AN H_{div}-BASED MIXED QUASI-REVERSIBILITY METHOD FOR SOLVING ELLIPTIC CAUCHY PROBLEMS

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Abstract. This work considers the Cauchy problem for a second order elliptic operator in a bounded domain. A new quasi-reversibility approach is introduced for approximating the solution of the ill-posed Cauchy problem in a regularized manner. The method is based on a well-posed mixed variational problem on $H^1 \times H_{\mathsf{div}}$, with the corresponding solution pair converging monotonically to the solution of the Cauchy problem and the associated flux, if they exist. It is demonstrated that the regularized problem can be discretized using Lagrange and Raviart–Thomas finite elements. The functionality of the resulting numerical algorithm is tested via three-dimensional numerical experiments based on simulated data. Both the Cauchy problem and a related inverse obstacle problem for the Laplacian are considered.

1. Introduction

We consider the Cauchy problem for a second order elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. Such problems have been studied extensively (see, e.g., [1, 4, 10, 24]), with the common understanding that they are 'exponentially' ill-posed. Cauchy problems for elliptic operators are encountered in many practical applications such as *electrocardiography* (ECG) [23] and *plasma physics* [7, 20], and they are also closely related to the inverse source problems arising from, e.g., *electroencephalography* (EEG) and *magnetoencephalography* (MEG) [25]. In addition, Cauchy problems play an important role in *inverse obstacle problems* (cf. [12]), which are studied, e.g., in connection with inclusion detection by *electrical impedance tomography* (EIT) when only one pair of boundary current and voltage is used for probing the examined body [8, 26].

Several methods have been proposed to regularize the Cauchy problem; see, e.g., [2, 3, 5, 6, 15, 28] and the references therein. Among these approaches, the variants of the *quasi-reversibility* (QR) method [30] comprise a technique with some useful properties. The main idea of the original QR method [30] is to approach the ill-posed second order Cauchy problem by a family of well-posed fourth order problems depending on a (small) regularization parameter. It is a non-iterative technique which can be applied numerically using *finite elements methods* (FEM), and thus it is adaptable for solving the Cauchy problem in complicated domains. However, the main drawback of the original QR method is the fourth order nature of the regularizing variational problems: one has to use C^1 finite elements, which are difficult to handle and seldom available in numerical solvers, especially in three dimensions.

The first attempt to get rid of this technical difficulty was introduced in [9], where a QR method based on a mixed variational regularization of the Cauchy problem was proposed. The method involves two unknown $H^1(\Omega)$-functions, one of which approximates the solution of the Cauchy problem and the other, loosely speaking, the error in the corresponding right-hand side (cf. (3.3)). In particular, the second member of the regularized solution does not carry any explicit information on the solution of the underlying Cauchy problem. (The approach of [9] has also other minor drawbacks as explained in Section 3 below.)

The purpose of this article is to introduce a new QR method that is based on a mixed variational formulation on $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$. As for the previous mixed QR method in [9], the first member of the regularized solution approximates the actual solution of the Cauchy problem, but in our case also the second member provides explicit information on the solution: it gives an estimate for the corresponding flux. As the regularization parameter tends to zero, the solution of our mixed QR problem converges monotonically to the solution of the Cauchy problem, if such exists, and diverges otherwise. According to our knowledge, *monotonic* convergence has not previously been shown for any QR formulation; our technique for proving this property can straightforwardly be adapted for the original QR method of

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[30] as well. Moreover, we briefly consider how the regularization parameter of the mixed QR method should be chosen in the case of noisy data and demonstrate how the method can be discretized by combining Lagrange and Raviart–Thomas finite elements, which are standard tools supported by many finite element solvers.

In order to illustrate the functionality of our mixed QR method, we test it numerically in a realistic three-dimensional setting; according to our knowledge, this is the first implementation of any QR method in three dimensions. Both the Laplace operator and a constant-coefficient, anisotropic operator of the divergence form are considered. In addition, we incorporate our method as a part of the level set algorithm for solving the inverse (Dirichlet) obstacle problem for the Laplace equation introduced in [12]. According to our numerical studies, the new QR formulation produces acceptable solutions both for Cauchy problems and for the inverse obstacle problem, even with considerable amounts of noise.

The paper is organized as follows. Section 2 introduces the considered Cauchy problem together with the related function spaces and terminology. In Section 3, we present the standard formulation of the QR method from [30] and the mixed one proposed in [9]. We highlight their main qualities and drawbacks. Section 4 is dedicated to the formulation of the new mixed QR method, its properties and discretization. Finally, Section 5 presents the numerical experiments and Section 6 lists the concluding remarks.

2. The setting

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary (cf. [22]). We denote by $\nu \in L^\infty(\partial\Omega, \mathbb{R}^d)$ the exterior unit normal of $\partial\Omega$, and suppose that $\partial\Omega$ is divided into two open subsets of positive measure Γ and Γ_c such that $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma}_c$. Let $A = [a_{ij}] : \Omega \to \mathbb{R}^{d\times d}$ be a real matrix valued function such that $A \in W^{1,\infty}(\Omega)^{d\times d}$ and

$$A^{\mathrm{T}} = A, \qquad \xi^{\mathrm{T}} A \xi \geq c|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d$$

almost everywhere in Ω. The Cauchy problem we are interested in is defined as follows:

Definition 2.1 (Cauchy problem)**.** For $(f, g_D, g_N) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $u \in H^1(\Omega)$ such that

$$\begin{cases} \nabla \cdot A\nabla u = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial \nu_A} = g_N & \text{on } \Gamma. \end{cases}$$

Here,

$$H^{1/2}(\Gamma) = \left\{ \phi_{|\Gamma} \ : \ \phi \in H^1(\Omega) \right\}$$

and $H^{-1/2}(\Gamma)$ is defined as the dual of (see, e.g., [19])

$$\tilde{H}^{1/2}(\Gamma) = \left\{ \phi \in L^2(\Gamma) \ : \ \exists v \in H^1(\Omega) \text{ s.t. } v_{|\Gamma} = \phi, \ v_{|\Gamma_c} = 0 \right\}.$$

Moreover,

$$\frac{\partial}{\partial \nu_A} : v \mapsto \sum_{i,j=1}^{d} \left(a_{ij} \frac{\partial v}{\partial x_j} \nu_i \right)_{|\Gamma}, \quad H^1(\Omega, \nabla \cdot A\nabla) \to H^{-1/2}(\Gamma) \tag{1}$$

is the linear and bounded conormal derivative operator defined on (cf., e.g., [22])

$$H^1(\Omega, \nabla \cdot A\nabla) := \left\{ v \in H^1(\Omega) \ : \ \nabla \cdot A\nabla v \in L^2(\Omega) \right\}.$$

Since Hadamard, it is well-known that the Cauchy problem of Definition 2.1 is severely ill-posed: although it has at most one solution (see, e.g., [31]), it may have none, and if a solution exists, it does not depend continuously on the data (f, g_D, g_N) in any reasonable topology. Therefore, regularization is needed to stabilize the problem.

3. Previous quasi-reversibility methods

For simplicity, we focus in this section on the special case that A is the identity matrix, i.e., we consider the Cauchy problem for the Poisson equation. We will return to the general setting in Section 4, where the new mixed formulation of the QR method will be introduced.

3.1. **Standard formulation.** For the standard formulation of the QR method, we need to make some extra smoothness assumption on $\partial\Omega$, g_D and g_N. More precisely, we suppose that $\partial\Omega$ is of the class $C^{1,1}$, $g_D \in H^{3/2}(\Gamma)$ and $g_N \in H^{1/2}(\Gamma)$; under this assumption on the regularity of $\partial\Omega$, the trace operation

$$H^2(\Omega) \ni v \mapsto \big(v, \frac{\partial v}{\partial \nu}\big) \in H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$$

is linear, bounded and surjective [22]. The QR method, as introduced in [30], relies on the following problem:

Definition 3.1 (Standard QR problem)**.** For $\varepsilon > 0$, find $u \in H^2(\Omega)$, with $u_{|\Gamma} = g_D$ and $(\partial u)/(\partial\nu)_{|\Gamma} = g_N$, such that for all $v \in H^2(\Omega)$, with $v_{|\Gamma} = 0$ and $(\partial v)/(\partial\nu)_{|\Gamma} = 0$,

$$\int_\Omega \Delta u \, \Delta v \, dx + \varepsilon \int_\Omega \Big(\sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} + \nabla u \cdot \nabla v + uv \Big) \, dx = \int_\Omega f \, \Delta v \, dx. \tag{2}$$

It follows directly from the Lax–Milgram theorem that the standard QR problem has a unique solution.

Proposition 3.1. *For any $\varepsilon > 0$, the standard QR problem of Definition 3.1 admits a unique solution $u_\varepsilon \in H^2(\Omega)$.*

Moreover, one can deduce the following convergence result [30, 18]:

Theorem 3.2. *Suppose that the Cauchy problem admits a (unique) solution $u \in H^2(\Omega)$. Then, the solution to the standard QR problem u_ε converges to u in $H^2(\Omega)$ as ε tends to zero, and it holds that*

$$\|\Delta u_\varepsilon - f\|_{L^2(\Omega)} \leq \sqrt{\varepsilon}\|u\|_{H^2(\Omega)}.$$

By nature, the QR method is non-iterative. Moreover, in case the data (g_D, g_N) are corrupted by noise, a method to set the regularization parameter $\varepsilon > 0$ as a function of the amplitude of the noise, say $\delta > 0$, based on the Morozov's discrepancy principle and the duality in optimization has been developed in [11]; the solution $u_{\varepsilon(\delta)}$ of the QR problem tends to the exact solution of the Cauchy problem when the amplitude of the noise goes to zero (cf. Section 4.3). Unfortunately, the standard QR formulation of Definition 3.1 has also two drawbacks, which are linked.

First of all, one needs relatively smooth data to ensure that (2) has a solution. To make matters worse, the convergence of the approximate solution u_ε toward the exact solution u has only been proved if $u \in H^2(\Omega)$, although it is well-known that in some cases u is only in $H^1(\Omega, \Delta)$ because the Cauchy data is *a priori* known to be $H^{3/2} \times H^{1/2}$-smooth only on a subset of $\partial\Omega$.

However, the main issue with the standard QR formulation is arguably related to its discretization: to obtain an approximate solution of the Cauchy problem, one has to discretize the variational formulation (2) using, e.g., some FEM. Since (2) is a fourth order problem, it cannot be discretized using standard C^0 finite elements. In order to obtain a conforming discretization of the problem, one has to use C^1 finite elements, which are difficult to handle and seldom available in numerical solvers, especially for three-dimensional problems; see [16] for a description of such elements in the two-dimensional case. According to our knowledge, a conforming discretization of the variational equation (2) has never been performed. However, (2) has been successfully discretized using non-conforming finite elements [12], namely the Fraeijs de Veubeke 1 elements [29], which are simpler than C^1 elements, but unfortunately rarely implemented in numerical solvers as well. It should also be mentioned that the QR problem has been successfully discretized using difference schemes [30] and splines [17], but these approaches are typically limited to simple geometries.

To overcome this technical difficulty with the discretization, we will propose a novel QR method based on a mixed variational formulation. The leading idea is to introduce an additional unknown, which deals with the second order derivatives, resulting in a lower order problem that can be discretized using standard finite elements. However, our mixed QR formulation is not the first of its kind, as indicated in the following section.

3.2. **A previous mixed formulation.** According to our knowledge, the first (and thus far only) mixed QR approach has been proposed in [9]. It relies on the following problem, for which the assumptions on $\partial\Omega$, g_D and g_N are as listed in Section 2.

Definition 3.3 (Mixed QR problem)**.** For ε, $\delta > 0$, find $u \in H^1(\Omega)$, with $u_{|\Gamma} = g_D$, and $\lambda \in H^1(\Omega)$, with $\lambda_{|\Gamma_c} = 0$, such that

$$\begin{cases} \varepsilon \displaystyle\int_\Omega (\nabla u \cdot \nabla v + u\, v)\, dx + \int_\Omega \nabla\lambda \cdot \nabla v\, dx = 0 & \text{for all } v \in H^1(\Omega),\ v_{|\Gamma} = 0, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla \mu\, dx - \int_\Omega \lambda\, \mu\, dx - \delta \int_\Omega (\nabla\lambda \cdot \nabla\mu + \lambda\, \mu)\, dx & \\ \qquad = \displaystyle\int_\Omega f\, \mu\, dx + \langle g_N, \mu\rangle_{H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)} & \text{for all } \mu \in H^1(\Omega),\ \mu_{|\Gamma_c} = 0. \end{cases}$$

The following theorem follows directly from the material in [9].

