The most accurate model for real-life electrical impedance tomography is the complete electrode model, which takes into account electrode shapes and (usually unknown) contact impedances at electrode-object interfaces. When the electrodes are small, however, it is tempting to formally replace them by point sources. This simplifies the model considerably and completely eliminates the effect of contact impedance.

In this work we rigorously justify such a point electrode model for the important case of having difference measurements (“relative data”) as data for the reconstruction problem. We do this by deriving the asymptotic limit of the complete model for vanishing electrode size. This is supplemented by an analogous result for the case that the distance between two adjacent electrodes also tends to zero, thus providing a physical interpretation and justification of the so-called backscattering data introduced by two of the authors.

Keywords: Electric impedance tomography; complete electrode model; point electrode model; localized current sources; elliptic boundary value problems.

AMS Subject Classification: 35Q60, 65N15, 35J25

1. Introduction

The aim of electrical impedance tomography is to produce images of the admittance within an electrically conducting object (such as the human body) from
boundary measurements of current and voltage, cf. the overview articles of Barber and Brown,\textsuperscript{1} Cheney, Isaacson and Newell,\textsuperscript{5} Borcea,\textsuperscript{3,4} Lionheart,\textsuperscript{15} Bayford,\textsuperscript{2} and the book edited by Holder.\textsuperscript{11} To alleviate modelling errors and measurement noise, many practical applications of impedance tomography utilize difference measurements (sometimes called “relative data”): For a given set of boundary current patterns, the measured voltages are compared with a set of reference potentials to generate an image of the corresponding admittance change inside the object. Common examples are time-difference and frequency-difference measurements.

The most accurate mathematical (forward) model for impedance tomography is known as the complete electrode model. This model takes into account both the shunting effect on the conducting electrodes and the contact impedance between the electrodes and the imaged object. It has been experimentally verified to be capable of predicting real-life measurements up to measurement precision, cf. Cheng, Isaacson, Newell and Gisser,\textsuperscript{6} and Somersalo, Cheney and Isaacson.\textsuperscript{20}

In many practical applications, the size of the electrodes seems negligibly small compared to the total boundary area and to the inevitable modelling errors, such as inaccurate positioning of the electrodes; of the many possible examples, consider, e.g., the geophysical applications of impedance tomography in Refs. 17, 18 and 19. It is therefore tempting to formally replace small electrodes by point electrodes modelled by delta distributions. For difference data this has the additional effect of eliminating the (usually unknown) contact impedances.

In this work we will give a mathematically rigorous justification for using this kind of point electrode model by deriving it as an asymptotic limit of the complete electrode model when the electrodes’ diameter $h$ tends to zero. More precisely, we will show that the relative approximation error decays like $h^2$, if the electrodes are replaced by point sources located at their centers. The precise formulation of this main contribution of our paper is given in Sec. 2.4 below. It is supplemented by an analogous result for the case that the diameter of the electrodes, i.e., $h$, is as small as the distance between two adjacent electrodes: We prove that in the limit $h \rightarrow 0$ the corresponding real-world measurements converge to the so-called backscattering data\textsuperscript{8,9} introduced by two of the authors before. For backscatter data, however, we can only prove a convergence rate $O(h)$ as $h \rightarrow 0$.

For completeness, it should be mentioned that the connection between the complete electrode model and the so-called continuum model of impedance tomography (see, e.g., Ref. 3) has previously been studied in Refs. 12, 13 and 14. However, the philosophy of these articles differs from the approach of this work: In Refs. 12, 13 and 14 it has been investigated in what sense the current-to-voltage map of the complete electrode model approximates the Neumann-to-Dirichlet boundary operator of the continuum model as the electrodes get smaller, their number increases, and their coverage of the object boundary is getting better and better. In the present work, the locations and the number of the electrodes are fixed and the only thing that is altered is the electrode size.

The outline of this work is as follows. In Sec. 2 we give the precise mathe-
mathematical specifications of the two relevant electrode models, and comment on their well-posedness for relative data; in a separate subsection we summarize all the geometrical assumptions on the finite size electrodes that we are going to impose when we let their diameter $h$ go to zero. Afterwards, in Sec. 2.4, we present the main result of this work, the proof of which is postponed to Sec. 3. Finally, in Sec. 4 we provide our asymptotic result for backscatter data.

2. The Setting and Main Result

In what follows, we assume that $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, is a bounded domain (i.e., an open and connected set) with $C^\infty$-boundary and connected complement. The outer unit normal of $\partial \Omega$ is denoted by $\nu$. Throughout, let $\sigma_0 \in C^\infty(\overline{\Omega}; \mathbb{C}^{n \times n})$ be a smooth and (real) symmetric background admittance. We assume that the true admittance inside $\Omega$ is a compactly supported perturbation of $\sigma_0$, i.e.,

$$\sigma = \sigma_0 + \kappa,$$

where $\kappa \in L^\infty(\Omega; \mathbb{C}^{n \times n})$ is (real) symmetric and supported away from the boundary $\partial \Omega$. Furthermore, both $\sigma$ and $\sigma_0$ are assumed to satisfy (see, e.g., Ref. 3)

$$\text{Re}(\sigma \xi \cdot \overline{\xi}) \geq c \|\xi\|_{\mathbb{C}^n}^2, \quad |\sigma \xi \cdot \overline{\xi}| \leq C \|\xi\|_{\mathbb{C}^n}^2, \quad c, C > 0,$$

for all $\xi \in \mathbb{C}^n$. Here, and throughout this work, $0 < c < C$ denote generic constants ($c$ a small one, $C$ a large one) that are independent of $h$ and that may change from one occasion to the next.

2.1. Complete electrode model

To begin with, let us recapitulate the complete electrode model, where we employ superscripts $h$ to serve as a measure for the size of the diameter of the electrodes. Later we will drive $h$ to zero in the asymptotic analysis.

