C$).

2.2. Fréchet differentiability of the measurement map

In this section, we summarize some Fréchet differentiability results for the measurement map $R : C^M_0 \rightarrow C^M/C$ with respect to the model parameters in (1); for more details, see [8, 9, 15, 23, 32]. We start by perturbing $\partial \Omega$ and introducing the corresponding shape derivative.

Assuming that the admittivity is defined in some neighborhood of $\Omega$, the measurement map of (6) can be considered as a function of two variables,

$$
R : (I, h) \mapsto U(I, h), \quad C^M_0 \times B_d \rightarrow C^M/C,
$$

where $B_d \subset [C^1(\partial \Omega)]^n$ is an origin-centered open ball of radius $d > 0$ and \((u(I, h), U(I, h))\) is the solution of (1) when $\partial \Omega$ and $E_m$, $m = 1, \ldots, M$, are replaced by the perturbed versions

$$
\partial \Omega^h = \{x + h(x) \mid x \in \partial \Omega\}, \quad E_m^h = \{x + h(x) \mid x \in E_m\},
$$

respectively. If $d > 0$ is chosen small enough, the above definitions are unambiguous in the sense that $\partial \Omega^h$ defines a $C^1$ boundary of a bounded domain $\Omega^h$ (cf., e.g., [9]); in the following, we will implicitly assume that this is the case. For the proof of the next theorem, we refer to [9].

**Theorem 2.1.** Suppose that the (real) symmetric admittivity $\sigma$ belongs to $C^1(\overline{\Omega}, C^{n \times n})$ and $\partial E_m, m = 1, \ldots, M$, are smooth in case $n = 3$. Then $R : C^M_0 \times B_d \rightarrow C^M/C$ is Fréchet differentiable with respect to its second variable at the origin.
Recall that this means there exists a (bi)linear and bounded map $U'(I, 0): [C^1(\partial \Omega)]^n \rightarrow \mathbb{C}^M / \mathbb{C}$ such that
\[
\lim_{0 \neq h \to 0} \frac{1}{\|h\|_{C^1}} \|U(I, h) - U(I, 0) - U'(I, 0)h\|_{\mathbb{C}^M / \mathbb{C}} = 0, \quad h \in [C^1(\partial \Omega)]^n,
\]
for any $I \in \mathbb{C}^M$. Moreover, if $I, \tilde{I} \in \mathbb{C}^M$ are electrode current patterns and $(u, U), (\tilde{u}, \tilde{U})$ are the respective solutions of (1), then $U' = U''(I, 0)h$ can be sampled via the relation
\[
U' \cdot \tilde{I} = -\int_{\partial \Omega} (h \cdot \nu)(\sigma \nabla u \cdot \nabla \tilde{u}) \, dS - \sum_{m=1}^{M} \frac{1}{z_m} \int_{E_m} (h \cdot \nu)(U_m - u) (\tilde{U}_m - \tilde{u}) \, dS - \sum_{m=1}^{M} \frac{1}{z_m} \int_{\partial E_m} (h \cdot \nu_{\partial E_m})(U_m - u)(\tilde{U}_m - \tilde{u}) \, ds,
\]
which can be deduced from [9, eq. (3.5)] using the boundary conditions of (1). Here, $\kappa \in C^\infty(\partial \Omega)$ is the sum of the principal curvatures of $\partial \Omega$ and $\nu_{\partial E_m}$ is the outward unit normal of $\partial E_m$ that lies in the tangent bundle of $\partial \Omega$. Take note that one gets a sampling formula for the derivative with respect to electrode shapes and locations by restricting to tangential perturbations $h \cdot \nu \equiv 0$ (cf. [8]). Observe also that the integrals on the right hand side of (8) are well-defined because, under the assumptions of Theorem 2.1, $u \in H^{2-\epsilon}(\Omega) / \mathbb{C}$ for any $\epsilon > 0$, and thus by the trace theorem,
\[
u_{\partial \Omega} \in H^{3/2-\epsilon}(\partial \Omega) / \mathbb{C}, \quad u|_{\partial \Omega} \in H^{1-\epsilon}(\partial \Omega) / \mathbb{C}, \quad \nabla u|_{\partial \Omega} \in [H^{1/2-\epsilon}(\partial \Omega)]^n;
\]
see [8, Remark 1].