Theorem 3.4. *For all ε, $\delta > 0$, the mixed QR problem of Definition 3.3 has a unique solution $(u_{\varepsilon,\delta}, \lambda_{\varepsilon,\delta})$ in $H^1(\Omega) \times H^1(\Omega)$. Furthermore, if $\delta > 0$ is defined as a function of $\varepsilon > 0$ so that*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta(\varepsilon)} = 0$$

and if the Cauchy problem has a (unique) solution $u \in H^1(\Omega)$, then

$$(u_{\varepsilon,\delta(\varepsilon)}, \lambda_{\varepsilon,\delta(\varepsilon)}) \xrightarrow[H^1 \times H^1]{\varepsilon \to 0} (u, 0).$$

Obviously, this mixed formulation of the QR method carries some properties that we were looking for. First of all, it does not require any additional smoothness assumptions on the boundary, the Cauchy data or the exact solution of the Cauchy problem. Secondly, the variational form appearing in Definition 3.3 can be discretized using standard C^0 Lagrange finite elements (cf. [9]), as the solution pair lies in $H^1(\Omega)^2$.

Unfortunately, this formulation has also some mild flaws. The first one concerns the additional unknown $\lambda_{\varepsilon,\delta}$, which can in a way be considered known: it can be interpreted as an estimate for $\Delta u - f$, with u being the solution of the original Cauchy problem (assuming that it exists). In other words, $\lambda_{\varepsilon,\delta}$ approximates the zero function, and does not provide any additional information on the Cauchy problem in hand.

Arguably, the most important drawback of the above mixed formulation of the QR method is that there currently exists no method for choosing the regularization parameters (ε, δ) in case of noisy data. In particular, the method developed in [11] for the standard QR formulation cannot be used for this mixed formulation. As the Cauchy problem is severely ill-posed and its solution thus very sensitive to noise, this can be considered a major issue.

Due to the above described difficulties with the mixed QR problem of Definition 3.3, we propose in the following section a new mixed QR formulation. Our aim is to circumvent the flaws of the original mixed QR approach without loosing its good qualities (no additional smoothness assumption, discretization with standard finite elements).

4. A new H_{div}-based mixed quasi-reversibility method

Recall the original Cauchy problem of Definition 2.1 and assume for now that it has a solution. If we define $\mathbf{p} := A\nabla u$, it is clear that $\mathbf{p} \in L^2(\Omega)^d$. Furthermore, as $\nabla \cdot A\nabla u = f$ belongs to $L^2(\Omega)$, we actually have $\mathbf{p} \in H_{\mathsf{div}}(\Omega)$, with the standard definition

$$H_{\mathsf{div}}(\Omega) := \left\{ \mathbf{q} \in L^2(\Omega)^d \ : \ \nabla \cdot \mathbf{q} \in L^2(\Omega) \right\}.$$

Hence, we can rewrite the Cauchy problem in the following, equivalent form:

Definition 4.1 (Reformulated Cauchy problem)**.** For $(f, g_D, g_N) \in L^2(\Omega) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, find $(u, \mathbf{p}) \in H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ such that

$$\begin{cases} A\nabla u = \mathbf{p} & \text{in } \Omega, \\ \nabla \cdot \mathbf{p} = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma, \\ \mathbf{p} \cdot \nu = g_N & \text{on } \Gamma. \end{cases} \tag{3}$$

The leading idea behind our new mixed QR method is to regularize the reformulated Cauchy problem of Definition 4.1.

4.1. **Formulation and basic properties.** To begin with, let us introduce a few auxiliary sets and spaces:

$$V := \{v \in H^1(\Omega) \;:\; v_{|\Gamma} = g_D\}, \qquad V_0 := \{v \in H^1(\Omega) \;:\; v_{|\Gamma} = 0\},$$
$$D := \{\mathbf{q} \in H_{\mathsf{div}}(\Omega) \;:\; (\mathbf{q}\cdot\nu)_{|\Gamma} = g_N\}, \qquad D_0 := \{\mathbf{q} \in H_{\mathsf{div}}(\Omega) \;:\; (\mathbf{q}\cdot\nu)_{|\Gamma} = 0\},$$

which are well defined due to trace theorems in $H^1(\Omega)$ and $H_{\mathsf{div}}(\Omega)$ (cf., e.g., [19, 22]). We consider the following QR problem:

Definition 4.2 (New mixed QR problem)**.** For $\varepsilon > 0$, find $(u, \mathbf{p}) \in V \times D$ such that for all $(v, \mathbf{q}) \in V_0 \times D_0$
(4)

$$\begin{cases} \displaystyle\int_\Omega (A\nabla u - \mathbf{p})\cdot A\nabla v\,dx + \varepsilon\int_\Omega(\nabla u\cdot\nabla v + u\,v)\,dx = 0, \\ \displaystyle -\int_\Omega (A\nabla u - \mathbf{p})\cdot\mathbf{q}\,dx + \int_\Omega(\nabla\cdot\mathbf{p})(\nabla\cdot\mathbf{q})\,dx + \varepsilon\int_\Omega\big((\nabla\cdot\mathbf{p})(\nabla\cdot\mathbf{q}) + \mathbf{p}\cdot\mathbf{q}\big)dx = \int_\Omega f(\nabla\cdot\mathbf{q})\,dx. \end{cases}$$

It follows relatively straightforwardly from the Lax–Milgram theorem that the above mixed variational problem has a unique solution.

Proposition 4.1. *For all $\varepsilon > 0$, the new mixed QR problem of Definition 4.2 has a unique solution $(u_\varepsilon, \mathbf{p}_\varepsilon) \in V \times D$.*

Proof. Since the mappings $v \in H^1(\Omega) \mapsto v_{|\Gamma} \in H^{1/2}(\Gamma)$ and $\mathbf{q} \in H_{\mathsf{div}}(\Omega) \mapsto (\mathbf{q}\cdot\nu)_{|\Gamma} \in H^{-1/2}(\Gamma)$ are linear, continuous and surjective [19, 22], they have continuous right inverses. Hence, for all $(g_D, g_N) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$, there exists $(u_D, \mathbf{p}_N) \in V \times D$ such that

$$\|u_D\|_{H^1(\Omega)} \le c\|g_D\|_{H^{1/2}(\Gamma)}, \qquad \|\mathbf{p}_N\|_{H_{\mathsf{div}}(\Omega)} \le c\|g_N\|_{H^{-1/2}(\Gamma)}, \tag{5}$$

where $c > 0$ does not depend on the data (g_D, g_N). In consequence, by adding the two equations in (4), we can rewrite the new mixed QR problem in the following equivalent form: find $(\tilde{u}, \tilde{\mathbf{p}}) \in V_0 \times D_0$ that satisfies

$$a_\varepsilon\big((\tilde{u}, \tilde{\mathbf{p}}), (v, \mathbf{q})\big) = L_\varepsilon(v, \mathbf{q}) \tag{6}$$

for all $(v, \mathbf{q}) \in V_0 \times D_0$, with

$$\begin{aligned} a_\varepsilon\big((\tilde{u}, \tilde{\mathbf{p}}), (v, \mathbf{q})\big) \;:=& \int_\Omega (A\nabla\tilde{u} - \tilde{\mathbf{p}})(A\nabla v - \mathbf{q})\,dx + \int_\Omega(\nabla\cdot\tilde{\mathbf{p}})(\nabla\cdot\mathbf{q})\,dx \\ &+ \varepsilon\int_\Omega(\nabla\tilde{u}\cdot\nabla v + \tilde{u}\,v)\,dx + \varepsilon\int_\Omega\big((\nabla\cdot\tilde{\mathbf{p}})(\nabla\cdot\mathbf{q}) + \tilde{\mathbf{p}}\cdot\mathbf{q}\big)dx \end{aligned}$$

and

$$L_\varepsilon(v, \mathbf{q}) := \int_\Omega f(\nabla\cdot\mathbf{q})\,dx - a_\varepsilon\big((u_D, \mathbf{p}_N), (v, \mathbf{q})\big).$$

The product space $V_0 \times D_0$, endowed with the scalar product

$$\big((u, \mathbf{p}), (v, \mathbf{q})\big)_{H^1\times H_{\mathsf{div}}} := \int_\Omega(\nabla u\cdot\nabla v + u\,v)\,dx + \int_\Omega\big((\nabla\cdot\mathbf{p})(\nabla\cdot\mathbf{q}) + \mathbf{p}\cdot\mathbf{q}\big)dx,$$

is a Hilbert space. Furthermore, a_ε and L are obviously bilinear and linear functionals on $(V_0 \times D_0)^2$ and $V_0 \times D_0$, respectively, and for all $(u, \mathbf{p}) \in V_0 \times D_0$ and $(v, \mathbf{q}) \in V_0 \times D_0$ it holds that

$$a_\varepsilon\big((v, \mathbf{q}), (v, \mathbf{q})\big) \ge \varepsilon\|(v, \mathbf{q})\|^2_{H^1\times H_{\mathsf{div}}},$$
$$\big|a_\varepsilon\big((u, \mathbf{p}), (v, \mathbf{q})\big)\big| \le \big(1 + 2\|A\|_\infty + \|A\|^2_\infty + \varepsilon\big)\,\|(u, \mathbf{p})\|_{H^1\times H_{\mathsf{div}}}\,\|(v, \mathbf{q})\|_{H^1\times H_{\mathsf{div}}},$$
$$|L_\varepsilon(v, \mathbf{q})| \le C\|(v, \mathbf{q})\|_{H^1\times H_{\mathsf{div}}},$$

where

$$C = \|f\|_{L^2(\Omega)} + c\big(1 + 2\|A\|_\infty + \|A\|^2_\infty + \varepsilon\big)\big(\|g_D\|_{H^{1/2}(\Gamma)} + \|g_N\|_{H^{-1/2}(\Gamma)}\big).$$

Hence, an application of the Lax-Milgram theorem [13] proves the unique existence of a solution $(\tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon)$ to (6), leading in turn to the existence of a unique solution

$$(u_\varepsilon, \mathbf{p}_\varepsilon) = (\tilde{u}_\varepsilon + u_D, \tilde{\mathbf{p}}_\varepsilon + \mathbf{p}_N) \in V \times D \tag{7}$$

to (4). □

Our main theorem states that $(u_\varepsilon, \mathbf{p}_\varepsilon)$ converges to the unique solution of the original Cauchy problem if such exists, and diverges otherwise.

Theorem 4.3. *If the Cauchy problem of Definition 2.1 has a unique solution $u \in H^1(\Omega)$, then $(u_\varepsilon, \mathbf{p}_\varepsilon)$ converges to $(u, A\nabla u)$ in $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ as $\varepsilon > 0$ tends to 0. If the Cauchy problem has no solution, then $\|(u_\varepsilon, \mathbf{p}_\varepsilon)\|_{H^1 \times H_{\mathsf{div}}} \to \infty$ as $\varepsilon > 0$ goes to 0.*

The mixed formulation (4) can thus be considered as a regularized version of the ill-posed Cauchy problem. Therefore, by solving the mixed QR problem, we obtain an approximation of the solution to the Cauchy problem, if such exists. Note that the additional unknown $\mathbf{p}_\varepsilon$ is an estimate of $A\nabla u$, and hence it provides additional information on the solution u.

Remark 4.4. The new mixed QR method can also be viewed as a compatibility test for the data (f, g_D, g_N). Indeed, if one wants to know if there exists a solution to the Cauchy problem, one can solve (4) for various $\varepsilon > 0$ and consider the behavior of the function $\mathbb{R}_+ \ni \varepsilon \mapsto \|(u_\varepsilon, \mathbf{p}_\varepsilon)\|_{H^1 \times H_{\mathsf{div}}}$ when ε tends to zero. If this function blows up, the Cauchy problem has no solution. However, it is probably difficult to use this criterion in practice — especially with noisy data —, as the blow-up may be very slow.

To prove Theorem 4.3, we divide it into two propositions, because the proofs of the two cases require slightly different arguments.

Proposition 4.2. *Suppose that the Cauchy problem of Definition 2.1 has a (unique) solution $u \in H^1(\Omega)$. Then,*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{H^1(\Omega)} = 0, \qquad \lim_{\varepsilon \to 0} \|\mathbf{p}_\varepsilon - A\nabla u\|_{H_{\mathsf{div}}(\Omega)} = 0,$$

and the estimates

$$\Big(\|A\nabla u_\varepsilon - \mathbf{p}_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{p}_\varepsilon - f\|^2_{L^2(\Omega)}\Big)^{1/2} \leq \sqrt{\varepsilon}\, \|(u, A\nabla u)\|_{H^1 \times H_{\mathsf{div}}}$$

hold.