Within the complete electrode model the boundary of $\Omega$ is assumed to be partly covered by $M$ electrodes, which are taken to be ideal conductors, and which are identified with the open, simply connected, and mutually disjoint parts $e^h_m \subset \partial \Omega$, $m = 1, \ldots, M$, of the surface that they cover. The union of these electrodes is denoted by $E^h$. All electrodes may be used both for current injection and voltage measurement, and the corresponding electrode net currents and voltages are denoted by $\{I^h_m\}, \{U^h_m\} \subset \mathbb{C}$, respectively. Due to the principle of charge conservation, the total current vector $I = [I^h_m]_{m=1}^M$ belongs to the space

$$\mathcal{C}_0^M := \left\{ Z = [Z_m]_{m=1}^M \in \mathbb{C}^M \left| \sum_{j=1}^M Z_j = 0 \right. \right\}.$$

During electrode measurements, a thin and highly resistive layer is formed at the electrode-object interface.\(^6\) It is characterized by the contact impedances $\{z_m\}$ that in our analysis are assumed to be complex numbers with positive real parts.
The corresponding forward problem is as follows: Given a current pattern \( I \in \mathbb{C}^M \), find \((u^h, U^h) \in (H^1(\Omega) \oplus \mathbb{C}^M)/\mathbb{C} =: \mathcal{H}\) that satisfies

\[
\begin{align*}
\nabla \cdot \sigma \nabla u^h &= 0 & \text{in } \Omega, \\
\nu \cdot \sigma \nabla u^h &= 0 & \text{on } \partial \Omega \setminus \mathcal{E}^h, \\
u \cdot \sigma \nabla u^h &= U^h_m & \text{on } \mathcal{E}_m^h, \quad m = 1, \ldots, M, \\
\int_{\mathcal{E}_m^h} \nu \cdot \sigma \nabla u^h \, dS &= I_m, \quad m = 1, \ldots, M,
\end{align*}
\]

in an appropriate weak sense (cf. Ref. 20). Note that in the factor space \( \mathcal{H} \) the quotient is taken with respect to constant shifts of both, \( u^h \) and \( U^h \), simultaneously. This reflects the freedom in the choice of the ground level of potential. By slight abuse of notation, we will subsequently identify complex numbers with the corresponding constant functions (over an appropriate domain), and refer to equivalence classes of factor spaces with respect to \( \mathbb{C} \) to identify elements (be it functions, numbers, or tuples of both of them) that only differ by additive shifts. Unless there is a possibility of confusion, we also do not distinguish between equivalence classes and representative elements of them.

With this understanding the equations in (2.3) uniquely determine the electromagnetic potential \( u^h \) within \( \Omega \), and the potentials \( \{U^h_m\} \) on the electrodes, and there holds

\[
\|(u^h, U^h)\|_\mathcal{H}^2 = \inf_{c \in \mathbb{C}} \left( \|u^h - c\|_{H^1(\Omega)}^2 + \sum_{m=1}^M \|U^h_m - c\|_{L^2(\mathcal{E}_m^h)}^2 \right) \\
\leq C \sum_{m=1}^M |I_m|^2/|v_m^h|.
\] (2.4)

Furthermore, we have a similar inequality for the flux of the component \( u^h \) of the solution across the boundary, i.e.,

\[
\|
\nu \cdot \sigma \nabla u^h\|_{L^2(\partial \Omega)}^2 \leq C \sum_{m=1}^M |I_m|^2/|v_m^h|.
\] (2.5)

Both in (2.4) and (2.5), the constant \( C = C(\Omega, \sigma, \{z_m\}) > 0 \) is independent of the electrode configuration. We refer to the material in Ref. 20, Theorem 2.3 of Ref. 12, and Lemma 2.1 of Ref. 13 for a proof of these results.

Real-life electrode measurements of impedance tomography provide a noisy version of the current-to-voltage map

\[
R^h : I \mapsto U^h, \quad \mathbb{C}^M \to \mathbb{C}^M/\mathbb{C}.
\] (2.6)

Accordingly, we denote by \((u^h_0, U^h_0) \in \mathcal{H}\) the reference potential for the background admittance \( \sigma_0 \), i.e., the solution of (2.3) with \( \sigma \) replaced by \( \sigma_0 \). \( R^h_0 : I \mapsto U^h_0 \) is the corresponding reference measurement operator.
2.2. Point electrode model

Alternatively, we consider a very simplistic electrode model with electrodes of infinitesimal size at the points \( x_m \in \partial \Omega, \ m = 1, \ldots, M \), where boundary currents are treated as isolated delta distributions. In other words, the corresponding forward problem reads

\[
\nabla \cdot \sigma \nabla u = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma \nabla u = f \quad \text{on } \partial \Omega,
\]

where

\[
f = \sum_{m=1}^{M} I_m \delta_{x_m} \in H^{(1-n)/2-\varepsilon}(\partial \Omega), \quad \text{for any } \varepsilon > 0,
\]

with \( I = [I_m]_{m=1}^{M} \in \mathbb{C}_o^M \) being the same as in Sec. 2.1, and \( \delta_{x_m} \) being the Dirac delta distribution on \( \partial \Omega \) supported in \( x_m \). It follows from the standard theory of elliptic boundary value problems that (2.7)–(2.8) has a unique solution \( u \in H^{(4-n)/2-\varepsilon}(\Omega)/C \) satisfying

\[
\|u\|_{H^{(4-n)/2-\varepsilon}(\Omega)/C} \leq C \|f\|_{H^{(1-n)/2-\varepsilon}(\partial \Omega)} \leq C \|I\|_{\mathbb{C}_o^M},
\]

for any \( \varepsilon > 0 \) and some \( C = C_\varepsilon > 0 \); see, e.g., Ref. 16 and Eq. (A.5) of Ref. 9.

Since the Dirichlet boundary value of \( u \) is (only) in \( H^{(3-n)/2-\varepsilon}(\partial \Omega)/C \) (cf. the trace theorems in Chapter 2 of Ref. 16), the boundary potential is not well defined at the discrete point \( x_m \) — unless \( I_m \) equals zero —, and thus there is no natural way of defining counterparts of the voltages \( U^h \) and the measurement operator \( R^h \) of the complete electrode model within this point electrode setting. However, there does exist a counterpart for difference measurements, i.e., for the relative voltages \( U^h - U^{h_0} \), and the relative current-to-voltage map \( R^h - R^{h_0} \).

To this end, consider the reference potential \( u_0 \in H^{(4-n)/2-\varepsilon}(\Omega)/C \) that solves (2.7)–(2.8) for the background admittance \( \sigma_0 \) and set \( w := u - u_0 \). Then the vector of point evaluations \( W := [w(x_m)]_{m=1}^{M} \in \mathbb{C}_o^M \) is well-defined; see Lemma 2.1 below. In our main result we prove that \( W \) provides an approximation of the corresponding relative voltages \( U^h - U^{h_0} \) of the complete electrode model, if the diameter of the finite size electrodes is small. An immediate corollary is that the measurement operator

\[
A : I \mapsto W : \mathbb{C}_o^M \to \mathbb{C}_o^M /C,
\]

approximates the corresponding relative measurement map \( R^h - R^{h_0} \) of the complete electrode model.

**Lemma 2.1.** The relative potential \( w = u - u_0 \) satisfies the estimate

\[
\|w\|_{H^{l}(\partial \Omega)/C} \leq C \|I\|_{\mathbb{C}_o^M}
\]

for any \( r \in \mathbb{R} \) and \( C = C(r) > 0 \). In particular, \( w|_{\partial \Omega} \) belongs to \( C^{\infty}(\partial \Omega)/C \).