Obviously, the measurement map $R$ can even be treated as a function of four variables by writing
\[
R : \{ (I, \sigma, z, h) \mapsto U(I, \sigma, z, h), \quad \mathcal{D} := \mathbb{C}^M_* \times \Sigma \times \mathbb{C}^M_+ \times \mathcal{B}_d \rightarrow \mathbb{C}^M / \mathbb{C},
\]
where $\mathbb{C}_+ = \{ w \in \mathbb{C} \mid \text{Re } w > 0 \}$ is the right half of the complex plain,
\[
\Sigma = \{ \sigma \in C^1(\overline{N_d(\Omega)}, \mathbb{C}^{n \times n}) \mid \sigma = \sigma^T \text{ satisfies (2) for some } \varsigma_- > 0 \}
\]
is a set of plausible admittivities and $N_d(\Omega)$ denotes the open $d$-neighborhood of $\Omega$ containing all $\Omega^h$ with $h \in \mathcal{B}_d$. The differentiability of $U = U(I, \sigma, z, h)$ with respect to its second member is known even for considerably less regular admittivities (see, e.g., [15, 23]), and that with respect to the contact impedances is straightforward to establish and has been utilized in many numerical algorithms (cf., e.g., [32]). We collect the needed differentiability results in the following corollary.

**Corollary 2.2.** Under the assumptions of Theorem 2.1, the measurement map of the CEM,
\[
R : \mathcal{D} \rightarrow \mathbb{C}^M / \mathbb{C},
\]
is Fréchet differentiable in the set $\mathbb{C}^M_* \times \Sigma \times \mathbb{C}^M_+ \times \{0\} \subset \mathcal{D}$. 
Proof. The assertion follows easily by utilizing Theorem 2.1, the linearity of \( R \) with respect to its first variable and the continuity of the Fréchet derivatives of \( R \) with respect to its second and third variables in the whole of \( D \) (cf. [15]).

By resorting to more general ways to perturb the boundary \( \partial \Omega \) [10], it is reasonable to expect that one can establish the differentiability of \( R \) with respect to a suitable shape parameter also away from the origin. However, as the derivative at the origin is enough for formulating our reconstruction algorithm in Section 3, we do not elaborate on this matter any further.

The numerical computation of the (partial) Fréchet derivatives of \( R \) with respect to \( \sigma \) and \( z \) has been considered in many previous works [15, 23, 32], and we will compute the needed shape derivatives (and the derivatives with respect to the electrode locations) with the help of (8) [8, 9]. However, there are unfortunately some instability issues related to (8), as discussed in the following section.

2.3. Regularity of the interior potential \( u \)

According to our numerical experiments, the sampling formula (8) for the shape derivative becomes numerically unstable when the minimal contact resistance \( \zeta_- \) from (3) approaches zero. This can be considered somewhat unfortunate since good contacts, i.e., small values of \( \{z_m\}_{m=1}^M \), are typically what one aims for in practical applications. Obviously, the utilization of the reciprocals \( z_m^{-1} \) in (8) explains part of the problem, but also the solutions to (1) employed in (8) seem to lose their regularity as \( \zeta_- \) goes to zero. This claim is concretized in the two-dimensional setting for \( \sigma \equiv 1 \) and \( \zeta_- = z_1 = \cdots = z_M = \zeta_+ \in \mathbb{R} \) by the following theorem that is based on the material in [7]. This simple setting is enough for motivating the choices we make for our reconstruction algorithm in Section 3, as there is no reason to expect more regular behavior for varying admittivities and contact impedances or in three dimensions.

**Proposition 2.3.** Assume that \( n = 2 \) and let \( (u, U) \in \mathcal{V} \) be the solution of (1) for the input current \( I \in \mathbb{C}^M_\circ, \sigma \equiv 1 \) and \( 0 < z_1 = \cdots = z_M =: \zeta \leq 1 \). Then, for any \( \epsilon, \delta > 0 \) such that \( \epsilon \leq \delta + 1/2 \) it holds that

\[
\|u\|_{H^{2-\epsilon}(\Omega)/\mathbb{C}} \leq C\zeta^{\epsilon-1/2-\delta}|I|, \tag{9}
\]

where \( C > 0 \) depends only on \( \epsilon, \delta \) and the measurement geometry, not on \( \zeta \).

**Proof.** Without loss of generality, we may assume that \( \epsilon \in (0,1) \) and \( \delta > 0 \) is such that \( |\epsilon - \delta| \leq 1/2 \); for more general \( \epsilon \) and \( \delta \), the assertion always follows from (9) with some choice of parameters in this range.