Proof. Because u is the solution of the Cauchy problem, we know that $u_{|\Gamma} = g_D$, $\nu \cdot A\nabla u_{|\Gamma} = g_N$ and $A\nabla u \in H_{\mathsf{div}}(\Omega)$. Hence, choosing $v = u_\varepsilon - u \in V_0$ and $\mathbf{q} = \mathbf{p}_\varepsilon - A\nabla u \in D_0$ in (4) leads to

$$\begin{cases} \displaystyle\int_\Omega (A\nabla u_\varepsilon - \mathbf{p}_\varepsilon) \cdot (A\nabla u_\varepsilon - A\nabla u)\, dx + \varepsilon \int_\Omega \big(\nabla u_\varepsilon \cdot \nabla(u_\varepsilon - u) + u_\varepsilon\, (u_\varepsilon - u)\big)\, dx = 0, \\ \displaystyle -\int_\Omega (A\nabla u_\varepsilon - \mathbf{p}_\varepsilon) \cdot (\mathbf{p}_\varepsilon - A\nabla u)\, dx + \int_\Omega (\nabla \cdot \mathbf{p}_\varepsilon)\, \big(\nabla \cdot (\mathbf{p}_\varepsilon - A\nabla u)\big)\, dx \\ \displaystyle\qquad + \varepsilon \int_\Omega \big((\nabla \cdot \mathbf{p}_\varepsilon)(\nabla \cdot (\mathbf{p}_\varepsilon - A\nabla u)) + \mathbf{p}_\varepsilon \cdot (\mathbf{p}_\varepsilon - A\nabla u)\big) dx = \int_\Omega f\, \nabla \cdot (\mathbf{p}_\varepsilon - A\nabla u)\, dx. \end{cases}$$

Summing these equalities and using the fact that $\nabla \cdot A\nabla u = f$, it follows that

$$\|A\nabla u_\varepsilon - \mathbf{p}_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{p}_\varepsilon - f\|^2_{L^2(\Omega)} + \varepsilon\big((u_\varepsilon, \mathbf{p}_\varepsilon), (u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\big)_{H^1 \times H_{\mathsf{div}}} = 0. \tag{8}$$

Therefore, we have

$$\big((u_\varepsilon, \mathbf{p}_\varepsilon), (u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\big)_{H^1 \times H_{\mathsf{div}}} \leq 0, \tag{9}$$

which, in particular, means that

$$\|(u_\varepsilon, \mathbf{p}_\varepsilon)\|_{H^1 \times H_{\mathsf{div}}} \leq \|(u, A\nabla u)\|_{H^1 \times H_{\mathsf{div}}}. \tag{10}$$

Equation (9) also leads to

$$\|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|^2_{H^1 \times H_{\mathsf{div}}} \leq -\big((u, A\nabla u), (u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\big)_{H^1 \times H_{\mathsf{div}}}, \tag{11}$$

which in turn implies that

$$\|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|_{H^1 \times H_{\mathsf{div}}} \leq \|(u, A\nabla u)\|_{H^1 \times H_{\mathsf{div}}}. \tag{12}$$

Finally, by applying (10) and (12) to (8), we deduce the needed convergence estimates:

$$\begin{aligned} \|A\nabla u_\varepsilon - \mathbf{p}_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla \cdot \mathbf{p}_\varepsilon - f\|^2_{L^2(\Omega)} &\leq \varepsilon\big|\big((u_\varepsilon, \mathbf{p}_\varepsilon), (u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\big)_{H^1 \times H_{\mathsf{div}}}\big| \\ &\leq \varepsilon\|(u_\varepsilon, \mathbf{p}_\varepsilon)\|_{H^1 \times H_{\mathsf{div}}}\|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|_{H^1 \times H_{\mathsf{div}}} \\ &\leq \varepsilon\|(u, A\nabla u)\|^2_{H^1 \times H_{\mathsf{div}}}. \end{aligned} \tag{13}$$

According to (10), the family of solutions $(u_\varepsilon, \mathbf{p}_\varepsilon)$ is bounded in $H^1(\Omega)\times H_{\mathsf{div}}(\Omega)$, and thus there exists a sequence of real positive numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to zero such that the corresponding solutions of (4), i.e., $(u_n, \mathbf{p}_n) := (u_{\varepsilon_n}, \mathbf{p}_{\varepsilon_n})$, weakly converge to some $(w, \mathbf{r})$ in $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$. Since the operations

$$H^1(\Omega) \times H_{\mathsf{div}}(\Omega) \ni (v, \mathbf{q}) \mapsto A\nabla v - \mathbf{q} \in L^2(\Omega)^d, \qquad H_{\mathsf{div}}(\Omega) \ni \mathbf{q} \mapsto \nabla\cdot\mathbf{q} \in L^2(\Omega),$$
$$H^1(\Omega) \ni v \mapsto v_{|\Gamma} \in H^{1/2}(\Gamma), \qquad H_{\mathsf{div}}(\Omega) \ni \mathbf{q} \mapsto (\mathbf{q}\cdot\nu)_{|\Gamma} \in H^{-1/2}(\Gamma)$$

are linear and bounded, they are also weakly continuous. In consequence, by taking the weak limit of $(u_n, \mathbf{p}_n)$ and employing (13) with $\varepsilon = \varepsilon_n$, we deduce that $(w, \mathbf{r})$ satisfies

$$\begin{cases} A\nabla w = \mathbf{r} & \text{in } \Omega, \\ \nabla\cdot\mathbf{r} = f & \text{in } \Omega, \\ w = g_D & \text{on } \Gamma, \\ \mathbf{q}\cdot\nu = g_N & \text{on } \Gamma, \end{cases}$$

which means that $(w, \mathbf{r})$ is the solution of (3), i.e., $(w, \mathbf{r}) = (u, A\nabla u)$. Hence, the standard argument *ad absurdum* allows us to conclude that $(u_\varepsilon, \mathbf{p}_\varepsilon)$ weakly converges to $(u, A\nabla u)$ in $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ as $\varepsilon > 0$ goes to zero. Due to (11), weak convergence implies strong convergence, which completes the proof. □

Proposition 4.3. *Suppose that the Cauchy problem of Definition 2.1 has no solution. Then,*

$$\lim_{\varepsilon\to 0} \|(u_\varepsilon, \mathbf{p}_\varepsilon)\|_{H^1\times H_{\mathsf{div}}} = \infty.$$

Proof. Assume that $\|(u_\varepsilon, \mathbf{p}_\varepsilon)\| \le C$, for some $C > 0$ independent of $\varepsilon > 0$. Then, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers converging to 0 and $(w, \mathbf{r}) \in H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ such that $(u_n, \mathbf{p}_n) := (u_{\varepsilon_n}, \mathbf{p}_{\varepsilon_n})$ weakly converges to $(w, \mathbf{r})$ as n goes to infinity. Defining $\boldsymbol{\omega} = A\nabla w - \mathbf{r} \in L^2(\Omega)^d$ and $\xi = \nabla\cdot\mathbf{r} - f \in L^2(\Omega)$, and taking the limit of (4), with $\varepsilon = \varepsilon_n$ and $(u, \mathbf{p}) = (u_n, \mathbf{p}_n)$, as n goes to infinity, it follows that

$$\begin{cases} \displaystyle\int_\Omega \boldsymbol{\omega}\cdot A\nabla v\, dx = 0 & \text{for all } v \in V_0, \\ \displaystyle -\int_\Omega \boldsymbol{\omega}\cdot\mathbf{q}\, dx + \int_\Omega \xi(\nabla\cdot\mathbf{q})\, dx = 0 & \text{for all } \mathbf{q} \in D_0. \end{cases} \tag{14}$$

As $C_0^\infty(\Omega)^d \subset D_0$, the second equation of (14) means, in particular, that $\nabla\xi = -\boldsymbol{\omega} \in L^2(\Omega)^d$ in the sense of distribution, and thus $\xi \in H^1(\Omega)$. In a similar manner, the first equation then implies that $\nabla\cdot A\nabla\xi = 0$ in Ω, as A is symmetric by assumption and $C_0^\infty(\Omega) \subset V_0$. We are therefore allowed to use the (generalized) Green's formula [32]

$$\int_\Omega \big((\nabla\cdot A\nabla\xi)v + A\nabla\xi\cdot\nabla v\big)dx = \langle A\nabla\xi\cdot\nu, v\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} \qquad \text{for all } v \in H^1(\Omega).$$

Together with the first part of (14), this leads to

$$\langle A\nabla\xi\cdot\nu, v\rangle_{H^{-1/2}(\Gamma_c), \tilde{H}^{1/2}(\Gamma_c)} = 0 \qquad \text{for all } v \in V_0,$$

i.e., $A\nabla\xi\cdot\nu = 0$ on Γ_c. By the Green's formula, we also have

$$\int_\Omega \big((\nabla\cdot\mathbf{q})\,\xi + \mathbf{q}\cdot\nabla\xi\big)dx = \langle \mathbf{q}\cdot\nu, \xi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)}$$

for all $\mathbf{q} \in H_{\mathsf{div}}(\Omega)$. Consequently, the second equation of (14) and the established connection between ξ and $\boldsymbol{\omega}$ indicate that

$$\langle \mathbf{q}\cdot\nu, \xi\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0 \qquad \text{for all } \mathbf{q} \in D_0,$$

which is just a more complicated way of writing $\xi = 0$ on Γ_c.

We have altogether concluded that $\xi \in H^1(\Omega)$ satisfies

$$\begin{cases} \nabla\cdot A\nabla\xi = 0 & \text{in } \Omega, \\ \xi = 0 & \text{on } \Gamma_c, \\ A\nabla\xi\cdot\nu = 0 & \text{on } \Gamma_c, \end{cases}$$

and thus ξ must vanish by the unique solvability of this Cauchy problem. By the construction, we thus have $\nabla\cdot\mathbf{r} = f$ and $A\nabla w = \mathbf{r}$ in Ω. Using the weak continuity of the trace operators $H^1(\Omega) \ni v \mapsto v_{|\Gamma} \in$

$H^{1/2}(\Gamma)$ and $H_{\mathsf{div}}(\Omega) \ni \mathbf{q} \mapsto (\mathbf{q}\cdot\nu)_{|\Gamma} \in H^{-1/2}(\Gamma)$, it also follows that

$$\begin{cases} w = \lim_{n\to\infty} u_n = g_D & \text{on } \Gamma, \\ \mathbf{r}\cdot\nu = \lim_{n\to\infty} \mathbf{p}_n\cdot\nu = g_N & \text{on } \Gamma. \end{cases}$$

In other words, $(w, \mathbf{r})$ is a solution of the reformulated Cauchy problem (3), which is a contradiction. □

4.2. **Monotonic convergence.** In this section, we are interested in the following question: Assume that the original Cauchy problem of Definition 2.1 has a solution u and suppose (a bit unrealistically) that we have access to its noiseless Cauchy data. Consider two possible values for the regularization parameter ε_1 and ε_2, with $\varepsilon_1 < \varepsilon_2$. Which parameter should we use in the new mixed QR method of Definition 4.2? Indeed, even though we know that u_ε tends to u and $\mathbf{p}_\varepsilon$ to $A\nabla u$ as ε goes to zero, we have not yet provided any result stating that the convergence is monotonic. We will tackle this imperfection and demonstrate that the function $\mathbb{R}_+ \ni \varepsilon \mapsto \|(u-u_\varepsilon, A\nabla u - \mathbf{p}_\varepsilon)\|_{H^1\times H_{\mathsf{div}}}$ is strictly increasing. In the rest of this section, we continue to implicitly assume that the original Cauchy problem has a (unique) solution $u \in H^1(\Omega)$.

4.2.1. *Smoothness of the map $\varepsilon \mapsto (u_\varepsilon, \mathbf{p}_\varepsilon)$.* Let us define an auxiliary function

$$F : \varepsilon \mapsto (u_\varepsilon, \mathbf{p}_\varepsilon), \quad \mathbb{R}_+ \to H^1(\Omega)\times H_{\mathsf{div}}(\Omega)$$

where $(u_\varepsilon, \mathbf{p}_\varepsilon)$ is the unique solution of (4). We will demonstrate that F is smooth, starting with continuity.