**Proof.** Let us fix \( \varepsilon > 0 \) and \( r \geq 3/2 \); obviously, the latter choice can be made without loss of generality. Let \( \Omega_0 \) and \( \Omega_1 \) be auxiliary \( C^{\infty} \)-domains with connected
complements, such that \( \text{supp } \kappa \subset \Omega_0 \), \( \overline{\Omega}_0 \subset \Omega_1 \) and \( \overline{\Omega}_1 \subset \Omega \). In addition, let \( D \) be a smooth neighborhood of \( \partial \Omega_1 \) with the property \( D \subset \Omega \setminus \overline{\Omega}_0 \). Since \( \nabla \cdot \sigma_0 \nabla w = 0 \) in \( \Omega \setminus \overline{\Omega}_0 \), it follows from (2.9) and a slight modification of Lemma A.1 in Ref. 9, i.e., from interior regularity for elliptic equations, that

\[
\| w \|_{H^{r+1/2}(D)/C} \leq C \| w \|_{H^{(4-n)/2}(-\Omega \setminus \overline{\Omega}_0)/C} \leq C \| I \|_{C^M}.
\]

In particular, using the trace theorem, we see that \( w \) satisfies

\[
\nabla \cdot \sigma_0 \nabla w = 0 \text{ in } \Omega \setminus \overline{\Omega}_1, \quad \nu \cdot \sigma_0 \nabla w = 0 \text{ on } \partial \Omega, \quad \nu \cdot \sigma_0 \nabla w = g \text{ on } \partial \Omega_1
\]

for some mean-free \( g \) with \( \| g \|_{H^{-1}(\partial \Omega_1)} \leq C \| I \|_{C^M} \). Hence, the claim follows from the combination of Remark 7.2 in Chapter 2 of Ref. 16 and the trace theorem. \( \square \)

2.3. Geometrical assumptions

We now list our assumptions on the interplay between the two electrode models introduced in Secs. 2.1 and 2.2 above. As before, we denote by \( e_h \), \( m = 1, \ldots, M \), the finite size electrodes, and by \( x_m \) the positions of the corresponding point electrodes. As already mentioned, we associate with \( h \) the size of the electrodes from the complete electrode model, and we let \( h \) float within some interval \( 0 < h < h_0 \), where \( h_0 > 0 \) is kept fixed.

To be precise, we assume throughout that there is a fixed convex reference domain \( Q \subset \mathbb{R}^{n-1} \) with \( |Q| = 1 \) and \( 0 \in Q \), such that, for each positive parameter \( h < h_0 \), the electrode \( e_h \subset \partial \Omega \), with \( m \) fixed, is given by a one-to-one parameterization

\[ e_h = X_h(Q_h) \quad \text{with} \quad Q_h = hQ, \]

which stands for

\[ e_h = \{ x = X_h(s) \mid s \in Q_h \}. \]

We assume that \( X_h \) is a diffeomorphism between \( Q_h \) and \( e_h \), i.e., both \( X_h \) and its inverse are infinitely times continuously differentiable, and that there are universal constants \( 0 < c < C < \infty \), independent of \( h \) and \( m \), such that the surface element \( dS \) on \( e_h \) satisfies

\[ dS = \sigma_h(s) \, ds \quad \text{with} \quad c \leq \sigma_h(s) \leq C, \quad (2.11) \]

where \( ds \) is the Lebesgue’s volume element of \( Q_h \). To be precise, by the first part of (2.11) we mean that

\[ \int_{e_h} g \, dS = \int_{Q_h} (g \circ X_h) \sigma_h \, ds \]

for any integrable function \( g \) on \( e_h \), i.e., \( \sigma_h \) is the local stretching factor corresponding to the parameterization \( X_h \). In addition, we need some extra control over the first and second order derivatives of \( X_h \), namely we require that

\[ \| X_h \|_{C^2(Q_h^*)} \leq C. \quad (2.12) \]
It is easy to see that under these assumptions there holds

\[ \| \psi \circ X^h_m \|_{H^1(Q^h)} \leq C \| \psi \|_{H^1(e^h_m)} \]  

(2.13)

for every \( \psi \in H^1(e^h_m) \), and

\[ |\Gamma| \leq C \| (X^h_m)^{-1}(\Gamma) \| \]  

(2.14)

for any smooth curve \( \Gamma \subset e^h_m \). (In fact, for (2.13) and (2.14) to hold the assumption on the second order derivatives of \( X^h_m \) is redundant; however, such an assumption is needed in the proof of Lemma 3.2 below.) We point out that because of the above stipulations the area covered by \( e^h_m \) is given by

\[ |e^h_m| = \int_{e^h_m} dS = \int_{Q^h} \sigma^h_m(s) \ ds = h^{n-1} \int_Q \sigma^h_m(hs) \ ds \leq C h^{n-1} \]

(2.15)

Finally, concerning the interplay between the two electrode models, we assume that \( x_m \in e^h_m \), more precisely, that

\[ x_m = X^h_m(0) \]

and, to enable an \( O(h^2) \) approximation property to be established below we require that

\[ \int_{Q^h} s \sigma^h_m(s) \ ds = 0, \]

(2.16)

i.e., that the origin (the preimage of \( x_m \) under \( X^h_m \)) is a (weighted) center of mass of \( Q^h \).

Some interpretation of the above assumptions may be useful:

- For \( n = 2 \), i.e., in two space dimensions, the boundary of \( \Omega \) is a closed curve, and it is most natural to assume that \( e^h_m \) are electrodes of length \( |e^h_m| = h \), say. In this case one can choose \( Q = [-1/2, 1/2] \), \( Q^h = [-h/2, h/2] \), and let \( X^h_m \) be, for all \( h > 0 \), an arc length parameterization of the boundary. In this case \( \sigma^h_m = 1 \), and the condition (2.16) is equivalent to saying that the position \( x_m = X^h_m(0) \) of the point electrode is half way (along the boundary of \( \Omega \)) between the two end points of the electrode \( e^h_m \).

- In three space dimensions, one can think of \( Q^h \) being a planar reference shape for each of the electrodes, that is shrinking with decreasing size parameter \( h \). Think of these electrodes as being elastic, so that they can be attached to the surface \( \partial \Omega \) around \( x_m \). The corresponding deformation is determined by \( X^h_m \), with

\[ \sigma^h_m(s) = \left| \frac{\partial X^h_m(s)}{\partial s_1} \times \frac{\partial X^h_m(s)}{\partial s_2} \right|, \quad s = (s_1, s_2), \]

being the local stretching factor. In order to satisfy condition (2.16), one has to make sure that the position of the point electrode is some kind of center of \( e^h_m \) (which is not the center of mass, though, as the latter does not usually sit on \( \partial \Omega \)).
2.4. The main result

We are now ready to formulate the main result of this paper. Recall that, given a current pattern $I$, we denote by $U_h$ and $U_h^0$ the voltages on the finite size electrodes corresponding to the admittance $\sigma$ of (2.1) and to the background admittance $\sigma_0$, respectively. Similarly, in the framework of point electrodes, $u$ and $u_0$ stand for the respective potentials of Sec. 2.2, and $W := \left[ (u - u_0)(x_m) \right]_{m=1}^{M} \in \mathbb{C}^M$ contains the relative voltages on the point electrodes.