We denote the characteristic function of \( E_m \) by \( \chi_m \) and define

\[
g_U = \sum_{m=1}^M U_m \chi_m \in L^2(\partial \Omega)/\mathbb{C}.
\]
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Considering the first three equations of (1) and choosing $\epsilon = 1 - s$ and $\delta = 1/2 - t$ in [7, Proposition 8.1], it follows that

$$\|u - c\|_{H^{2-\epsilon}(\Omega)} \leq C\|g_U - c\|_{H^1(E)} \zeta^{\epsilon-1/2-\delta}, \quad c \in \mathbb{C},$$

where the constant $C > 0$ does not depend on $\zeta$ and we have picked a stronger norm for $g_U - c$ than any of those appearing on the right-hand side of [7, eq. (8.1)]. Taking into account that $g_U$ is constant on each electrode, we obtain

$$\|u - c\|_{H^{2-\epsilon}(\Omega)} \leq C \left( \sum_{m=1}^{M} (U_m - c)^2 \right)^{1/2} \zeta^{\epsilon-1/2-\delta}, \quad c \in \mathbb{C},$$

where again $C > 0$ does not depend on $\zeta$. Denoting by $c_0 \in \mathbb{C}$ the mean value of the components in $U$, we deduce that

$$\|u\|_{H^{2-\epsilon}(\Omega)/\mathbb{C}} \leq \|u - c_0\|_{H^{2-\epsilon}(\Omega)} \leq C \inf_{c \in \mathbb{C}} \left( \sum_{m=1}^{M} (U_m - c)^2 \right)^{1/2} \zeta^{\epsilon-1/2-\delta}.$$

The assertion now follows from the estimate (5). \qed

The constant of continuity in (9) blows up for any $0 < \epsilon \leq 1/2$ as $\zeta > 0$ tends to zero independently of the choice of $\delta > 0$, which suggests that the regularity of the CEM forward solution is gradually lost when the contact impedances approach zero. This is ill-fated for our reconstruction algorithm for two reasons: Firstly, according to standard finite element theory, it is to be expected that the discrepancy between the exact and an approximate finite element solution of (1) (with a fixed mesh and basis functions) increases when the higher Sobolev norms of the first component of the exact solution to (1) get larger. Secondly, the interpretation of the sampling formula (8) becomes nontrivial when the contact impedances tend to zero: Even if one utilizes the third equation of (1) to get rid of $1/z_m$ in (8) by replacing $\check{U}_m - \check{u}$ with $z_m \nu \cdot \sigma \nabla \check{u}$, the limits of the integrals on the right-hand side of (8) when $z_m$ goes to zero seem ambiguous, since one expects the corresponding limits of $u$ and $\check{u}$ only belong to $H^{3/2-\delta}(\partial \Omega)$, $\delta > 0$, if they exist. (Note that the right-hand side of (9) stays bounded for $\epsilon = 1/2 + \delta$ for arbitrarily small $\delta > 0$.) In particular, one cannot expect more boundary regularity from the gradients of $u$ and $\check{u}$ in the limit $z_m \to 0$ than $\nabla u|_{\partial \Omega}, \nabla u|_{\partial \Omega} \in [H^{-\delta}(\partial \Omega)]^n$, $\delta > 0$ (assuming these traces can be defined, cf. [25]), meaning that the first integral on the right-hand side of (8) necessarily becomes ambiguous.

The treatment of the previous paragraph is only speculative because it is not obvious that (9) is optimal and, on the other hand, that there exists a limit for $(u, U)$ as the contact impedances tend to zero. These doubts can be partly erased by considering the
so-called shunt model forward problem [6],

\[
\begin{align*}
\nabla \cdot \sigma \nabla u^0 &= 0 & \text{in } \Omega, \\
\nu \cdot \sigma \nabla u^0 &= 0 & \text{on } \partial\Omega \setminus \overline{E}, \\
u^0 &= U^0_m & \text{on } E_m, \\
\int_{E_m} \nu \cdot \sigma \nabla u^0 \, dS &= I_m, & m = 1, \ldots, M,
\end{align*}
\]

(10)

which assumes perfect contacts. An interior potential \(u^0\) that satisfies (10) is in general only in \(H^{3/2-\delta}(\Omega)\), \(\delta > 0\) (see, e.g., [7]). Furthermore, it can be argued that the solution pair of (1) tends to a solution \((u^0, U^0)\) of (10) as the contact impedances vanish, with the convergence of the interior potential component in \(H^{3/2-\delta}(\Omega)\), \(\delta > 0\). Under the assumptions of Proposition 2.3 and working with electrode potential inputs, i.e., prescribing \(U\) instead of \(I\) and deleting the last formulas of (1) and (10), this claim follows directly from [7, Corollary 4.4]. However, for our choice of electrode current inputs, nonconstant admittivity and varying contact impedances such a conclusion requires some extra argumentation that is left for future studies.