Proposition 4.4. *It holds that $F \in C^0(\mathbb{R}_+, H^1(\Omega)\times H_{\mathsf{div}}(\Omega))$.*

Proof. Let $\varepsilon \in \mathbb{R}_+$ and $h \in \mathbb{R}$ be such that also $\varepsilon + h \in \mathbb{R}_+$. Due to (6) and (7), we have

$$\begin{cases} a_{\varepsilon+h}\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (v,\mathbf{q})\big) = a_\varepsilon\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (v,\mathbf{q})\big) + h\,\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (v,\mathbf{q})\big)_{H^1\times H_{\mathsf{div}}} = (f, \nabla\cdot\mathbf{q})_{L^2}, \\ a_\varepsilon\big((u_\varepsilon, \mathbf{p}_\varepsilon), (v,\mathbf{q})\big) = (f, \nabla\cdot\mathbf{q})_{L^2} \end{cases}$$

for all $v \in V_0$ and $\mathbf{q} \in D_0$. Choosing $v = u_{\varepsilon+h} - u_\varepsilon$, $\mathbf{q} = \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon$, and subtracting the two equalities, it follows that

$$\begin{aligned} \|A\nabla(u_{\varepsilon+h} - u_\varepsilon) - (\mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\|_{L^2}^2 + \|\nabla\cdot(\mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\|_{L^2}^2 + \varepsilon\,\|u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon\|_{H^1\times H_{\mathsf{div}}}^2 \\ = -h\,\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}}. \end{aligned}$$

Omitting the first two terms on the left-hand side and applying the Cauchy–Schwarz inequality, we have

$$\|(u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\|_{H^1\times H_{\mathsf{div}}} \le \frac{h}{\varepsilon}\|(u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h})\|_{H^1\times H_{\mathsf{div}}} \le \frac{h}{\varepsilon}\|(u, A\nabla u)\|_{H^1\times H_{\mathsf{div}}}, \tag{15}$$

where the latter inequality follows from (10). □

Remark 4.5. Since $(u_\varepsilon, \mathbf{p}_\varepsilon)$ converges to $(u, A\nabla u)$ as ε goes to zero, we can extend F to be a continuous function on $\mathbb{R}_+ \cup \{0\}$ by setting $F(0) = (u, A\nabla u)$.

Let us then consider the following problem: For $\varepsilon > 0$, find $(u, \mathbf{p}) \in V_0 \times D_0$ such that

$$a_\varepsilon\big((u,\mathbf{p}), (v,\mathbf{q})\big) = -\big((u_\varepsilon, \mathbf{p}_\varepsilon), (v,\mathbf{q})\big)_{H^1\times H_{\mathsf{div}}} \tag{16}$$

for all $(v, \mathbf{q}) \in V_0 \times D_0$.

Proposition 4.5. *The problem* (16) *has a unique solution $(u_\varepsilon^1, \mathbf{p}_\varepsilon^1) \in V_0 \times D_0$ that satisfies*

$$\big\|(u_\varepsilon^1, \mathbf{p}_\varepsilon^1)\big\|_{H^1\times H_{\mathsf{div}}} \le \frac{1}{\varepsilon}\|(u, A\nabla u)\|_{H^1\times H_{\mathsf{div}}}. \tag{17}$$

Proof. The result follows straightforwardly from the Lax–Milgram theorem and the estimate (10). □

Proposition 4.6. *It holds that $F \in C^1(\mathbb{R}_+, H^1(\Omega)\times H_{\mathsf{div}}(\Omega))$, with the corresponding derivative defined via $F'(\varepsilon) = (u_\varepsilon^1, \mathbf{p}_\varepsilon^1)$, where $(u_\varepsilon^1, \mathbf{p}_\varepsilon^1) \in V_0 \times D_0$ is the unique solution of* (16).

Proof. For all $\varepsilon > 0$, $h \in \mathbb{R}$ such that $\varepsilon + h > 0$, and $(v, \mathbf{q}) \in V_0 \times D_0$, we have

$$\begin{aligned} a_{\varepsilon+h}\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (v,\mathbf{q})\big) &= (f, \nabla\cdot\mathbf{q})_{L^2}, \\ -a_\varepsilon\big((u_\varepsilon, \mathbf{p}_\varepsilon), (v,\mathbf{q})\big) &= -(f, \nabla\cdot\mathbf{q})_{L^2}, \\ -h\,a_\varepsilon\big((u_\varepsilon^1, \mathbf{p}_\varepsilon^1), (v,\mathbf{q})\big) &= h\,\big((u_\varepsilon, \mathbf{p}_\varepsilon), (v,\mathbf{q})\big)_{H^1\times H_{\mathsf{div}}}. \end{aligned}$$

Choosing $v = v_{\varepsilon,h} := u_{\varepsilon+h} - u_\varepsilon - h u_\varepsilon^1 \in V_0$ and $\mathbf{q} = \mathbf{q}_{\varepsilon,h} := \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon - h\mathbf{p}_\varepsilon^1 \in D_0$, and summing the three equalities, one obtains that

$$\|A\nabla v_{\varepsilon,h} - \mathbf{q}_{\varepsilon,h}\|_{L^2}^2 + \|\nabla\cdot\mathbf{q}_{\varepsilon,h}\|_{L^2}^2 + \varepsilon\|(v_{\varepsilon,h}, \mathbf{q}_{\varepsilon,h})\|_{H^1\times H_{\mathsf{div}}}^2 = -h\big((u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon), (v_{\varepsilon,h}, \mathbf{q}_{\varepsilon,h})\big)_{H^1\times H_{\mathsf{div}}},$$

and thus

$$\frac{1}{|h|}\big\|(u_{\varepsilon+h} - u_\varepsilon - hu_\varepsilon^1, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon - h\mathbf{p}_\varepsilon^1)\big\|_{H^1\times H_{\mathsf{div}}} \leq \frac{1}{\varepsilon}\|(u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\|_{H^1\times H_{\mathsf{div}}} \xrightarrow{h\to 0} 0.$$

It remains to be shown that the derivative $\varepsilon \mapsto (u_\varepsilon^1, \mathbf{p}_\varepsilon^1)$ is continuous as a map from $\mathbb{R}_+$ to $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$. For all $(v, \mathbf{q}) \in V_0 \times D_0$, it holds that

$$\begin{aligned} a_{\varepsilon+h}\big((u_{\varepsilon+h}^1, \mathbf{p}_{\varepsilon+h}^1), (v, \mathbf{q})\big) &= -\big((u_{\varepsilon+h}, \mathbf{p}_{\varepsilon+h}), (v, \mathbf{q})\big)_{H^1\times H_{\mathsf{div}}} \\ a_\varepsilon\big((u_\varepsilon^1, \mathbf{p}_\varepsilon^1), (v, \mathbf{q})\big) &= -\big((u_\varepsilon, \mathbf{p}_\varepsilon), (v, \mathbf{q})\big)_{H^1\times H_{\mathsf{div}}}. \end{aligned}$$

Therefore, choosing $v = v_{\varepsilon,h}^1 := u_{\varepsilon+h}^1 - u_\varepsilon^1 \in V_0$, $\mathbf{q} = \mathbf{q}_{\varepsilon,h}^1 := \mathbf{p}_{\varepsilon+h}^1 - \mathbf{p}_\varepsilon^1 \in D_0$, and subtracting the two equalities, we obtain

$$a_\varepsilon\big((v_{\varepsilon,h}^1, \mathbf{q}_{\varepsilon,h}^1), (v_{\varepsilon,h}^1, \mathbf{q}_{\varepsilon,h}^1)\big) = -\big((u_{\varepsilon+h} - u_\varepsilon + h\, u_{\varepsilon+h}^1, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon + h\, \mathbf{p}_{\varepsilon+h}^1), (v_{\varepsilon,h}^1, \mathbf{q}_{\varepsilon,h}^1)\big)_{H^1\times H_{\mathsf{div}}},$$

meaning, in particular, that

$$\varepsilon\big\|(v_{\varepsilon,h}^1, \mathbf{q}_{\varepsilon,h}^1)\big\|_{H^1\times H_{\mathsf{div}}} \leq \|(u_{\varepsilon+h} - u_\varepsilon, \mathbf{p}_{\varepsilon+h} - \mathbf{p}_\varepsilon)\|_{H^1\times H_{\mathsf{div}}} + h\,\big\|(u_{\varepsilon+h}^1, \mathbf{p}_{\varepsilon+h}^1)\big\|_{H^1\times H_{\mathsf{div}}}.$$

In consequence, the bounds (15) and (17) provide the estimate

$$\big\|(v_{\varepsilon,h}^1, \mathbf{q}_{\varepsilon,h}^1)\big\|_{H^1\times H_{\mathsf{div}}} \leq \frac{2\,h}{\varepsilon^2}\|(u, A\nabla u)\|_{H^1\times H_{\mathsf{div}}},$$

which completes the proof. □

For $\varepsilon > 0$, we now define a sequence of function pairs $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m) \in H^1(\Omega)\times H_{\mathsf{div}}(\Omega)$, $m \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, recursively:

- Set $(u_\varepsilon^0, \mathbf{p}_\varepsilon^0) = (u_\varepsilon, \mathbf{p}_\varepsilon)$.
- For $m \in \mathbb{N}_0$, define $(u_\varepsilon^{m+1}, \mathbf{p}_\varepsilon^{m+1})$ to be the unique element of $V_0 \times D_0$ that satisfies

$$a_\varepsilon\big((u_\varepsilon^{m+1}, \mathbf{p}_\varepsilon^{m+1}), (v, \mathbf{q})\big) = -(m+1)\big((u_\varepsilon^m, \mathbf{p}_\varepsilon^m), (v, \mathbf{q})\big)_{H^1\times H_{\mathsf{div}}} \tag{18}$$

  for all $(v, \mathbf{q}) \in V_0 \times D_0$.

As in the case of (16), the fact that the sequence $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m)$ is well defined follows from a simple application of the Lax–Milgram theorem, which also produces the estimate

$$\big\|(u_\varepsilon^m, \mathbf{p}_\varepsilon^m)\big\| \leq \frac{m!}{\varepsilon^m}\|(u, A\nabla u)\|_{H^1\times H_{\mathsf{div}}}$$

as a recursive by-product. In particular, it turns out that the sequence $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m)$, $m \in \mathbb{N}_0$, defines the derivatives of F.

Theorem 4.6. *It holds that $F \in C^\infty(\mathbb{R}_+, H^1(\Omega) \times H_{\mathsf{div}}(\Omega))$, and for all $m \in \mathbb{N}$, $F^{(m)}(\varepsilon) = (u_\varepsilon^m, \mathbf{p}_\varepsilon^m)$, where $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m) \in V_0 \times D_0$ is the (recursively defined) unique solution of* (18).

Proof. The claim follows from induction and similar arguments as those in the proof of Proposition 4.6. □

4.2.2. *Proof of monotonic convergence.* To begin with, note that the unique solvability of (4) and the linear dependence of $(u_\varepsilon^{m+1}, \mathbf{p}_\varepsilon^{m+1})$ on $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m)$, $m \in \mathbb{N}_0$, provide us with the following lemma.

Lemma 4.7. *Suppose that the (compatible) data of the Cauchy problem verifies $(f, g_D, g_N) \neq (0, 0, 0)$. Then $(u_\varepsilon^m, \mathbf{p}_\varepsilon^m) \neq (0, \mathbf{0})$ for all $m \in \mathbb{N}$ and $\varepsilon > 0$.*

Now, we are ready to formulate and prove the main result of this section.

Theorem 4.8. *The function $\mathbb{R}_+ \ni \varepsilon \mapsto \|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|_{H^1\times H_{\mathsf{div}}}$ is strictly increasing.*

Proof. Obviously, it is sufficient to prove that $g : \mathbb{R}_+ \ni \varepsilon \mapsto \frac{1}{2}\|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|^2_{H^1\times H_{\mathsf{div}}}$ is strictly increasing.