**Theorem 2.1.** Under the assumptions from Sec. 2.3,
\[
\| (U^h - U_0^h) - W \|_{\mathbb{C}^M / \mathbb{C}} \leq C h^2 \| I \|_{\mathbb{C}^M},
\]
where $C > 0$ is independent of $h \in (0, h_0)$ and $I \in \mathbb{C}^M$.

For the corresponding measurement maps of the complete electrode model and the point electrode model,
\[
R^h : I \mapsto U^h, \quad R_0^h : I \mapsto U_0^h, \quad \text{and} \quad A : I \mapsto W,
\]
we deduce that $A$ is an accurate approximation of $R^h - R_0^h$, provided that the parameter $h$, which measures the diameter of the electrodes, is relatively small.

**Corollary 2.1.** Under the assumptions from Sec. 2.3,
\[
\| (R^h - R_0^h) - A \|_{L(\mathbb{C}^M, \mathbb{C}^M / \mathbb{C})} \leq C h^2,
\]
where $C > 0$ is independent of $h \in (0, h_0)$.

**Remark 2.1.** Suppose that the point electrode locations $\{x_m\}$ are not the centers of the corresponding finite size electrodes $\{e_h^m\}$ in the sense of (2.16). In such a case Theorem 2.1 and Corollary 2.1 are no longer valid. However, as long as only $x_m \in e_h^m$, $m = 1, \ldots, M$, one can still obtain the weaker convergence rate
\[
\| (R^h - R_0^h) - A \|_{L(\mathbb{C}^M, \mathbb{C}^M / \mathbb{C})} \leq Ch. \tag{2.17}
\]
In fact, the proof of Lemma 3.2 below could be shortened considerably, and also the geometrical assumptions about the finite size electrodes could be weakened, if the aim was only to prove an $O(h)$-estimate.

Furthermore, it is easy to see from the proof of Theorem 2.1 that its assertion remains valid, if one or all $x_m$ deviate from the center of the respective electrode(s) by $O(h^2)$.

3. Proof of the Main Result

This section is devoted to proving Theorem 2.1. For a current pattern $I \in \mathbb{C}^M$, let $(u^h, U^h)$ and $(u_0^h, U_0^h)$ be the solution pairs of (2.3) corresponding to the admittance $\sigma$ from (2.1) and the background admittance $\sigma_0$, respectively, and set $(w^h, W^h) := (u^h - u_0^h, U^h - U_0^h)$.
We begin with a refinement of the inequality (2.5) for the complete electrode model, which – in contrast to (2.5) – is only valid on the electrodes.

**Lemma 3.1.** The component \( u^h \) of the solution to (2.3) satisfies

\[
\| \nu \cdot \sigma \nabla u^h \|^2_{H^1(E^h)} \leq C \sum_{m=1}^{M} \frac{|I_m|^2}{|e_m^h|},
\]

where \( C > 0 \) is independent of the electrode configuration, and of \( h \), in particular.

**Proof.** Because \( \sigma \) is smooth in a neighborhood of \( \partial \Omega \), the Neumann-to-Dirichlet map corresponding to the first equation of (2.3) is bounded from the subspace of \( L^2(\partial \Omega) \)-functions with zero integral mean to \( H^1(\partial \Omega)/C \) (cf., e.g., Theorem A.3 of Ref. 9), and thus (2.5) gives

\[
\| u^h \|_{H^1(\partial \Omega)/C}^2 \leq C \sum_{m=1}^{M} \frac{|I_m|^2}{|e_m^h|}.
\]

Hence, it follows from the third equation of (2.3) that (cf. Remark 6 on p. 146 of Ref. 7)

\[
\| \nu \cdot \sigma \nabla u^h \|^2_{H^1(E^h)} \leq C \sum_{m=1}^{M} \left( \| u^h \|^2_{H^1(e_m^h)/C} + \| U_m^h - u^h \|^2_{L^2(e_m^h)} \right)
\]

\[
\leq C \left( \sum_{m=1}^{M} \frac{|I_m|^2}{|e_m^h|} + \sum_{m=1}^{M} \| U_m^h - u^h \|^2_{L^2(e_m^h)} \right).
\]

For the second term on the right hand side we have

\[
\sum_{m=1}^{M} \| U_m^h - u^h \|^2_{L^2(e_m^h)} \leq C \sum_{m=1}^{M} \left( \| U_m^h - c \|^2_{L^2(e_m^h)} + \| c - u^h \|^2_{L^2(e_m^h)} \right)
\]

\[
\leq C \left( \| u^h - c \|^2_{H^1(\Omega)} + \sum_{m=1}^{M} \| U_m^h - c \|^2_{L^2(e_m^h)} \right),
\]

where we have applied the trace theorem. By taking the infimum over \( c \in \mathbb{C} \), and using (2.4) it thus follows that

\[
\sum_{m=1}^{M} \| U_m^h - u^h \|^2_{L^2(e_m^h)} \leq C \sum_{m=1}^{M} \frac{|I_m|^2}{|e_m^h|},
\]

which completes the proof. \( \square \)

Next we provide a first result (in a comparatively weak norm) on how well \( u^h \) approximates the potential \( u \) of Sec. 2.2 for point electrodes on \( \partial \Omega \).
Lemma 3.2. Under the assumptions from Sec. 2.3 we can find for every \( \epsilon > 0 \) some \( C_\epsilon > 0 \) such that

\[
\| \nu \cdot \sigma \nabla (u^h - u) \|_{H^{-(n+3)/2}(\Omega)} \leq C_\epsilon h^2 \| I \|_{C^M}
\]

for every \( 0 < h < h_0 \) and all \( I \in C^M_\partial \).