To sum up, both numerical and theoretical considerations suggest that it is not recommendable to use the sampling formula (8) if the contact impedances are ‘too small’. For this reason, we enforce relatively large contact impedances at the beginning of our iterative reconstruction algorithm, and start to fine-tune their values only after the reconstruction of the body shape has converged qualitatively (cf. Section 4).

**Remark 2.4.** The justification of (8) in numerical computations is not self-evident even if the contact impedances are not small. Indeed, when (1) is solved numerically by, e.g., some finite element method (FEM), the convergence of the approximate solution toward the exact one is typically established only in \(V = (H^1(\Omega) \oplus \mathbb{C}^M)/\mathbb{C}\). However, a guaranteed convergence of the right-hand side of (8) when \((u, U)\) and \((\tilde{u}, \tilde{U})\) are replaced by their finite element counterparts seems to require convergence of the employed FEM at the very least in \((H^{3/2}(\Omega) \oplus \mathbb{C}^M)/\mathbb{C}\) (cf. the trace theorems in [25]).

In our numerical tests, we work with isotropic admittivities, meaning that the normal derivative of \(u\) in (8) can be replaced with \((U_m - u)/(z_m\sigma)\). In particular, the right-hand side of (8) can then be evaluated for finite element approximations of \((u, U)\) and \((\tilde{u}, \tilde{U})\) based on standard piecewise polynomial \(C^0\) basis functions; in our algorithmic implementation, we just use such evaluations in place of the exact integrals in (8). As the reconstructions of Section 4 demonstrate, this approach seems adequate from a practical point of view.

### 3. Reconstruction algorithm for cylindrical objects

In this section we introduce our reconstruction method that is built within the Bayesian paradigm and based on an iterative Gauss–Newton scheme. The algorithm is a modified version of the one introduced in [9]. In what follows, we assume that admittivities are real-valued and isotropic, and contact impedances are positive real numbers.
We approximate the object shape by cylinders of the form $\Omega_\alpha = D_\alpha \times (0, d) \subset \mathbb{R}^3$, where $D_\alpha \subset \mathbb{R}^2$ is a two-dimensional star-shaped domain and $d > 0$ is the known height of the examined object. The boundary $\partial D_\alpha$ is parametrized as

$$\partial D_\alpha = \{ \gamma_\alpha(\theta) \mid \theta \in [0, 2\pi) \} \subset \mathbb{C} \simeq \mathbb{R}^2,$$

where $\gamma_\alpha(\theta) = r_\alpha(\theta)e^{i\theta}$ and

$$r_\alpha(\theta) = \alpha_0 + \sum_{j=1}^{N_1} (\alpha_j \cos j\theta + \alpha_{j+N_1} \sin j\theta).$$

Here, the parameter vector $\alpha = [\alpha_0, \ldots, \alpha_{2N_1}]^T \in \mathbb{R}^{2N_1+1}$ is assumed to be such that $\partial D_\alpha$ does not intersect itself. We parametrize the electrodes by $E_m = \gamma_\alpha([\phi_m, \phi_m + \eta_m]) \times (0, d)$, where $\eta_m = \eta_m(\alpha, \phi_m)$ is such that the electrodes are of some known fixed width. In particular, the electrode configuration is fully characterized by the list of initial polar angles $\phi = [\phi_1, \ldots, \phi_M]^T$ given $\alpha$.

To introduce a computational model for the admittivity, we fix a large enough cuboid divided into a voxel grid $Q = \bigcup_{j=1}^{N_2} Q_j$ such that the modelled object is contained in its interior. The admittivity distribution is then imaged as piecewise constant,

$$\sigma(x) = \sum_{j=1}^{N_2} \sigma_j \chi_j(x), \quad \sigma_j > 0,$$

with $\chi_j$ being the characteristic function of the voxel $Q_j$. Altogether the to-be-determined parameters are $\sigma, z, \alpha, \phi$, meaning that we have ended up with a problem of $N := 2N_1 + N_2 + 2M + 1$ unknowns.