A simple computation shows that

$$g'(\varepsilon) = \big((u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u), (u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}}$$

and

$$g''(\varepsilon) = \big((u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u), (u^2_\varepsilon, \mathbf{p}^2_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}} + \big\|(u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big\|^2_{H^1\times H_{\mathsf{div}}}.$$

Because $(u^2_\varepsilon, \mathbf{p}^2_\varepsilon)$ solves (18) with $m = 1$, and $(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u) \in V_0 \times D_0$, we have

$$\begin{aligned}\varepsilon\big((u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u), (u^2_\varepsilon, \mathbf{p}^2_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}} =& -2\big((u^1_\varepsilon, \mathbf{p}^1_\varepsilon), (u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\big)_{H^1\times H_{\mathsf{div}}}\\ &- (A\nabla u^2_\varepsilon - \mathbf{p}^2_\varepsilon, A\nabla u_\varepsilon - \mathbf{p}_\varepsilon)_{L^2} - (\nabla\cdot\mathbf{p}^2_\varepsilon, \nabla\cdot\mathbf{p}_\varepsilon - f)_{L^2}.\end{aligned}$$

Since $(u_\varepsilon, \mathbf{p}_\varepsilon)$ solves (4) and $(u^2_\varepsilon, \mathbf{p}^2_\varepsilon) \in V_0 \times D_0$, we can further deduce that

$$\varepsilon\,((u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u), (u^2_\varepsilon, \mathbf{p}^2_\varepsilon))_{H^1\times H_{\mathsf{div}}} = -2\,g'(\varepsilon) + \varepsilon\,\big((u^2_\varepsilon, \mathbf{p}^2_\varepsilon), (u_\varepsilon, \mathbf{p}_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}}.$$

Altogether, we have thus far obtained that

$$\varepsilon\,g''(\varepsilon) + 2\,g'(\varepsilon) = \varepsilon\,\big\|(u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big\|^2_{H^1\times H_{\mathsf{div}}} + \varepsilon\,\big((u^2_\varepsilon, \mathbf{p}^2_\varepsilon), (u_\varepsilon, \mathbf{p}_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}}.$$

Now, utilizing the fact that $(u^1_\varepsilon, \mathbf{p}^1_\varepsilon)$ is the solution of (18) with $m = 0$, it follows that

$$\big((u^2_\varepsilon, \mathbf{p}^2_\varepsilon), (u_\varepsilon, \mathbf{p}_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}} = -a_\varepsilon\big((u^1_\varepsilon, \mathbf{p}^1_\varepsilon), (u^2_\varepsilon, \mathbf{p}^2_\varepsilon)\big).$$

Since $(u^2_\varepsilon, \mathbf{p}^2_\varepsilon)$ is in turn the solution of problem (18) with $m = 1$, we obtain

$$\big((u^2_\varepsilon, \mathbf{p}^2_\varepsilon), (u_\varepsilon, \mathbf{p}_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}} = 2\,\big\|(u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big\|^2_{H^1\times H_{\mathsf{div}}}.$$

In consequence,

$$\big[\varepsilon^2 g'(\varepsilon)\big]' = \varepsilon^2\,g''(\varepsilon) + 2\varepsilon\,g'(\varepsilon) = 3\varepsilon^2\,\big\|(u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big\|^2_{H^1\times H_{\mathsf{div}}}.$$

Because $(u^1_\varepsilon, \mathbf{p}^1_\varepsilon) \neq (0, \mathbf{0})$, it holds that $\big[\varepsilon^2 g'(\varepsilon)\big]' > 0$, i.e., $\varepsilon^2 g'(\varepsilon)$ is a strictly increasing function. Moreover,

$$|\varepsilon^2 g'(\varepsilon)| = \varepsilon^2\Big|\big((u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u), (u^1_\varepsilon, \mathbf{p}^1_\varepsilon)\big)_{H^1\times H_{\mathsf{div}}}\Big|,$$

and thus the Cauchy–Schwarz inequality and (17) imply that

$$|\varepsilon^2 g'(\varepsilon)| \le \varepsilon\|(u_\varepsilon - u, \mathbf{p}_\varepsilon - A\nabla u)\|_{H^1\times H_{\mathsf{div}}}\|(u, A\nabla u)\|_{H^1\times H_{\mathsf{div}}} \xrightarrow{\varepsilon\to 0} 0.$$

Hence, we have $\varepsilon^2 g'(\varepsilon) > 0$ for all $\varepsilon > 0$, indicating that also $g'(\varepsilon) > 0$. This completes the proof. $\square$

4.3. **Noisy data.** In this section, we focus on the important case of noisy data. It is assumed that the (reformulated) Cauchy problem of Definition 4.1 has a unique solution $(u, \mathbf{p}) \in H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$, but one has only access to the noisy data $f^\delta \in L^2(\Omega)$, $g^\delta_D \in H^{1/2}(\Gamma)$ and $g^\delta_N \in H^{-1/2}(\Gamma)$ that satisfy

$$\|f^\delta - f\|_{L^2(\Omega)} \le \delta, \qquad \|g^\delta_D - g_D\|_{H^{1/2}(\Gamma)} \le \delta, \qquad \|g^\delta_N - g_N\|_{H^{-1/2}(\Gamma)} \le \delta$$

for some $\delta > 0$. Let us denote by $(u^\delta_\varepsilon, \mathbf{p}^\delta_\varepsilon)$ the solution of the mixed QR problem (4), when the exact data (f, g_D, g_N) are replaced by their noisy counterparts $(f^\delta, g^\delta_D, g^\delta_N)$; the pair $(u_\varepsilon, \mathbf{p}_\varepsilon)$ still denotes the mixed QR solution for the noiseless data. The aim is to answer the following question: For a certain amplitude of noise $\delta > 0$, how should the regularization parameter $\varepsilon := \varepsilon(\delta)$ be chosen in order to assure the convergence of $(u^\delta_{\varepsilon(\delta)}, \mathbf{p}^\delta_{\varepsilon(\delta)})$ to $(u, \mathbf{p})$ when the amplitude of noise goes to zero.

Proposition 4.7. *It holds that*

$$\big\|(u^\delta_\varepsilon - u, \mathbf{p}^\delta_\varepsilon - \mathbf{p})\big\|_{H^1\times H_{\mathsf{div}}} \le C(\varepsilon)\frac{\delta}{\sqrt{\varepsilon}} + C'\delta + \|(u_\varepsilon - u, \mathbf{p}_\varepsilon - \mathbf{p})\|_{H^1\times H_{\mathsf{div}}}, \tag{19}$$

where

$$C(\varepsilon) := 1 + C'\sqrt{1 + 2\|A\|_\infty + \|A\|^2_\infty + \varepsilon}$$

and $C' > 0$ depends only on the geometry of Ω.

Proof. Let $(u_D, \mathbf{p}_N)$ and $(\tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon)$ be as in the proof of Proposition 4.1 and let us introduce a similar decomposition for the mixed QR solution corresponding to the noisy data:

$$u_\varepsilon^\delta = \tilde{u}_\varepsilon^\delta + u_D^\delta, \qquad \mathbf{p}_\varepsilon^\delta = \tilde{\mathbf{p}}_\varepsilon^\delta + \mathbf{p}_N^\delta, \tag{20}$$

with

$$\|u_D^\delta\|_{H^1(\Omega)} \le c\|g_D^\delta\|_{H^{1/2}(\Gamma)}, \qquad \|\mathbf{p}_N^\delta\|_{H_{\mathsf{div}}(\Omega)} \le c\|g_N^\delta\|_{H^{-1/2}(\Gamma)} \tag{21}$$

and $(\tilde{u}_\varepsilon^\delta, \tilde{\mathbf{p}}_\varepsilon^\delta) \in V_0 \times D_0$ satisfying

$$a_\varepsilon\big((\tilde{u}_\varepsilon^\delta, \tilde{\mathbf{p}}_\varepsilon^\delta), (v, \mathbf{q})\big) = \int_\Omega f^\delta (\nabla \cdot \mathbf{q})\, dx - a_\varepsilon\big((u_D^\delta, \mathbf{p}_N^\delta), (v, \mathbf{q})\big)$$

for all $(v, \mathbf{q}) \in V_0 \times D_0$. By subtracting (6), it follows that

$$a_\varepsilon\big((\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon), (v, \mathbf{q})\big) = \int_\Omega (f^\delta - f)(\nabla \cdot \mathbf{q})\, dx - a_\varepsilon\big((u_D^\delta - u_D, \mathbf{p}_N^\delta - \mathbf{p}_N), (v, \mathbf{q})\big). \tag{22}$$

Choosing $(v, \mathbf{q}) = (\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)$ in (22), we obtain

$$\big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_\varepsilon^2 = \int_\Omega (f^\delta - f)(\nabla \cdot (\tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon))\, dx - a_\varepsilon\big((u_D^\delta - u_D, \mathbf{p}_N^\delta - \mathbf{p}_N), (\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big),$$

where $\|(\cdot,\cdot)\|_\varepsilon$ denotes the ε-dependent norm on $H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ induced by the bilinear form a_ε. Due to the definition of a_ε and the assumption on the noise level, it obviously holds that

$$\int_\Omega (f^\delta - f)(\nabla \cdot (\tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon))\, dx \le \big\|f^\delta - f\big\|_{L^2(\Omega)} \big\|\nabla \cdot (\tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_{L^2(\Omega)} \le \delta \big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_\varepsilon,$$

and

$$\begin{aligned} \big\|(u_D^\delta - u_D, \mathbf{p}_N^\delta - \mathbf{p}_N)\big\|_\varepsilon^2 &\le (1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon) \big\|(u_D^\delta - u_D, \mathbf{p}_N^\delta - \mathbf{p}_N)\big\|_{H^1 \times H_{\mathsf{div}}}^2 \\ &\le 2\, c^2\, (1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon)\, \delta^2, \end{aligned} \tag{23}$$

where the last step follows from the fact that an estimate of the type (5) and (21) naturally holds also for the differences $u_D^\delta - u_D$ and $\mathbf{p}_N^\delta - \mathbf{p}_N$ by virtue of the linearity of the right inverses for the associated trace maps.

Together with the Cauchy–Schwarz inequality, the previous three formulas induce the estimate

$$\big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_\varepsilon^2 \le \delta \left(1 + \sqrt{2}\, c\, \sqrt{1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon}\right) \big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_\varepsilon\,,$$

which in turn leads to

$$\sqrt{\varepsilon}\big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_{H^1 \times H_{\mathsf{div}}} \le \big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_\varepsilon \le \delta \left(1 + \sqrt{2}\, c\, \sqrt{1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon}\right).$$

Combining this with (20) and the corresponding decomposition for $(u_\varepsilon, \mathbf{p}_\varepsilon)$, we finally have

$$\begin{aligned} \big\|(u_\varepsilon^\delta - u_\varepsilon, \mathbf{p}_\varepsilon^\delta - \mathbf{p}_\varepsilon)\big\|_{H^1 \times H_{\mathsf{div}}} &\le \big\|(\tilde{u}_\varepsilon^\delta - \tilde{u}_\varepsilon, \tilde{\mathbf{p}}_\varepsilon^\delta - \tilde{\mathbf{p}}_\varepsilon)\big\|_{H^1 \times H_{\mathsf{div}}} + \big\|(u_D^\delta - u_D, \mathbf{p}_N^\delta - \mathbf{p}_N)\big\|_{H^1 \times H_{\mathsf{div}}} \\ &\le \frac{\delta}{\sqrt{\varepsilon}} \left(1 + \sqrt{2}\, c\, \sqrt{1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon}\right) + \sqrt{2}\, c\, \delta, \end{aligned}$$

where the second term on the right-hand side of the first inequality is estimated as in (23). The claim now follows from the triangle inequality. □

The following corollary is an immediate consequence of Proposition 4.7 and Theorem 4.3.

Corollary 4.9. *For any choice of the regularization parameter* $\varepsilon = \varepsilon(\delta)$ *such that*

$$\lim_{\delta \to 0} \varepsilon(\delta) = \lim_{\delta \to 0} \frac{\delta^2}{\varepsilon(\delta)} = 0, \tag{24}$$

it holds that

$$\big(u_{\varepsilon(\delta)}^\delta, \mathbf{p}_{\varepsilon(\delta)}^\delta\big) \to (u, \mathbf{p}) \qquad \text{in } H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$$

as δ *goes to zero.*

According to Corollary 4.9, any choice of $\varepsilon = \varepsilon(\delta)$ that satisfies (24) is admissible. For example, one could set $\varepsilon(\delta) := \delta^\alpha$ with any $\alpha \in (0, 2)$. However, for a fixed noise level, there naturally exist choices of regularization parameter that are better (in some sense) than others. The task is then to choose the *optimal* parameter — and also the considered optimality criterion. In the following, we will briefly describe two methods that could be used to set the regularization parameter.