Proof. 1. Let \( \phi \in C^\infty(\partial \Omega) \) be fixed, and denote \( \nu \cdot \sigma \nabla u^h \rvert_{\partial \Omega} \) by \( f^h \). Note that it follows from Lemma 3.1 and (2.15) that

\[
\| f^h \|_{H^1(e^h_m)} \leq Ch^{1-n} \| I \|_{C^M}.
\]

According to the boundary conditions of (2.3) and (2.7), we can therefore rewrite

\[
\| \nu \cdot \sigma \nabla (u^h - u), \phi \rvert_{\partial \Omega} \|_{C^M} = \left| \int_{\partial \Omega} f^h \phi \, dS - \sum_{m=1}^M I_m \phi(x_m) \right|
\]

\[
= \left| \sum_{m=1}^M \int_{e^h_m} \left( f^h - I_m / |e^h_m| \right) (\phi - \phi(x_m)) \, dS \right|
\]

\[
\leq \sum_{m=1}^M \left| \int_{e^h_m} \left( f^h - I_m / |e^h_m| \right) (\phi - \phi(x_m)) \, dS \right| + \sum_{m=1}^M \left| I_m / |e^h_m| \right| \int_{e^h_m} (\phi - \phi(x_m)) \, dS
\]

and hence,

\[
\| \nu \cdot \sigma \nabla (u^h - u), \phi \rvert_{\partial \Omega} \|_{C^M} \leq \sum_{m=1}^M \| f^h - I_m / |e^h_m| \|_{L^2(e^h_m)} \| \phi - \phi(x_m) \|_{L^2(e^h_m)}
\]

\[
+ \sum_{m=1}^M \left| I_m / |e^h_m| \right| \int_{e^h_m} (\phi - \phi(x_m)) \, dS.
\]

The terms that enter on the right-hand side of (3.2) will now be treated separately for any fixed \( m \in \{1, \ldots, M\} \).

2. To begin with, we remark that, according to (2.3), \( \psi = f^h - I_m / |e^h_m| \) has vanishing integral mean over \( e^h_m \), i.e.,

\[
0 = \int_{e^h_m} \psi(x) \, dS = \int_{Q_e^h} \psi(X^h_m(s)) \sigma^h_m(s) \, ds = h^{n-1} \int_{Q} \psi(X^h_m(hs)) \sigma^h_m(hs) \, ds.
\]

Therefore the Poincaré-Friedrichs inequality for the domain \( Q \) yields

\[
\int_Q \psi(X^h_m(hs)) \sigma^h_m(hs) \, ds \leq C \int_Q \nabla_s \left( \psi(X^h_m(hs)) \sigma^h_m(hs) \right) \, ds
\]

\[
\leq C h^2 h^{1-n} \int_{Q_e^h} \left| \nabla_s \left( \psi(X^h_m(s)) \sigma^h_m(s) \right) \right|^2 \, ds,
\]
and hence, as $\sigma_m^h$ is bounded by $c$ from below,

$$
\|f^h - I_m/|e_m^h|\|_{L^2(e_m^h)}^2 \leq \frac{1}{c} \int_{Q^h} |\psi(X_m^h(s))\sigma_m^h(s)|^2 \, ds
$$

$$
= \frac{1}{c} h^{n-1} \int_{Q^h} |\psi(X_m^h(sh))\sigma_m^h(sh)|^2 \, ds
$$

$$
\leq Ch^2 \int_{Q^h} \|\nabla_s(\psi(X_m^h(s))\sigma_m^h(s))\|^2 \, ds.
$$

Due to (2.12), $\sigma_m^h$ is uniformly bounded in $C^1(Q^h)$ with respect to $h$, and we can continue by using (2.13) to obtain

$$
\|f^h - I_m/|e_m^h|\|_{L^2(e_m^h)}^2 \leq Ch^2 \|\psi \circ X_m^h\|_{H^1(Q^h)}^2 \leq Ch^2 \|\psi\|_{H^1(e_m^h)}^2
$$

and hence, assuming that $Ch^2 \leq 1/2$ and using (3.1), we conclude that

$$
\|f^h - I_m/|e_m^h|\|_{L^2(e_m^h)} \leq Ch\|f^h\|_{H^1(e_m^h)} \leq Ch^{3/2-n/2}\|I\|_{C^1}.
$$

3. Due to (2.14), for any $x \in e_m^h$ there exists a smooth curve $\Gamma \subset e_m^h$ connecting $x$ and $x_m$ such that

$$
|\Gamma| \leq Ch.
$$

Indeed, one can construct such a $\Gamma$ by taking the line segment between the points $(X_m^h)^{-1}(x)$ and $(X_m^h)^{-1}(x_m)$ in $Q^h$, and mapping it back onto $e_m^h$ with $X_m^h$. As a consequence,

$$
|\varphi(x) - \varphi(x_m)| \leq Ch\|\varphi\|_{C^1(\partial \Omega)},
$$

and hence, by virtue of (2.15) there holds

$$
\|\varphi - \varphi(x_m)\|_{L^2(e_m^h)} \leq Ch^2|\varphi(x_m)|_{C^1(\partial \Omega)} \leq Ch^{n+1}|\varphi(x_m)|_{C^1(\partial \Omega)}. \tag{3.4}
$$

4. According to (2.16), the quadrature formula

$$
\int_{e_m^h} \varphi \, ds = \int_{Q^h} (\varphi \circ X_m^h) \sigma_m^h \, ds \approx \varphi(x_m)|e_m^h| \tag{3.5}
$$

is exact whenever $\varphi \circ X_m^h$ is a polynomial of degree less or equal to one. Because $Q^h$ is convex, we can expand

$$
(\varphi \circ X_m^h)(s) = \varphi(x_m) + s \cdot \nabla_s(\varphi \circ X_m^h)(0) + r(s),
$$

where

$$
|r(s)| \leq C|s|^2\|\varphi\|_{C^2(\partial \Omega)}
$$

because of (2.12). Hence,

$$
\int_{e_m^h} \varphi \, ds - \varphi(x_m)|e_m^h| = \int_{Q^h} r(s)\sigma_m^h(s) \, ds \leq Ch^{n+1}\|\varphi\|_{C^2(\partial \Omega)}. \tag{3.6}
$$
5. Inserting the three estimates (3.3), (3.4), and (3.6), together with (2.15) into (3.2), we finally arrive at

\[ \left| \langle \nu \cdot \sigma \nabla (u^h - u), \varphi \rangle \right| \leq C h^2 \| I \|_{CM} \| \varphi \|_{C^2(\partial \Omega)}. \]

Now, if \( \varepsilon > 0 \) then \( H^{(n+3)/2+\varepsilon}(\partial \Omega) \) is continuously embedded in \( C^2(\partial \Omega) \) according to the Sobolev embedding theorem, cf. Hebey, and hence, there is \( C_\varepsilon > 0 \) such that

\[ \left| \langle \nu \cdot \sigma \nabla (u^h - u), \varphi \rangle \right| \leq C_\varepsilon h^2 \| I \|_{CM} \| \varphi \|_{H^{(n+3)/2+\varepsilon}(\partial \Omega)}. \]

Because \( C^\infty(\partial \Omega) \) is dense in \( H^{(n+3)/2+\varepsilon}(\partial \Omega) \) (see, e.g., Sec. 7.3 of Chapter 1 in Ref. 16), we deduce that

\[ \| \nu \cdot \sigma \nabla (u^h - u) \|_{H^{-(n+3)/2-\varepsilon}(\partial \Omega)} \leq C_\varepsilon h^2 \| I \|_{CM}, \]

which completes the proof.