Let $V$ be the measured (noisy) electrode potential vector obtained by applying one or several linearly independent current patterns. In the case of more than one input current, $V$ is formed by piling the separate measurements into a single column vector (cf. [9, eq. (5.3)]). With the input current(s) fixed, we denote the second component(s) of the corresponding FEM solution(s) to (1) by $U^{\text{FE}}(\sigma, z, \alpha, \phi)$, which is of the same size as $V$ and corresponds to a given set of feasible model parameters $\{\sigma, z, \alpha, \phi\}$. To be a bit more precise, the parametrized domain $\Omega_\alpha$ is discretized into tetrahedrons with appropriate mesh refinements at the boundaries of the electrodes, and then an approximate FEM solution to (1) is computed using a piecewise quadratic basis for the interior potential and a piecewise linear conductivity obtained by linearly interpolating a given $\sigma$ of the form (12). The FEM solver used in this work is an adaptation of the implementation in [31].

After making suitable probabilistic interpretations (an additive Gaussian noise model and Gaussian priors, cf. [9, Section 5.2]), we can seek a maximum a posteriori (MAP) estimate for the model parameters $\sigma, z, \alpha, \phi$ as a minimizer of the Tikhonov-type functional

$$|V - U^{\text{FE}}(\sigma, z, \alpha, \phi)|^2_{\Gamma^{-1}} + |\sigma - \sigma_*|^2_{\Gamma^{-1}} + \gamma_z^{-2}|z - z_*|^2 + \gamma_\alpha^{-2}|r_\alpha - r_\alpha_*|^2_{H^s(0,2\pi)} + \gamma_\phi^{-2}|\phi - \phi_*|^2,$$

where $\Gamma$ is the regularization parameter.
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where we have used the standard notation $|x|_A^2 = x^T A x$. Here, $\sigma_*, z_*, \alpha_*, \phi_*$ are the prior mean values for the corresponding parameters. Moreover, $\Gamma_n$ is the covariance matrix of the assumed additive zero-mean Gaussian measurement noise model, and $\Gamma_\sigma$ is our choice of the prior covariance matrix for the voxel values of the admittivity,

$$(\Gamma_\sigma)_{ij} = \kappa^2 \exp\left(-\frac{|x_i - x_j|^2}{2\lambda^2}\right), \quad \kappa, \lambda > 0,$$

where $x_j$ is the center point of the voxel $Q_j$, $\kappa^2$ is the variance of the voxel admittivity values, and $\lambda$ is the correlation length that controls the degree of spatial smoothness [24]. Finally, $\gamma_\zeta$ and $\gamma_\phi$ are the prior standard deviations of the contact impedances and electrode angles, respectively, and $\gamma_\alpha$ is that of $\alpha$ with respect to the $H^s(0, 2\pi)$-norm of $r_\alpha$. Notice that (our choice of) the Sobolev norm appearing in (13) can be computed efficiently via

$$\|r_\alpha\|_{H^s(0, 2\pi)}^2 = \alpha_0^2 + \sum_{j=1}^{N_1} j^{2s}(\alpha_j^2 + \alpha_{j+N_1}^2)$$

for any smoothness level $s \in \mathbb{R}$.

Denoting by $\beta = [\sigma, z, \alpha, \phi]^T \in \mathbb{R}^N$ the column vector of the model parameters and by $\beta_*$ the corresponding prior mean, we may write the minimization problem in the form

$$\arg\min_{\beta} \left\{ \left| L \left( \begin{bmatrix} UFE(\beta) \\ \beta \end{bmatrix} - \begin{bmatrix} V \\ \beta_* \end{bmatrix} \right) \right|^2 \right\}, \quad (14)$$

where $\beta$ runs over feasible model parameters and $L$ is a triangular square matrix obtained as a Cholesky factor of the block diagonal matrix defined by the inverses of the (prior) covariances of the noise and the model parameters. To solve (14), we apply a Gauss–Newton algorithm [28, 16] combined with the golden section line search. In the following, $J_\beta$ is a numerical approximation for the Jacobian of $UFE(\beta)$ at $\beta$ and we denote by $I \in \mathbb{R}^{N \times N}$ the identity matrix.
Algorithm 1 Gauss–Newton based algorithm for solving (14)

