Factorization and source support methods for electrical impedance tomography

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0 Practical issues

Information and material

These slides will be posted after/during the summer school on:

`http://wiki.helsinki.fi/display/mathstatKurssit/FICS+2010`

Printed versions will also be produced in somewhat random manner.

It is assumed that the audience is familiar with basic theory of functional analysis, Sobolev spaces, elliptic partial differential equations and complex analysis.

Background

The exists a vast amount of literature on the *factorization* and *source/scattering support* methods. The original ideas were presented in the context of inverse scattering in the papers

A. Kirsch, *Characterization of the shape of a scattering obstacle using the spectral data of the far field operator*, Inverse Problems, 14, 1489–1512 (1998), and

S. Kusiak and J. Sylvester, *The scattering support*, Commun. Pure Appl. Math., 56, 1525–1548 (2003),

respectively.

Here, we only consider these methods in the framework of *electrical impedance tomography* (EIT), and follow mainly the four articles

M. Brühl, *Explicit characterization of inclusions in electrical impedance tomography*, SIAM J. Math. Anal., 32, 1327–41 (2001),

M. Brühl and M. Hanke, *Numerical implementation of two noniterative methods fro locating inclusions by impedance tomography*, Inverse Problems, 16, 1029–1042 (2000),

B. Gebauer and N. Hyvönen, *Factorization method and irregular inclusions in electrical impedance tomography*, Inverse Problems, 23, 2159-2170 (2007), and

M. Hanke, N. Hyvönen, and S. Reusswig, *Convex source support and its application to electric impedance tomography*, SIAM J. Imag. Sci., 1, 364-378 (2008).

Timetable

The preliminary timetable is as follows:

Monday (1 hour): Short introduction to EIT and some monotonicity results.

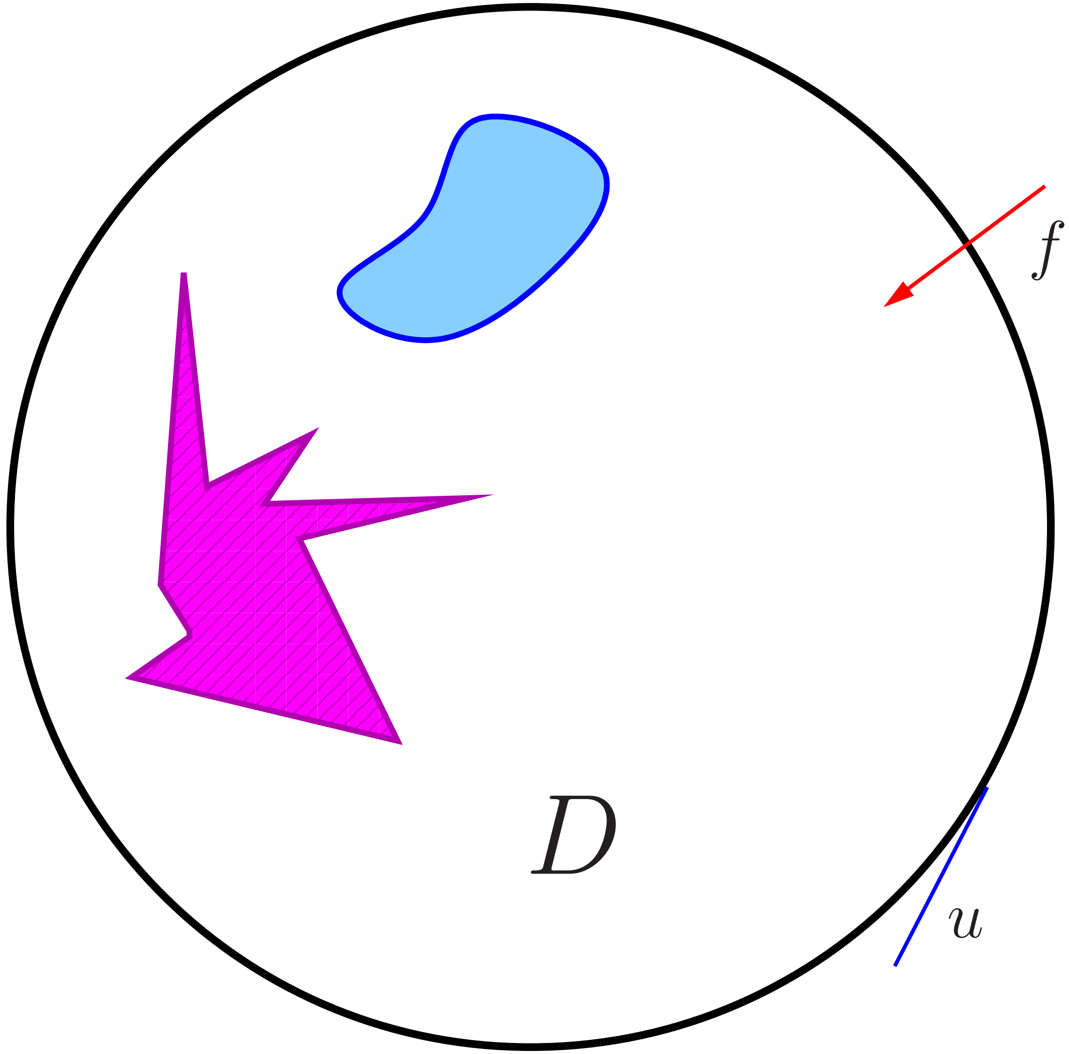
Tuesday (2 hours): Theory and algorithmic implementation of the factorization method.