Due to the regularity properties of elliptic partial differential equations, the approximation of Lemma 3.2 gets stronger if one concentrates on the behavior of the corresponding potentials at some distance from the boundary \( \partial \Omega \). This statement is made concrete by the following corollary.

**Corollary 3.1.** Let \( \Omega_0 \subset \mathbb{R}^n \) be a nonempty domain such that \( \overline{\Omega}_0 \subset \Omega \). Then, there holds that

\[ \| u^h - u \|_{H^1(\Omega_0)/C} \leq C h^2 \| I \|_{CM}, \]

and

\[ \| u^h \|_{H^1(\Omega_0)/C} + \| u \|_{H^1(\Omega_0)/C} \leq C \| I \|_{CM}, \]

where \( C = C(\Omega_0) > 0 \) is independent of \( h \).

**Proof.** Since \( u^h - u \) satisfies the conductivity equation for the admittance \( \sigma \) of (2.1), and since \( \sigma \) is smooth in some neighborhood of \( \partial \Omega \), it follows from Lemma 3.2 and the continuous dependence on the boundary data for the Neumann problem (cf. Eq. (A.5) of Ref. 9) that

\[ \| u^h - u \|_{H^{n/2-\varepsilon}(\Omega)/C} \leq C \varepsilon h^2 \| I \|_{CM} \]  

for some \( C_\varepsilon > 0 \).

Using a similar interior regularity argument as in the proof of Lemma 2.1, we see that \( u^h - u \) satisfies the Neumann problem

\[ \nabla \cdot \sigma \nabla (u^h - u) = 0 \quad \text{in} \ \Omega_0, \quad \nu \cdot \sigma \nabla (u^h - u) = g \quad \text{on} \ \partial \Omega_0 \]

for any smooth domain \( \Omega_0 \) with connected complement such that \( \text{supp} \ \kappa \subset \Omega_0 \) and \( \overline{\Omega}_0 \subset \Omega \), and for some mean-free \( g \) with \( \| g \|_{L^2(\partial \Omega_0)} \leq C(\Omega_0) h^2 \| I \|_{CM} \). Notice that a more general \( \Omega_0 \) can be enclosed by a domain with these properties. Hence, the
The estimate for $u$ is obtained in the exactly same manner, with (2.9) playing the role of (3.7). Finally, the claim about $u_h$ follows from the triangle inequality.}

We can now deduce Theorem 2.1 from Corollary 3.1 by a duality argument.

**Proof of Theorem 2.1.** Let $J \in \mathbb{C}_0^M$ be arbitrary and choose an auxiliary domain $\Omega_0$ such that $\text{supp} \kappa \subset \Omega_0$ and $\overline{\Omega_0} \subset \Omega$. We fix the ground level of potential, i.e., choose a representative of a quotient equivalence class, so that

$$J := W^h - W = ((R^h - R^h_0) - A)I \in \mathbb{C}_0^M,$$

(3.9)

where $W_m = w(x_m)$ is defined as in (2.10). We denote by $(v^h, U^h) \in \mathcal{H}$ the solution of the complete electrode problem (2.3) for this newly defined electrode current pattern $J = [J_m]_{m=1}^M$. The variational formulation for this problem in Proposition 3.1 of Ref. 20 gives

$$\sum_{m=1}^M J_m W^h_m = \int_\Omega \sigma \nabla v^h \cdot \nabla w^h \, dx + \sum_{m=1}^M \frac{1}{\varepsilon_m} \int_{e_m^h} (v^h - V^h_m)(u^h - W^h_m) \, dS$$

$$= - \int_{\Omega_0} \kappa \nabla u^h_0 \cdot \nabla v^h \, dx,$$

(3.10)

where the second step is a consequence of the very same variational formulation for the pairs $(u^h, U^h)$ and $(u^h_0, U^h_0)$, respectively, together with the definition of $(u^h, W^h)$.

Similarly, let $v \in H^{(4-n)/2-\varepsilon}(\Omega)/\mathbb{C}$ solve the forward problem

$$\nabla \cdot \sigma \nabla v = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma \nabla v = g \quad \text{on } \partial \Omega,$$

with the point current pattern

$$g = \sum_{m=1}^M J_m \delta_{x_m}.$$

We introduce a mean-free sequence $(g_k) \subset C^\infty(\partial \Omega)$ that converges towards $g$ in the topology of $H^{(1-n)/2-\varepsilon}(\partial \Omega)$ (cf., e.g., Sec. 7.3 of Chapter 1 in Ref. 16). As in the proof of Corollary 3.1, it follows from interior regularity arguments that the solutions $(v_k) \subset H^1(\Omega)/\mathbb{C}$ of

$$\nabla \cdot \sigma \nabla v_k = 0 \quad \text{in } \Omega, \quad \nu \cdot \sigma \nabla v_k = g_k \quad \text{on } \partial \Omega,$$

fulfil

$$\lim_{k \to \infty} \|v_k - v\|_{H^1(\Omega_0)/\mathbb{C}} = 0.$$

Since

$$\int_{\partial \Omega} g_k w \, dS = \int_\Omega \sigma \nabla v_k \cdot \nabla w \, dx = - \int_{\Omega_0} \kappa \nabla u_0 \cdot \nabla v_k \, dx,$$
and since $w$ is smooth on $\partial \Omega$ (cf. Lemma 2.1), we obtain that
\[
\sum_{m=1}^{M} J_{m} W_{m} = \lim_{k \to \infty} \int_{\partial \Omega} g_{k} w \, dS = -\int_{\Omega_{0}} \kappa \nabla u_{0} \cdot \nabla v \, dx.
\] (3.11)

Combining (3.9), (3.10) and (3.11), we deduce that
\[
\|W^{h} - W\|_{C^{M}}^{2} = \int_{\Omega_{0}} (\kappa \nabla u_{0} \cdot \nabla v - \kappa \nabla u_{0}^{h} \cdot \nabla v^{h}) \, dx
\leq C \left( \|u_{0} - u_{0}^{h}\|_{H^{1}(\Omega_{0}/C)} \|v\|_{H^{1}(\Omega_{0}/C)} + \|u_{0}^{h}\|_{H^{1}(\Omega_{0}/C)} \|v - v^{h}\|_{H^{1}(\Omega_{0}/C)} \right)
\leq C h^{2} \|W^{h} - W\|_{C^{M}} \|I\|_{C^{M}},
\]
where the last step follows by applying Corollary 3.1 to each of the four distributions
$u_{0} - u_{0}^{h}$, $v$, $u_{0}^{h}$ and $v - v^{h}$. In consequence, division by $\|W^{h} - W\|_{C^{M}}$ completes the proof. □

Concerning Remark 2.1 we note that the special definition (2.16) of $x_{m}$ as the weighted center of mass of $e_{m}^{h}$ is only used in the fourth part of the proof of Lemma 3.2: Assuming merely that $x_{m}$ belongs to $e_{m}^{h}$ reduces the accuracy of the quadrature formula (3.5), so that it is exact only for constants. Accordingly, this decreases the exponent of $h$ on the right-hand side of (3.6) by one, which carries over to (3.2) and, eventually, to the conclusion of Lemma 3.2. This first order convergence rate in $h$ then transports trivially to Corollary 3.1 and Theorem 2.1.