1: Choose $\beta^0 = \beta^*$ and set $j = 0$;
2: repeat
3: Compute $J = J_{\beta^j}$;
4: Determine $\Delta \beta$ by finding the least squares solution for
   \[
   L \begin{bmatrix} J \\ \mathbb{I} \end{bmatrix} \Delta \beta = L \begin{bmatrix} V - U_{FE}(\beta^j) \\ \beta^* - \beta^j \end{bmatrix};
   \]
5: Solve a one-dimensional version of (14) over the line segment
   \[
   \mathcal{L} = \{ \beta^j + t \Delta \beta \mid 0 \leq t \leq t_0 \}
   \]
   by the golden section line search ($t_0 > 0$ is such that all model parameters on the
   line segment are feasible);
6: Update $\beta^{j+1} = \beta^j + t^* \Delta \beta$, where $t^* \geq 0$ is the minimizing step length from
   step 5;
7: Update $j = j + 1$;
8: until the chosen stopping criterion is met

To be quite precise, in order to enable faster convergence of the geometry
parameters, sometimes the maximal step length $t_0 > 0$ in the above algorithm is allowed
to be so large that some of the parameter vectors on $\mathcal{L}$ correspond to negative voxel
values for the conductivity. In such a case, the parameters defining the conductivity on $\mathcal{L}$
are forced above a small positive threshold to maintain stability. Moreover, the precise
stopping criterion of Algorithm 1 bore no importance for any of the reconstructions
presented in Section 4.3, as the convergence of the iterations was always unambiguous.

The differentiability results related to the Jacobian $J_{\beta}$ were discussed in Section 2.2.
For the numerical computation of the derivatives of $U_{FE}$ with respect to the admittivity
and the contact impedances, we refer to [15, 32], respectively. The derivatives with
respect to the geometrical parameters, i.e., the electrode polar angles and boundary
parameters, are approximated by applying the sampling formula (8) to the relevant
finite element solutions of (1); see [9, Section 5.3] and Remark 2.4 for more details. The
assumption of fixed electrode widths is incorporated in the computations with the help
of the arclength formula for (11), the Leibniz differentiation formula and the chain rule
for total derivative.

4. Reconstructions from experimental data

The feasibility of the simultaneous reconstruction of the boundary shape and the internal
admittivity distribution is evaluated experimentally with water tank data corresponding
to different cross-section shapes and conductivities. For each target, we compute three
reconstructions corresponding to different levels of information about the measurement
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Figure 1: The employed water tank shapes. Tanks A.1 and A.2 are different realizations of the same deformable tank.

geometry (cf. [9]):

(i) The water tank shape is incorrectly assumed to be a disk with equispaced electrodes on its boundary. That is, the geometry is fixed and only $\sigma$ and $z$ are reconstructed by Algorithm 1; see Section 4.2 for details. (This approach corresponds to ignoring the incompleteness of the available geometric information.)

(ii) The water tank shape and the electrode locations are estimated from a photograph. The reconstruction is computed as in case (i). (These reconstructions correspond to a relatively accurate information on the measurement geometry.)

(iii) The full Algorithm 1 is applied to the simultaneous recovery of the unknown boundary shape and the admittivity distribution. Here, the geometrical setting of case (i) serves as the initial guess for $\alpha$ and $\phi$. (See Section 4.2 for a slight adjustment to Algorithm 1 caused by the instability issues described in Section 2.3.)

4.1. Measurements

The experiments were performed using the three tank shapes shown in Figure 1. Two different plastic tanks were used: one with a deformable cross-section (Tank A) and one with a fixed cross-section (Tank B). Sixteen identical metallic electrodes were attached to the interior lateral surface of each tank. The tanks were filled with Finnish tap water, and objects made of different materials (steel or plastic) were placed inside the tanks. All inclusions were homogeneous in the vertical direction and extended through the water surface. The water level was kept constant, i.e., up to the top of the rectangular electrodes, by controlling the amount of water in the tank. The geometric parameters of the used tanks are given in Table 1.

The measurements were conducted using the Kuopio Impedance Tomography (KIT4) device [22]. Low-frequency alternating current (frequency 1 kHz, amplitude 1 mA) was injected between one fixed electrode (grounded) and each of the other
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electrodes in turn. Corresponding to each current injection, the electrode potentials were measured against the ground. The phase information of the voltage measurements was ignored, and the electrode currents and potentials were interpreted as real vectors, with the aim being the reconstruction of an approximative real admittivity (i.e. conductivity).