In [15], a method based on a *balancing principle* is proposed to choose the regularization parameter for the standard QR method described in Section 3.1. If translated to the setting of the new mixed QR method and Proposition 4.7, the idea of [15] can be explained as follows: The right-hand side of the error estimate (19) is composed of two terms, namely $f_1(\varepsilon) = C(\varepsilon)\,\delta/\sqrt{\varepsilon} + C'\delta$ and $f_2(\varepsilon) = \|(u_\varepsilon - u, \mathbf{p}_\varepsilon - \mathbf{p})\|_{H^1\times H_{\mathsf{div}}}$, which have opposite behaviors with respect to ε. Indeed,

$$\lim_{\varepsilon\to 0} f_1(\varepsilon) = \infty, \qquad \lim_{\varepsilon\to \infty} f_1(\varepsilon) = 2C'\delta$$

and

$$\lim_{\varepsilon\to 0} f_2(\varepsilon) = 0, \qquad \lim_{\varepsilon\to \infty} f_2(\varepsilon) = \|(u, \mathbf{p})\|_{H^1\times H_{\mathsf{div}}},$$

where the first limits follow from Proposition 4.7 and the second ones straightforwardly from Theorem 4.3 and the estimates in the proof of Proposition 4.1. Therefore, if $\delta > 0$ is sufficiently small, it is clear that there exists a unique $\varepsilon_{\text{opt}} > 0$ such that $f_1(\varepsilon_{\text{opt}}) = f_2(\varepsilon_{\text{opt}})$ by virtue of Theorem 4.8 and the monotonicity of f_1. In a sense, such a parameter choice balances the error due to the noise in the data and the discrepancy between the exact and the regularized solution for the noiseless case. Unfortunately, ε_{opt} cannot be determined directly since f_2 is unknown. The authors of [15] use Carleman inequalities to obtain an estimate for f_2 *inside* the domain, i.e., far from the complementary boundary Γ_c where the boundary conditions are unknown. They are then in position to propose an iterative algorithm that produces $\tilde{\varepsilon}_{\text{opt}} \approx \varepsilon_{\text{opt}}$, even without knowing explicitly the constants in the estimate (19). It appears that the balancing principle of [15] could be applied to our new mixed QR formulation by taking advantage of the monotonic convergence guaranteed by Theorem 4.8, but such considerations are left for future studies.

Another technique for finding an *optimal* regularization parameter is proposed in [11]. The method is based on the well-known *Morozov discrepancy principle*, which states that $\varepsilon_{\text{opt}} > 0$ should be the (largest) regularization parameter for which the discrepancy in the data fit equals the noise level. The authors of [11] introduce an algorithm based on duality in optimization for computing such ε_{opt} exactly (up to numerical errors). The method has been tested successfully with the standard QR formulation. However, a closer look at [11] demonstrates that this dual optimization algorithm could also be adapted without difficulty to our mixed QR setting, leading to a method for automatically obtaining the regularization parameter that satisfies the Morozov discrepancy principle. The exact formulation of the algorithm, together with a proof of convergence and numerical tests, is left for future studies.

4.4. **Discretization.** The mixed QR method of Definition 4.2 can obviously be discretized with conforming finite elements. Indeed, the considered Sobolev spaces, namely $H^1(\Omega)$ and $H_{\mathsf{div}}(\Omega)$, are standard spaces appearing, e.g., in mathematical analysis of fluid mechanics and electromagnetism. Consequently, their finite element discretization has been studied extensively; see, e.g., [14, 16] and the references therein.

In this section, we assume that Ω is a two-dimensional polygonal domain (resp. a three-dimensional polyhedral domain). We define $\mathcal{T}_h$ to be a regular triangulation of $\overline{\Omega}$ in the sense of [16], such that the diameter of each triangle (resp. each tetrahedron) is bounded by $h > 0$. We assume that $\overline{\Gamma}$ is the union of edges (resp. faces) of some triangles (resp. tetrahedra) of $\mathcal{T}_h$. For $k \in \mathbb{N}$ and a triangle/tetrahedron $\mathcal{K}$ of $\mathcal{T}_h$, we define $P_k(\mathcal{K})$ to be the space of polynomial functions of degree lower or equal to k in $\mathcal{K}$. The standard Lagrange finite element space L_h^k is then defined to be the set of functions $v_h \in C^0(\overline{\Omega})$ such that $v_{h|\mathcal{K}} \in P_k(\mathcal{K})$ for any $\mathcal{K}$ of $\mathcal{T}_h$. It is well-known that $L_h^k \subset H^1(\Omega)$, and thus we can use Lagrange finite elements to approximate V and V_0.

To be more precise, we assume that $g_D \in H^{1/2}(\Gamma) \cap C^0(\overline{\Gamma})$, and define $g_{D,h}$ to be its interpolant over the traces of L_h^k-functions on Γ with some fixed $k \in \mathbb{N}$; here and in what follows, we define an interpolant as the element of some (context-dependent) finite element subspace with the same degrees of freedom as the function that is interpolated. We then set

$$V_h^k := \left\{ v_h \in L_h^k \ : \ v_h = g_{D,h} \text{ on } \Gamma \right\}, \qquad V_{0,h}^k := \left\{ v_h \in L_h^k \ : \ v_h = 0 \text{ on } \Gamma \right\} \subset V_0.$$

By assumption V_h^k is non-empty, as is $V_{0,h}^k$.

In order to obtain conforming approximations of D and D_0, which are subsets of $H_{\mathsf{div}}(\Omega)$, we use the well-known Raviart–Thomas RT_h^k finite elements (cf., e.g., [14]), which are defined as follows:

$$RT_h^k := \left\{ \mathbf{q}_h \in H_{\mathsf{div}}(\Omega) \ : \ \mathbf{q}_{h|\mathcal{K}} \in P_k(\mathcal{K})^d + xP_k(\mathcal{K}) \text{ for all } \mathcal{K} \in \mathcal{T}_h \right\}, \qquad k \in \mathbb{N}_0,$$

with $x \in \mathbb{R}^d$ being the spatial variable. If the Neumann data g_N is assumed to be in $L^2(\Gamma)$, we can introduce its interpolant $g_{N,h}$ over the space of the normal components on Γ of vector fields in RT_h^k, and then define

$$D_h^k := \left\{ \mathbf{q}_h \in RT_h^k \ : \ \mathbf{q}_h \cdot \nu = g_{N,h} \text{ on } \Gamma \right\},$$
$$D_{0,h}^k := \left\{ \mathbf{q}_h \in RT_h^k \ : \ \mathbf{q}_h \cdot \nu = 0 \text{ on } \Gamma \right\} \subset D_0.$$

Again, $D_h^k \neq \emptyset \neq D_{0,h}^k$.

With the help of these finite element spaces, we can now define the discretized version of the mixed QR problem.

Definition 4.10 (Mixed QR$_h$ problem)**.** For $\varepsilon > 0$ and some $k \in \mathbb{N}$, find $(u_h, \mathbf{p}_h) \in V_h^k \times D_h^{k-1}$ such that for all $(v, \mathbf{q}) \in V_{0,h}^k \times D_{0,h}^{k-1}$

(25)

$$\begin{cases} \displaystyle\int_\Omega (A\nabla u_h - \mathbf{p}_h) \cdot A\nabla v \, dx + \varepsilon \int_\Omega (\nabla u_h \cdot \nabla v + u_h \, v) \, dx = 0, \\ \displaystyle -\int_\Omega (A\nabla u_h - \mathbf{p}_h) \cdot \mathbf{q} \, dx + \int_\Omega (\nabla \cdot \mathbf{p}_h)(\nabla \cdot \mathbf{q}) \, dx + \varepsilon \int_\Omega \big((\nabla \cdot \mathbf{p}_h)(\nabla \cdot \mathbf{q}) + \mathbf{p}_h \cdot \mathbf{q}\big) dx = \int_\Omega f(\nabla \cdot \mathbf{q}) \, dx. \end{cases}$$

The unique solvability of (25) follows in exactly the same way as that of (4).

Proposition 4.8. *For all $\varepsilon > 0$, the mixed QR$_h$ problem of Definition 4.10 admits a unique solution $(u_{\varepsilon,h}, \mathbf{p}_{\varepsilon,h}) \in V_h^k \times D_h^{k-1}$.*

The main theorem of this section provides an estimate for the convergence of $(u_{\varepsilon,h}, \mathbf{p}_{\varepsilon,h})$ towards $(u_\varepsilon, \mathbf{p}_\varepsilon)$.

Theorem 4.11. *Suppose that the solution of (4), i.e., $(u_\varepsilon, \mathbf{p}_\varepsilon)$, belongs to $(H^{k+1}(\Omega) \cap C^0(\overline{\Omega})) \times H^{k+1}(\Omega)^d$. Then,*

$$\|(u_\varepsilon - u_{\varepsilon,h}, \mathbf{p}_\varepsilon - \mathbf{p}_{\varepsilon,h})\|_{H^1 \times H_{\mathsf{div}}} \le c\, h^k \sqrt{\frac{1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon}{\varepsilon}} \sqrt{\|u_\varepsilon\|_{H^{k+1}(\Omega)}^2 + \|\mathbf{p}_\varepsilon\|_{H^{k+1}(\Omega)^d}^2},$$

where $c > 0$ is independent of $h > 0$ and $\varepsilon > 0$.

Proof. It follows directly from the Cea's lemma [16] and the estimates in the proof of Proposition 4.1 that

$$\|(u_\varepsilon - u_{\varepsilon,h}, \mathbf{p}_\varepsilon - \mathbf{p}_{\varepsilon,h})\|_{H^1 \times H_{\mathsf{div}}} \le C(\varepsilon) \inf_{(v,\mathbf{q}) \in V_h^k \times D_h^{k-1}} \|(u_\varepsilon - v, \mathbf{p}_\varepsilon - \mathbf{q})\|_{H^1 \times H_{\mathsf{div}}},$$

where

$$C(\varepsilon) = \sqrt{\frac{1 + 2\|A\|_\infty + \|A\|_\infty^2 + \varepsilon}{\varepsilon}}.$$

As $u_\varepsilon \in H^{k+1}(\Omega) \cap C^0(\overline{\Omega})$, we can define its interpolant $\tilde{u}_{\varepsilon,h}$ in L_h^k. By definition, the trace of $\tilde{u}_{\varepsilon,h}$ on Γ is equal to $g_{D,h}$, making it an element of V_h^k. Furthermore, it holds that [16]

$$\|u_\varepsilon - \tilde{u}_{\varepsilon,h}\|_{H^1(\Omega)} \le ch^k |u_\varepsilon|_{H^{k+1}(\Omega)},$$

with a constant $c > 0$ that is independent of h and ε. Accordingly, if $\mathbf{p}_\varepsilon$ is an element of $H^{k+1}(\Omega)^d$, we can define its interpolant $\tilde{\mathbf{p}}_{\varepsilon,h}$ in RT_h^{k-1}, which by definition is an element of D_h^{k-1}, and we have [14]

$$\|\mathbf{p}_\varepsilon - \tilde{\mathbf{p}}_{\varepsilon,h}\|_{H_{\mathsf{div}}(\Omega)} \le ch^k \|\mathbf{p}_\varepsilon\|_{H^{k+1}(\Omega)},$$

where $c > 0$ is again independent of h and ε. The assertion now follows by combining the above three estimates. □

Remark 4.12. Theorem 4.11 concretizes the well-known fact that *the choice of the regularization parameter $\varepsilon > 0$ cannot be independent of the mesh size $h > 0$*. More precisely, it is useless to choose a very small ε if the size of the mesh h is not small as well.

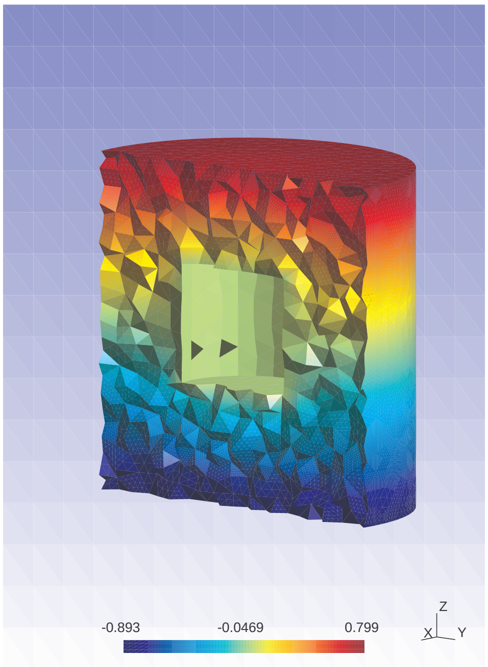

FIGURE 1. Left: The target solution of (26).