4. Electrode Dipoles and Backscatter Data

In this section we restrict our attention to two space dimensions, i.e., $n = 2$, and to the case where there are only two small electrodes attached close to each other on $\partial \Omega$. For any fixed $y \in \partial \Omega$ we let $X \in C^{\infty}(\mathbb{R}; \mathbb{R}^{2})$ be a counterclockwise $|\partial \Omega|$-periodic parameterization of $\partial \Omega$ with respect to arc length, such that $X(0) = y$, and
\[
\partial \Omega = \{X(s) \mid -|\partial \Omega|/2 \leq s < |\partial \Omega|/2\}.
\]
Then we define a pair of electrodes centered around $y$ via
\[
e_{+}^{h} = \{X(s) \mid h/2 < s < 3h/2\},
\]
and
\[
e_{-}^{h} = \{X(s) \mid X(-s) \in e_{+}^{h}\},
\]
where $h > 0$ is the length of the electrodes. The idea is to drive $1/(2h)$ units of current from $e_{+}^{h}$ to $e_{-}^{h}$ and measure the resulting potential difference.
In this setting, the forward solution of this problem is the unique pair \((u^h, U^h)\) \(\in H^1(\Omega) \oplus \mathbb{C}\) that satisfies weakly
\[
\begin{align*}
\nabla \cdot \sigma \nabla u^h &= 0 & \text{in } \Omega, \\
\nu \cdot \sigma \nabla u^h &= 0 & \text{on } \partial \Omega \setminus (\overline{e^h} \cup \overline{e}^h), \\
u^h + z \pm \nu \cdot \sigma \nabla u^h &= \pm U^h & \text{on } e^h_{\pm}, \\
\int_{e^h_{\pm}} \nu \cdot \sigma \nabla u^h \, dS &= \pm 1/(2^h),
\end{align*}
\] (4.1)
where we have fixed the ground level of potential in an obvious way. Let \((u^h_0, U^h_0)\) \(\in H^1(\Omega) \oplus \mathbb{C}\) be the solution of (4.1) when \(\sigma\) of (2.1) is replaced by the smooth background admittance \(\sigma_0\). As in Sec. 3 we set \((w^h, W^h) = (u^h - u^h_0, U^h - U^h_0)\), and define
\[
b^h = W^h/h.
\]

Our goal is to prove that \(b^h\) can be approximated by the corresponding backscatter data introduced in Refs. 8 and 9.\(^a\) Such data are defined via the following variant of the point electrode forward problem introduced in Sec. 2.2:
\[
\begin{align*}
\nabla \cdot \sigma \nabla u &= 0 & \text{in } \Omega, \\
\nu \cdot \sigma \nabla u &= -\delta' y & \text{on } \partial \Omega,
\end{align*}
\] (4.2)
where the (mean-free) dipole current \(\delta' y \in H^{-3/2-\varepsilon}(\partial \Omega)\), \(\varepsilon > 0\), is defined by virtue of
\[
\langle \delta' y, v \rangle = -\frac{\partial v(X(s))}{\partial s} \bigg|_{s=0}
\] (4.3)
for \(v \in H^{3/2+\varepsilon}(\partial \Omega)\). It follows, e.g., from the material in the appendix of Ref. 9 that (4.2) has a unique solution \(u \in H^{-\varepsilon}(\Omega)/\mathbb{C}\) for any \(\varepsilon > 0\) satisfying
\[
\|u\|_{H^{-\varepsilon}(\Omega)/\mathbb{C}} \leq C,
\] (4.4)
where \(C = C(\Omega, \sigma, \varepsilon) > 0\) is independent of \(y\). We denote by \(u_0\) the reference potential, i.e., the solution of (4.2) with \(\sigma\) replaced by \(\sigma_0\), and set \(w = u - u_0\). Then, the backscatter data of electrical impedance tomography at \(y\) is defined to be
\[
b = -\langle \delta' y, w \rangle.
\]
Take note that \(b\) is a well defined number because the dipole \(\delta' y\) does not see the ground level of potential and, furthermore,
\[
\|w\|_{H^{\varepsilon}(\partial \Omega)/\mathbb{C}} \leq C,
\] (4.5)

\(^a\)We mention that it has been shown in Ref. 8 that these backscatter data (as a function of the point \(y \in \partial \Omega\)) uniquely define a simply connected insulating obstacle within \(\Omega\), if the background admittance \(\sigma_0\) is constant.
for any $r \in \mathbb{R}$ and $C = C(r) > 0$ that can be chosen independently of $y \in \partial \Omega$. This estimate follows by repeating the argumentation of Lemma 2.1 and noticing that the appearing constants can be chosen so that they depend on $\kappa$, $\sigma_0$ and the geometry of $\Omega$, but not on $y$.

The main result of this section is as follows:

**Theorem 4.1.** There holds

$$|b^h - b| \leq Ch,$$

where $C > 0$ is independent of $h > 0$ and $y \in \partial \Omega$.

The rest of this section is devoted to proving Theorem 4.1. We start by presenting the counterpart of Corollary 3.1 in this new setting.

**Lemma 4.1.** Let $u^h$ and $u$ be given by (4.1) and (4.2), respectively. Furthermore, let $\Omega_0 \subset \mathbb{R}^2$ be a nonempty domain such that $\Omega_0 \subset \Omega$. Then there holds that

$$\|u^h - u\|_{H^1(\Omega_0)/C} \leq Ch,$$

and

$$\|u^h\|_{H^1(\Omega_0)/C} + \|u\|_{H^1(\Omega_0)/C} \leq C,$$

where $C = C(\Omega_0) > 0$ is independent of $h$ and $y \in \partial \Omega$.