Table 1: Dimensions of the used water tanks.

<table>
<thead>
<tr>
<th>Tank</th>
<th>circumference (m)</th>
<th>depth (m)</th>
<th>electrode width (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.86</td>
<td>0.07</td>
<td>0.02</td>
</tr>
<tr>
<td>B</td>
<td>1.06</td>
<td>0.05</td>
<td>0.02</td>
</tr>
</tbody>
</table>

4.2. Implementational issues

For a fixed geometry, i.e., in cases (i) and (ii), we choose the prior mean \((\sigma^*, z^*)\) to be the minimizer of the output least squares part of (13), i.e.,

\[
\| V - U^{FE}(\sigma, z) \|^2_{\Gamma_n^{-1}}
\]

over pairs of homogeneous admittivities and constant contact impedance vectors, assuming the measurement geometry is fixed according to the considered configuration. Algorithm 1 is then run with a fixed geometry until convergence is observed; this means that the second line of the Tikhonov functional (13) is ignored and the remaining part is treated as a function of \(\sigma\) and \(z\) only. The used noise covariance matrix \(\Gamma_n\), the parameters defining \(\Gamma_\sigma\) and the standard deviation of the contact impedances \(\gamma_z\) are given in the right-hand column Table 2.

In the full algorithm, we chose \(N_1 = 9\) as the number of the Fourier modes in the representation (11). As discussed in Section 2.3, we experienced instability in the computation of the shape derivatives, i.e., the derivatives with respect to the parameters \(\alpha\) and \(\phi\), if the contact impedances used in (8) were small — as they typically are in water tank experiments. Consequently, for case (iii) our implementation of Algorithm 1 consists of two stages:

I In the first stage, we enforce the components of \(z\) large enough by choosing the expectation \(z^*\) and the standard deviation \(\gamma_z\) as if we were anticipating bad contacts; in light of the material of Section 2.3, this choice can be considered as regularization. The prior mean for \(\sigma\) is a constant distribution with value \(0.016 (\Omega m)^{-1}\), which is in good agreement with cases (i) and (ii), and those for \(\alpha\) and \(\phi\) correspond to a discoidal cross-section with equiangled electrodes. Notice that neither \(\alpha^*\) nor \(\phi^*\) corresponds to the target value of the respective parameter in any of the measurement configurations, as the target electrodes are not equiangled although they are equispaced along the water tank boundary. The prior means and (co)variances of different parameters are listed in the left-hand column of Table 2. (In particular \(\Gamma_n\) and \(\Gamma_\sigma\) are the same as for cases (i) and (ii).) We run the full Algorithm 1 with these parameter choices until convergence is observed.
Table 2: Prior means and covariances for Algorithm 1.

<table>
<thead>
<tr>
<th></th>
<th>full algorithm</th>
<th>fixed-geometry version</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_*$</td>
<td>$0.016 ; (\Omega m)^{-1}$</td>
<td>best homogeneous</td>
</tr>
<tr>
<td>$z_*$</td>
<td>$0.08 ; \Omega m^2$</td>
<td>best homogeneous</td>
</tr>
<tr>
<td>$\alpha_*$</td>
<td>$[0.12 ; m, 0, \ldots, 0]^T \in \mathbb{R}^{19}$</td>
<td>–</td>
</tr>
<tr>
<td>$\phi_*$</td>
<td>equiangled</td>
<td>–</td>
</tr>
</tbody>
</table>

\[
\tau = (10^{-3} (\max_m V_m - \min_m V_m))^2 \mathbb{I} \quad \text{(full algorithm)}
\]

II In the second stage, we first fix $\phi$ to the value obtained in the first stage and scale $\alpha$ so that the circumference of the tank is adjusted to the known correct value. Subsequently, the reconstruction is fine-tuned by running the fixed-geometry version of Algorithm 1 — as in cases (i) and (ii) — until satisfactory convergence is observed.