5. NUMERICAL EXPERIMENTS

In this section, we test the new mixed QR method numerically. We start with the Cauchy problem and subsequently move on to the inverse obstacle problem for the Laplacian. All forward solutions are computed with FreeFem++ [27] coupled with the mesh generator Gmsh [21].

5.1. **Cauchy problem.** Let us denote by $\mathcal{D}(x,r)$ the open disk of radius $r>0$ centered at $x\in\mathbb{R}^2$. For the Cauchy problem, we choose $\Omega=\mathcal{C}_g\setminus\overline{\mathcal{C}}_s\subset\mathbb{R}^3$, i.e., the body of interest is the great cylinder $\mathcal{C}_g:=\mathcal{D}(0,1)\times]0,2[$ without the small cylinder $\mathcal{C}_s:=\mathcal{D}(0,0.4)\times]0.7,1.4[$. In all examples, Γ is a subset of $\partial\mathcal{C}_g$, for different parts of which we introduce the following shorthand notations:

$$\Gamma_l=\partial\mathcal{D}(0,1)\times]0,2[,\quad \Gamma_t=\mathcal{D}(0,1)\times\{2\},\quad \Gamma_b=\mathcal{D}(0,1)\times\{0\}\,.$$

In the first experiment, we set $A\equiv I$, i.e., consider the Cauchy problem for the Laplacian. The data are simulated by solving the boundary value problem

$$\begin{cases}\Delta u=0 & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\nu}=w & \text{on }\partial\mathcal{C}_g,\\ u=0 & \text{on }\partial\mathcal{C}_s,\end{cases}\tag{26}$$

where $w\equiv 1,-1$ and 0 on Γ_t, Γ_b and Γ_l, respectively, using the FEM with L^2_h Lagrange elements. The corresponding solution is visualized in Figure 1.

Let us first consider the case of exact data and $\Gamma=\partial\mathcal{C}_g$. In other words, the Dirichlet and Neumann traces of the FEM solution to (26) on the whole exterior boundary of Ω are used directly as the latter two components of the data $(0,g_D,g_D)$ for the new mixed QR method introduced in Section 4. We choose the value $\varepsilon=10^{-4}$ for the regularization parameter. In order to avoid an inverse crime, we use different meshes to solve the direct and inverse problems. The discrepancy between the first component of the solution to (25) and the FEM solution of (26) is visualized in Figure 2, both inside Ω and on the lateral boundary of the void $\mathcal{C}_s$. Apparently, the method works as desired. Here and in all the following numerical tests, we have used $k=1$ for the discretized problem of Definition 4.10.

In practise, the measurements always contain uncertainties, and thus it is essential to also test the new mixed QR method with noisy data. In addition, we make the Cauchy problem more demanding by using data only on $\Gamma=\Gamma_t\cup\Gamma_p$, i.e., only at the top and the bottom of Ω. The exact data $(0,g_D,g_N)$

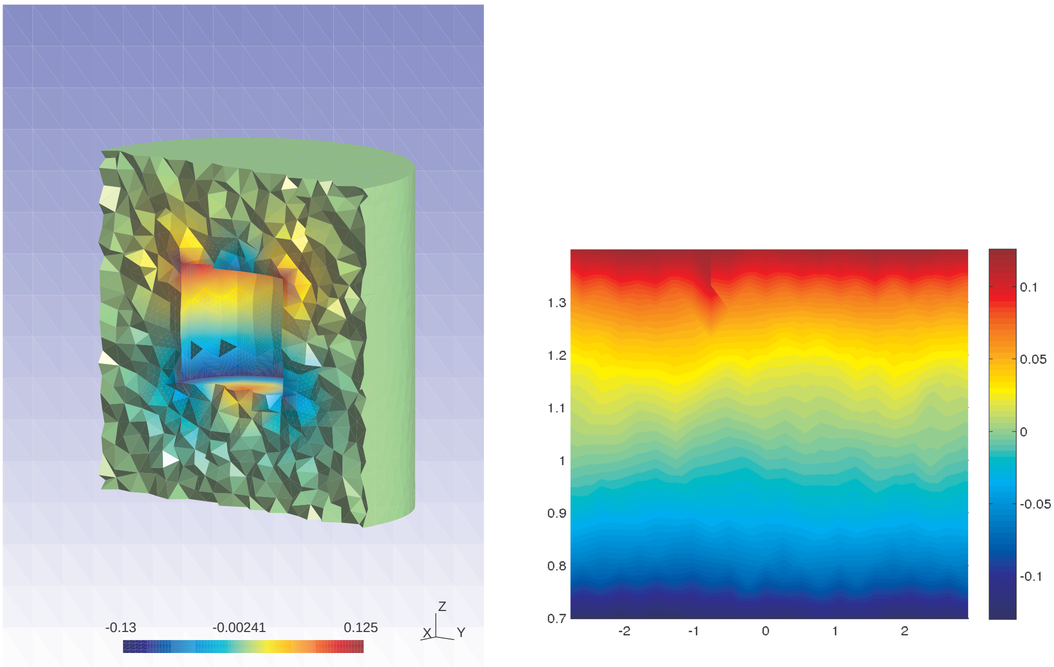

FIGURE 2. The discrepancy $u_\varepsilon - u$ in Ω (left) and on the lateral boundary of $\mathcal{C}_s$ (right) for $\Gamma = \partial\mathcal{C}_g$ and the Laplace operator. Relative $L^2(\Omega)$-error: 3.6%.

are replaced by $(0, g_D^\delta, g_N^\delta)$ such that

$$\|g_D - g_D^\delta\|_{L^2(\Gamma)} = 0.05\,\|u\|_{L^2(\Gamma)}, \quad \|g_N - g_N^\delta\|_{L^2(\Gamma)} = 0.05\,\|w\|_{L^2(\Gamma)} \tag{27}$$

due to introduction of additive noise (defined in a suitable way). We still use $\varepsilon = 10^{-4}$ as the regularization parameter. The corresponding discrepancy between the regularized solution provided by the new mixed QR method and the target solution of (26) is shown in Figure 3. Obviously, the reduction in the amount of data and the addition of noise have increased the discrepancy, but the proposed QR method still seems to function relatively well.

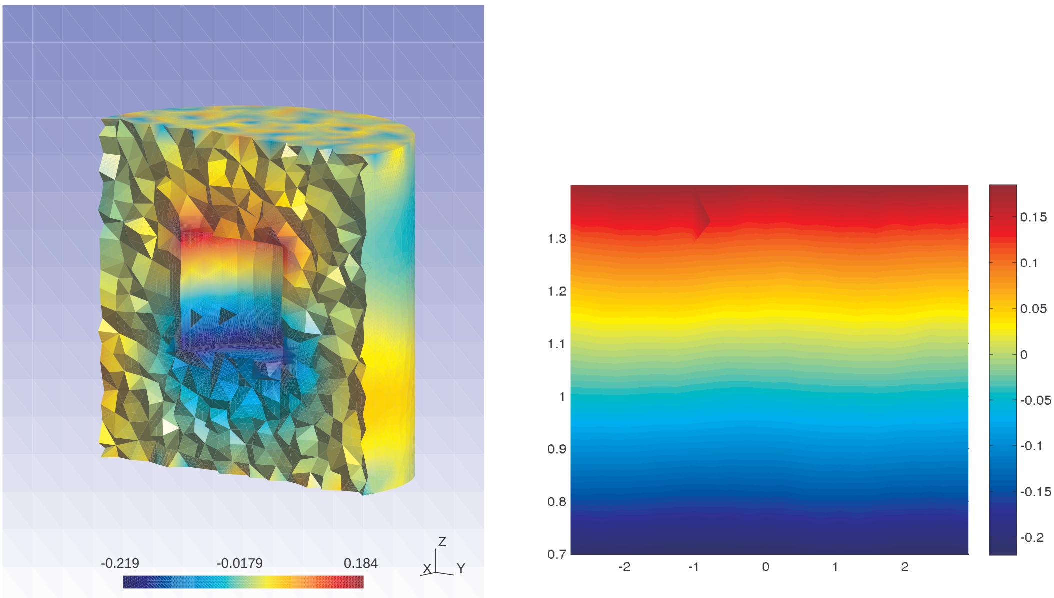

FIGURE 3. The discrepancy $u_\varepsilon - u$ in Ω (left) and on the lateral boundary of $\mathcal{C}_s$ (right) for $\Gamma = \Gamma_t \cup \Gamma_b$, the Laplace operator and 5% of noise in the data. Relative $L^2(\Omega)$-error: 9.0%.

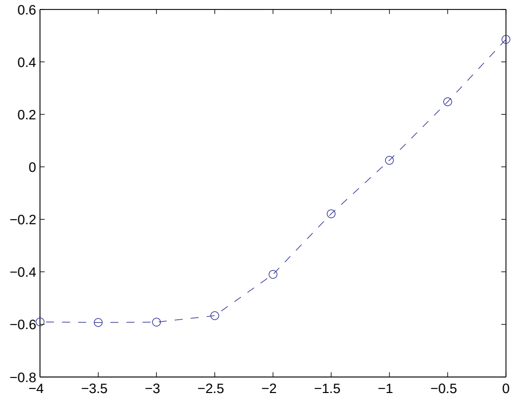

FIGURE 4. The logarithmic discrepancy (28) as a function of $\log \varepsilon$ for $u = x^2 + y^2 - 2\,z^2$, $\Gamma = \partial \mathcal{C}_g$ and $h \approx 0.05$.

We continue to consider the same geometrical setting and the Laplacian, but next choose $u = x^2 + y^2 - 2\,z^2$ to be the exact solution of the Cauchy problem. In other words, the explicit expressions for the Dirichlet and Neumann traces of this potential are employed to compute the latter two components of the (noiseless) Cauchy data $(0, g_D, g_N)$ on $\Gamma = \partial \mathcal{C}_g$; note that once again the first component of the data triplet vanishes as the chosen target function satisfies the Laplace equation. Figure 4 shows the logarithmic discrepancy

$$\log\left(\|(u_{\varepsilon,h} - u, \mathbf{p}_{\varepsilon,h} - \nabla u)\|_{H^1(\Omega)\times H_{\mathrm{div}}(\Omega)}\right) \tag{28}$$

between the chosen target potential and the solution of the mixed QR_h problem of Definition 4.10, with $h \approx 0.05$, as a function of $\log \varepsilon$. As predicted by Theorem 4.8 for the exact QR solution of (4), for fairly large values of the regularization parameter the discrepancy decreases monotonically when ε gets smaller. However, for small $\varepsilon > 0$ the error between the solutions of (4) and (25) seems to dominate (cf. Theorem 4.11), and the convergence stalls. It is to be expected that for a finer discretization, i.e., for a smaller mesh parameter $h > 0$, the monotonic convergence would continue for even smaller regularization parameters $\varepsilon > 0$.

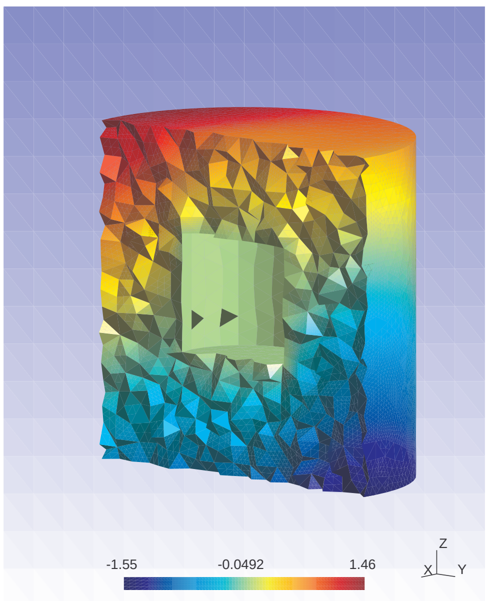

FIGURE 5. The target solution of (26) with I replaced by A of (29).