**Proof.** The leading idea of this proof is the same as in Lemma 3.2 and Corollary 3.1: We first show that $\nu \cdot \sigma \nabla u^h |_{\partial \Omega}$ provides an approximation of $\delta_y'$ in some weak Sobolev norm, after which the assertion follows by an interior regularity argument. It is straightforward to see that the constants in the estimates below can be chosen so that they depend on $\sigma$ and the geometry of $\Omega$, but not on $y \in \partial \Omega$.

A simple calculation utilizing the boundary conditions of (4.1) shows that

$$\int_{\partial \Omega} \nu \cdot \sigma \nabla u^h \varphi \, dS = \frac{1}{2h^2} \int_{e^h_+} \varphi \, dS + \int_{e^h_-} (\nu \cdot \sigma \nabla u^h - 1/(2h^2))(\varphi - \varphi(y)) \, dS$$

$$- \frac{1}{2h^2} \int_{e^h_-} \varphi \, dS + \int_{e^h_+} (\nu \cdot \sigma \nabla u^h + 1/(2h^2))(\varphi - \varphi(y)) \, dS$$

for $\varphi \in C^\infty(\partial \Omega)$. As $\pm 1/(2h^2)$ is the mean of $\nu \cdot \sigma \nabla u^h$ over $e^h_{\pm}$, the Poincaré inequality and Lemma 3.1 provide the estimate

$$\|\nu \cdot \sigma \nabla u^h \mp 1/(2h^2)\|_{L^2(e^h_{\pm})} \leq Ch\|\nu \cdot \sigma \nabla u^h\|_{H^1(e^h_{\pm})} \leq Ch^{-1/2}.$$

Furthermore, as in part 3 of Lemma 3.2, we have that for all $x \in e^h_{\pm}$,

$$|\varphi(x) - \varphi(y)| \leq Ch\|\varphi\|_{C^1(\partial \Omega)},$$

so that we get from the Sobolev embedding theorem

$$\|\varphi - \varphi(y)\|_{L^2(e^h_{\pm})} \leq Ch^{3/2}\|\varphi\|_{C^1(\partial \Omega)} \leq Ch^{3/2}\|\varphi\|_{H^{3/2+\epsilon}(\partial \Omega)}$$
for any \( \varepsilon > 0 \) and some \( C = C_\varepsilon > 0 \). Combining the above estimates, it follows from the Schwarz inequality that

\[
\left| \int_{\partial \Omega} \nu \cdot \sigma \nabla u^h \varphi \, dS - \frac{1}{2h^2} \int_{e^h_+} \varphi \, dS + \frac{1}{2h^2} \int_{e^h_-} \varphi \, dS \right| \leq Ch\|\varphi\|_{H^{3/2+\varepsilon}((\partial \Omega))},
\]

and hence, the triangle inequality gives

\[
\left| (\nu \cdot \sigma \nabla (u^h - u), \varphi) \right| \leq \left| \frac{\partial \varphi(X(s))}{\partial s} \right|_{s=0} - \frac{1}{2h^2} \int_{e^h_+} \varphi \, dS + \frac{1}{2h^2} \int_{e^h_-} \varphi \, dS \right| + Ch\|\varphi\|_{H^{3/2+\varepsilon}((\partial \Omega))}.
\]

Using Taylor’s theorem around \( s = 0 \) together with the Sobolev embedding theorem, it is straightforward to deduce that (cf. the appendix of Ref. 8)

\[
\left| \frac{\partial \varphi(X(s))}{\partial s} \right|_{s=0} - \frac{1}{2h^2} \int_{e^h_+} \varphi \, dS + \frac{1}{2h^2} \int_{e^h_-} \varphi \, dS \right| \leq Ch\|\varphi\|_{H^{5/2+\varepsilon}((\partial \Omega))}.
\]

Hence, as \( C(\partial \Omega) \) is dense in \( H^{5/2+\varepsilon}((\partial \Omega)) \), the estimate

\[
\| \nu \cdot \sigma \nabla (u^h - u) \|_{H^{-3/2-\varepsilon}(\partial \Omega)} \leq Ch, \quad C = C(\varepsilon) > 0,
\]

follows by taking the supremum over \( \varphi \) with \( \|\varphi\|_{H^{5/2+\varepsilon}((\partial \Omega))} = 1 \) in (4.6). With (4.7) in the role of Lemma 3.2 and (4.4) in that of (2.9), the assertion follows by repeating the argumentation from the proof of Corollary 3.1.

Now, we have gathered enough material to prove Theorem 4.1. The techniques used below are in essence the same as in the proof of Theorem 2.1.

**Proof of Theorem 4.1.** Let us fix \( y \in \partial \Omega \), but note that all constants in the following estimates can be chosen independently of \( y \). Moreover, we choose an auxiliary domain \( \Omega_0 \) such that \( \text{supp} \kappa \subset \Omega_0 \) and \( \Omega_0 \subset \Omega \).

The definition of \( b^h \) and the variational formulation of the forward problem (4.1) gives (cf. Proposition 3.1 of Ref. 20)

\[
b^h = 2(1/(2h))W^h \\
= \int_\Omega \sigma \nabla u^h \cdot \nabla u^h \, dx + \frac{1}{z_+} \int_{e^h_+} (u^h - U^h)(w^h - W^h) \, dS \\
+ \frac{1}{z_-} \int_{e^h_-} (u^h + U^h)(w^h + W^h) \, dS \\
= - \int_{\Omega_0} \kappa \nabla u_0^h \cdot \nabla u^h \, dx.
\]

On the other hand, after approximating \( -\delta_y^\varepsilon \) by a sequence of smooth mean-free functions \( g_k \) in the topology of \( H^{-3/2-\varepsilon}(\partial \Omega) \), exactly the same line of reasoning as in the second paragraph of the proof of Theorem 2.1 indicates that

\[
b = - \langle \delta_y^\varepsilon, w \rangle = \lim_{k \to \infty} \int_{\partial \Omega} g_k w \, dS = - \int_{\Omega_0} \kappa \nabla u_0 \cdot \nabla u \, dx.
\]
Combining this with (4.8) results in

$$|b^h - b| = \left| \int_{\Omega} (\kappa \nabla u_0 \cdot \nabla u - \kappa \nabla u_0^h \cdot \nabla u^h) \, dx \right| \leq C h,$$

where the last step follows with the same rationale that has been used for the last estimate in the proof of Theorem 2.1, with Lemma 4.1 playing the role of Corollary 3.1. This completes the proof.

\[\square\]

Acknowledgment

Martin Hanke acknowledges support by the German Research Foundation (DFG) under grant HA2121/6-2. The work of Bastian Harrach was supported by the German Federal Ministry of Education and Research (BMBF) under grant 03HBPAM2. The work of Nuutti Hyvönen was supported by the Academy of Finland (projects 115013, 135979 and 213476) and the Finnish Funding Agency for Technology and Innovation (project 40370/08).

References