It should be emphasized that the first stage of the algorithm already results in qualitatively feasible reconstructions of the admittivity and the measurement geometry, but in a too small geometric scale (this is the reason for scaling $\alpha$ in the beginning of the second stage). In particular, scaled versions of the reconstructions provided by the first stage could quite well be considered as the ‘final reconstructions’. What is more, the quality of the actual final reconstruction provided by the two-stage algorithm does not seem to depend heavily on the initial guesses (or prior means) for $\sigma$ and $z$ in the second stage. It is naturally possible to use a scaled version of the reconstruction from the first stage as the initial guess for $\sigma$ in the second stage, but all our reconstructions were anyway computed by running a fixed-geometry version of Algorithm 1 with homogeneous estimates for $\sigma$ and $z$ as the initial guesses. In other words, the second stage exactly corresponds to the reconstruction technique used in cases (i) and (ii) with the parameters listed in the right-hand column of Table 2.

4.3. Results and discussion

The reconstructions corresponding to the three water tank shapes and altogether nine admittivity phantoms are shown in Figures 2, 3 and 4. The admittivity is homogeneous in the vertical direction, which allows us to only consider cross-sections
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Figure 2: Tank A1. First column: photos of the targets (top: two plastic cuboids; middle: a hollow steel cylinder; bottom: a hollow steel cylinder and a plastic cuboid). Second column: reconstructions with an incorrect geometry. Third column: reconstructions with an (almost) accurate geometry. Fourth column: simultaneous reconstructions of the geometry and the conductivity by the full algorithm.

As obvious from the reconstructions in the second columns of Figures 2, 3 and 4, approach (i) that ignores the uncertainties in the measurement configuration results in intolerably bad admittivity reconstructions. On the other hand, in both cases (ii) and (iii) the reconstructions clearly carry useful information about the target admittivity. While approach (ii) reproduce the shapes of the inclusions more accurately, there are less artifacts close to the tank boundary in the reconstructions corresponding to case (iii). Two conclusions can be drawn: Firstly, even the ever-so-slight mismodelling of the measurement geometry in case (ii) is enough to affect the admittivity reconstruction close to the electrodes. Secondly, the smoothing prior for the admittivity tends to dominate in case (iii): the reconstructed inclusion shapes are too round, which is
Figure 3: Tank A.2. First column: photos of the targets (top: a plastic cuboid; middle: three thin steel cylinders; bottom: two plastic cuboids). Second column: reconstructions with an incorrect geometry. Third column: reconstructions with an (almost) accurate geometry. Fourth column: simultaneous reconstructions of the geometry and the conductivity by the full algorithm.

compensated for by producing slightly inaccurate shapes for the water tank (cf. lines 1 and 3 in Figures 2 and 3). This latter, arguably unwanted phenomenon could be tackled by tightening the prior for the geometry parameter $\alpha$ — or loosening that for $\sigma$ —, especially if more accurate a priori information on the body shape was available. (Recall that in all reconstructions for case (iii) the initial guess for the cross-section shape is a disk.)

To sum up, the full reconstruction algorithm utilized in case (iii) provides reasonable reconstructions of both the admittivity distribution and the body shape in all of our experiments. In particular, the results for (iii) are comparable to case (ii), which assumes almost accurate information on the measurement geometry, and far superior compared to case (i), which works in the geometry that serves as the initial guess for (iii).

5. Conclusions

Uncertainties in the outer boundary shape, electrode positions and contact impedances are the most significant sources of modeling errors in medical applications of EIT. Because absolute EIT reconstructions are highly intolerant to such inaccuracies, it has often been claimed that absolute EIT imaging is not applicable to medical diagnostics. However, if it was possible to recover from these errors by modeling the
uncertainties, absolute EIT imaging could produce useful quantitative information in medical applications — in contrast to the qualitative information provided by commonly used difference imaging.

In this work, we have introduced an iterative method that simultaneously reconstructs the admittivity, the exterior boundary shape and the contact impedances of a cylindrical three-dimensional object in the framework of the CEM of EIT (cf. [9]). Via experimental studies with water tank data, it was demonstrated that our method provides reasonable reconstructions even if the initial guess for the measurement geometry is substantially misspecified. The computation of the shape derivatives employed in the iterative Gauss–Newton-type output least squares algorithm was observed to suffer from instability if the contact impedances at the electrode-object interfaces were too small. This imperfection was tackled by computing the estimates in two steps: In the first stage, the boundary shape was reconstructed while enforcing artificially high contact impedances, which was observed to have a regulative effect. In the second stage, the body shape was fixed and the enforcement of the contact impedances was relaxed.
Acknowledgments

N. Hyvönen (decision 135979), A. Seppänen (the Centre of Excellence in Inverse Problems Research and decision 140280) and S. Staboulis (decision 141044) were supported by the Academy of Finland.

References

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