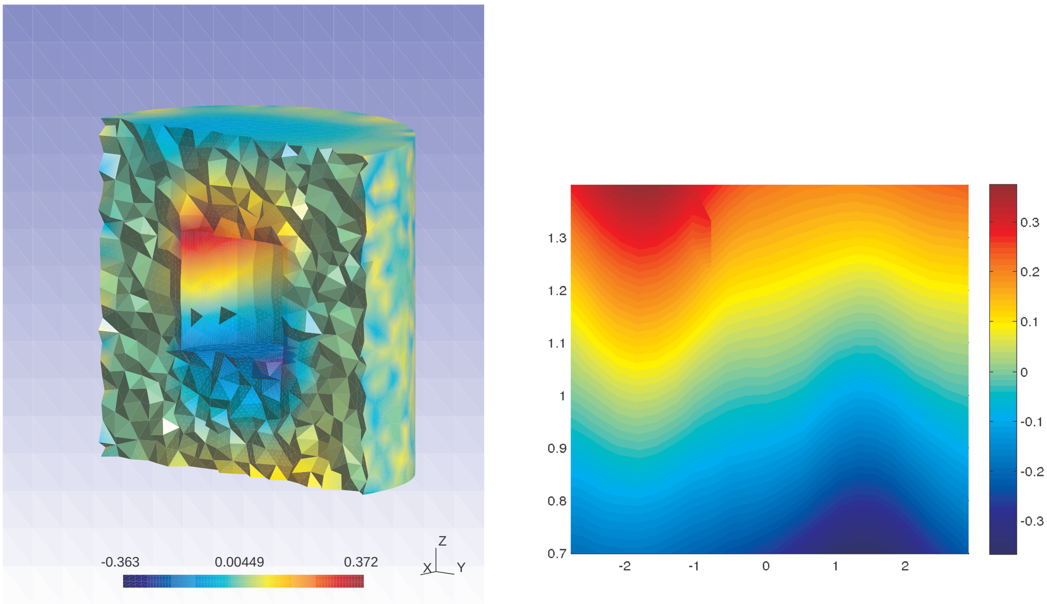

FIGURE 6. The discrepancy $u_\varepsilon - u$ in Ω (left) and on the lateral boundary of $\mathcal{C}_s$ (right) for $\Gamma = \Gamma_l$, the diffusion matrix from (29) and 5% of noise in the data. Relative $L^2(\Omega)$-error: 8.7%.

In our final numerical experiment with the Cauchy problem, we assume that the object of interest $\Omega = \mathcal{C}_g \setminus \overline{\mathcal{C}}_s$ is anisotropic. To be more precise, we choose the 'diffusion matrix' in the Cauchy problem of Definition 2.1 to be

$$A \equiv \begin{bmatrix} 1 & -0.0607 & 0.1344 \\ -0.0607 & 0.2 & 0.1051 \\ 0.1344 & 0.1051 & 1 \end{bmatrix}, \tag{29}$$

which is a randomly picked symmetric and positive definite matrix with the eigenvalues 0.179, 0.8855, and 1.1355. The latter two components of the data triplet $(0, g_D, g_N)$ on $\Gamma = \Gamma_l$ are simulated by solving the boundary value problem (26) with the Laplacian and the standard normal derivative replaced by $\nabla \cdot A\nabla$ and the conormal derivative $\partial u/\partial \nu_A$ from (1), respectively. Subsequently, 5% of noise is added to the data in the sense of (27). The corresponding target potential is shown in Figure 5.

The discrepancy between the target potential and the QR_h solution of (25), corresponding to $\varepsilon = 10^{-4}$ and the Cauchy data with 5% of noise on $\Gamma = \Gamma_l$, is visualized in Figure 6. Even in this noisy anisotropic case, the difference between the target solution and the one provided by our mixed QR method is of an order of magnitude smaller than the values of the target potential itself.

5.2. **Inverse obstacle problem.** Let us next focus on the Dirichlet obstacle problem for the Laplace equation. Suppose that we have access to the Cauchy data $(g_D, g_N) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ on $\partial\Omega$.[1] The problem we are interested in consists of finding an open set $\mathcal{O} \subset \Omega$ with a continuous boundary such that $\Omega \setminus \overline{\mathcal{O}}$ is connected and the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{\mathcal{O}}, \\ u = g_D & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = g_N & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\mathcal{O} \end{cases} \tag{30}$$

admits a solution $u \in H^1(\Omega \setminus \overline{\mathcal{O}}) \cap C^0(\Omega \setminus \overline{\mathcal{O}})$. It is well-known that this inverse obstacle problem is severely ill-posed but has at most one solution.

To solve the above introduced problem numerically in a regularized manner, we resort to the *exterior approach* introduced in [11, 12]. Here, we only outline a slightly simplified version of the reconstruction method for finding $\mathcal{O}$:

[1]In fact, we could deal with partial Cauchy data, i.e., data defined only on a part of $\partial\Omega$, in exactly the same way.

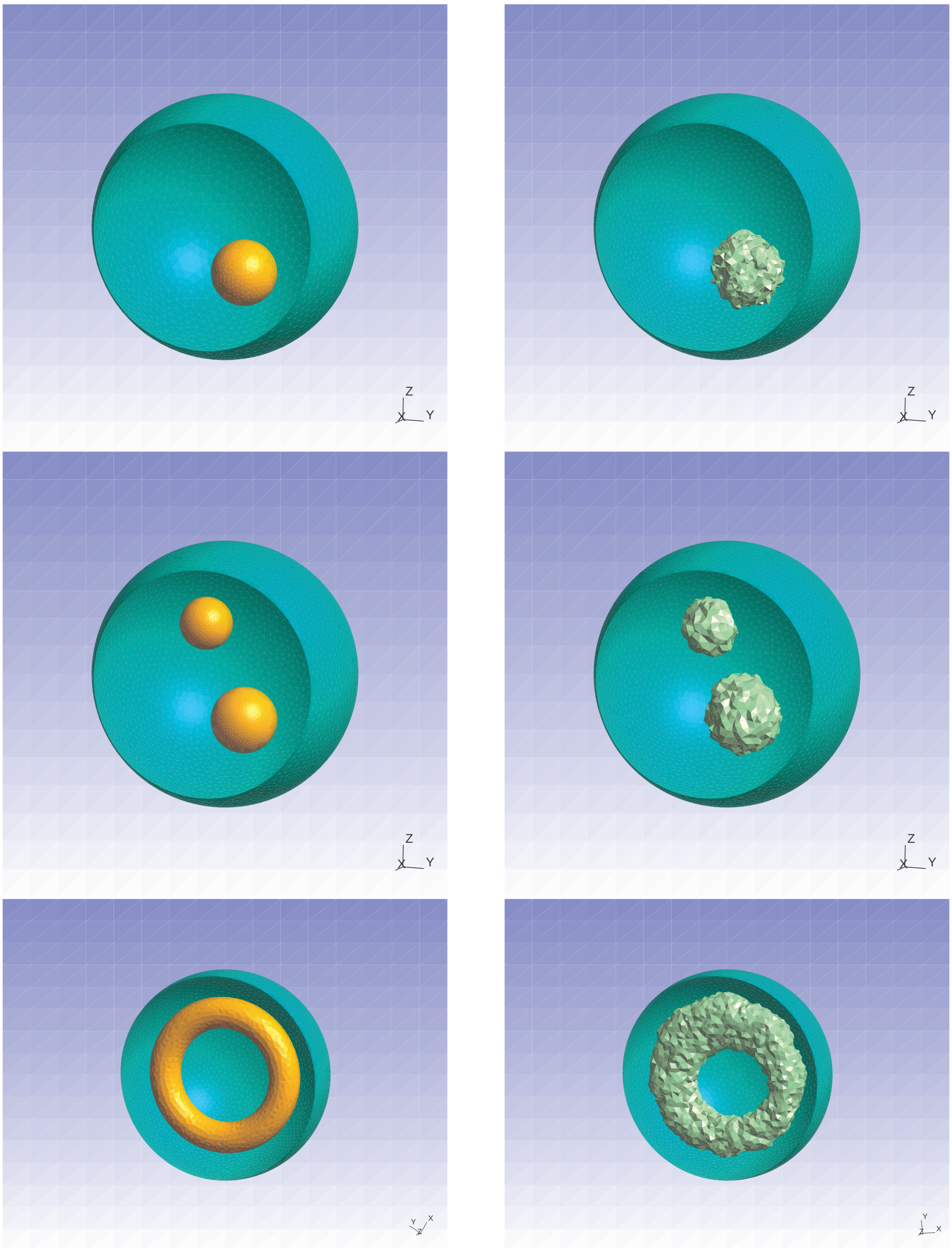

FIGURE 7. Performance of the exterior approach algorithm for three targets. Left: the obstacle used in data simulation. Right: the corresponding reconstruction.

I Initialization: Choose an initial guess ω_0 such that $\mathcal{O} \subset \omega_0 \subset \Omega$. Set $m = 0$.

II Iteration:

(1) Reconstruct a potential u in $\Omega \setminus \overline{\omega}_m$ from the Cauchy data available on $\partial\Omega$, i.e., solve the Cauchy problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \overline{\omega}_m, \\ u = g_D & \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = g_N & \text{on } \partial\Omega, \end{cases} \tag{31}$$

with your method of choice.

(2) For a sufficiently large constant C, solve the following Poisson problem in ω_m:

$$\begin{cases} \Delta v_m = C & \text{in } \omega_m, \\ v_m = |u| & \text{on } \partial\omega_m \end{cases} \tag{32}$$

using, e.g., some FEM.

(3) Define

$$\omega_{m+1} := \{x \in \omega_m \ : \ v_m(x) < 0\} .$$

(4) If the chosen stopping criterion is satisfied, terminate the iteration. Otherwise, increase m by one and return to step (1).

This algorithm builds a sequence of open sets $(\omega_m)_{m\in\mathbb{N}}$ verifying $\mathcal{O} \subset \omega_m$, and it is proved in [12, 18] that this sequence converges to the searched obstacle if, e.g., the boundaries of the open sets are smooth enough. For more information on the details of the exterior approach, such as the stopping criterion or the choice of the free parameters, we refer to [11, 12, 18].

In our implementation of the exterior approach, we use the mixed QR method introduced in Section 4 for solving (31). More precisely, we set $\varepsilon = 10^{-4}$ and utilize the combination of first order Lagrange and zeroth order Raviart–Thomas elements, i.e., we choose $k = 1$ in Definition 4.10. The Poisson problem (32) is solved by a FEM employing first order Lagrange elements.

Let Ω be the open unit ball; the initial guess ω_0 is chosen to be a smaller concentric ball of radius 0.8 in all our tests. Figure 7 presents the reconstructions produced by the above outlined exterior approach algorithm for three different target obstacles $\mathcal{O}$: a small ball, a union of two balls, and a torus. The corresponding Cauchy data were simulated by solving (with first order Lagrange elements) the well-posed boundary value problem obtained by deleting the second equation of (30), setting $g_N \equiv 1$, and choosing $\mathcal{O}$ accordingly. Afterwards, g_D was defined to be the Dirichlet trace of the corresponding FEM solution.

In each of the three cases, the qualitative shape of the obstacle is reproduced accurately; in particular, the homotopy classes of the obstacle and the corresponding reconstruction are the same in all three tests. Here, we considered only exact Cauchy data (not accounting for numerical inaccuracies), but according to our experience the performance of the method with noisy data is comparable to what is presented in [11, 12, 18] for the corresponding two-dimensional setting.

6. Concluding remarks

We have introduced a novel mixed QR method for regularizing the ill-posed Cauchy problem of Definition 2.1 without resorting to optimization schemes. Our method provides an approximate solution that converges monotonically to the exact one, if such exists; the technique for proving the monotonicity property in Section 4.2 can also be applied to the original QR method of [30]. Like the first mixed QR method in [9], our new formulation can be discretized using standard finite elements that are often available in numerical solvers. Furthermore, both components of the solution $(u_\varepsilon, \mathbf{p}_\varepsilon) \in H^1(\Omega) \times H_{\mathsf{div}}(\Omega)$ to the new mixed QR problem (4) provide information about the solution u of the original Cauchy problem: while u_ε approximates u, the vector field $\mathbf{p}_\varepsilon$ gives an estimate for the corresponding flux $A\nabla u$. The functionality of our new method was demonstrated via three-dimensional numerical studies considering both the Cauchy problem and a related inverse obstacle problem.

In our numerical experiments the choice of the regularization parameter $\varepsilon > 0$ was not considered in detail, but its value was just picked so that the resulting reconstructions appeared reasonable ($\varepsilon = 10^{-4}$ in most tests). It should be noted, however, that the new QR method seems surprisingly insensitive to the size of a smallish ε. Moreover, the method of [11] for choosing the regularization parameter as a function of the noise amplitude for the standard QR formulation can be straightforwardly adapted for our new method; see Section 4.3 for more information. As a consequence, it seems that the proposed mixed QR method can be coupled with a systematic technique for choosing the regularization parameter, assuming that there exists accurate enough information on the amount of measurement noise. Further considerations of this matter are left for future studies.

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