Thursday (2 hours): Numerical examples for the factorization method. Theory of the convex source support algorithm.

Friday (1 hour): Algorithmic implementation and numerical examples for the convex source support algorithm.

1. EIT with inclusions

Idealized EIT measurements with inclusions



Neumann-to-Dirichlet maps

Let $D \subset \mathbb{R}^n$, $n = 2$ or 3, be a simply connected domain with a conductivity $\sigma \in L^\infty(D)$, $\sigma > c_0 > 0$, such that $\overline{\Omega} := \operatorname{supp}(\sigma - 1)$ is a compact subset of D. We consider the Neumann problem

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D$$

where $f \in L^2_\diamond(\partial D) := \left\{ f \in L^2(\partial D) \mid \langle f, 1 \rangle = 0 \right\}$ is the applied boundary current density and ν is the exterior unit normal. These equations define the electromagnetic potential $u \in H^1(D)/\mathbb{R}$ uniquely.

The Neumann-to-Dirichlet, or current-to-voltage, map

$$\Lambda : f \mapsto u|_{\partial D}, \quad L^2_\diamond(\partial D) \to L^2_\diamond(\partial D),$$

is bounded, compact and self-adjoint. Note that we constantly identify $L^2_\diamond(\partial D)$ with $L^2(\partial D)/\mathbb{R}$ by choosing the ground level appropriately.

Similarly, we introduce the 'background' Neumann-to-Dirichlet map

$$\Lambda_0 : f \mapsto u_0|_{\partial D}, \quad L^2_\diamond(\partial D) \to L^2_\diamond(\partial D),$$

where $u_0 \in H^1(D)/\mathbb{R}$ is the unique solution of

$$\Delta u_0 = 0 \quad \text{in } D, \qquad \frac{\partial u_0}{\partial \nu} = f \quad \text{on } \partial D$$

for $f \in L^2_\diamond(\partial D)$.

Take note that the relative Neumann-to-Dirichlet map

$$\Lambda_0 - \Lambda : f \mapsto (u_0 - u)|_{\partial D}$$

is infinitely smoothening, i.e., $(\Lambda_0 - \Lambda)f = u_0 - u$ belongs to $C^\infty_\diamond(\partial D)$ for any $f \in L^2_\diamond(\partial D)$.

The task in hand

Throughout this course, the aim is to extract constructive information on the inhomogeneity, i.e., on the set $\overline{\Omega} = \mathrm{supp}(\sigma - 1)$, from (partial and noisy information on) the relative boundary map $\Lambda_0 - \Lambda$.

1.1 Two monotonicity lemmas

The lemmas and their proofs

Lemma. *Assume that* σ_1 *and* σ_2 *are feasible conductivities and such that* $\sigma_1 \leq \sigma_2$*. Then the corresponding relative Neumann-to-Dirichlet operator* $\Lambda_1 - \Lambda_2$ *is positive semi-definite, i.e.,*

$$\langle f, (\Lambda_1 - \Lambda_2) f \rangle_{L^2(\partial D)} \geq 0$$

for all $f \in L^2_\diamond(\partial D)$.

Proof. According to the fundamental variational theory of elliptic partial differential equations, the electromagnetic potential $u_1 \in H^1(D)/\mathbb{R}$ corresponding to σ_1 and a nonzero current density $f \in L^2_\diamond(\partial D)$ is the unique solution of the variational equation

$$\int_D \sigma_1 \nabla u_1 \cdot \nabla v \, dx = \int_{\partial D} f v \, dx \qquad \text{for all } v \in H^1(D)/\mathbb{R}, \tag{1}$$

as well as the unique minimizer of the energy functional

$$\frac{1}{2} \int_D \sigma_1 |\nabla v|^2 \, dx - \int_{\partial D} f v \, dx$$

in $H^1(D)/\mathbb{R}$. The corresponding minimum value is

$$\frac{1}{2} \int_D \sigma_1 |\nabla u_1|^2 \, dx - \int_{\partial D} f u_1 \, dx = -\frac{1}{2} \int_{\partial D} f u_1 \, dx = -\frac{1}{2} \int_{\partial D} f \, \Lambda_1 f \, dx$$

due to (1). (The above conclusions remain valid if σ_1, u_1 and Λ_1 are replaced by σ_2, u_2 and Λ_2, respectively.)

In consequence,

$$
\begin{aligned}
-\frac{1}{2}\int_{\partial D} f\,\Lambda_1 f\,dx &= \frac{1}{2}\int_D \sigma_1|\nabla u_1|^2\,dx - \int_{\partial D} f u_1\,dx \\
&\leq \frac{1}{2}\int_D \sigma_1|\nabla u_2|^2 dx - \int_{\partial D} f u_2\,dx \\
&\leq \frac{1}{2}\int_D \sigma_2|\nabla u_2|^2 dx - \int_{\partial D} f u_2\,dx \\
&= -\frac{1}{2}\int_{\partial D} f\,\Lambda_2 f\,dx,
\end{aligned}
$$

which proves the claim. $\square$

Lemma. *Assume that σ_1 and σ_2 are as in the previous lemma and let σ_0 be yet another feasible conductivity. If $\sigma_0 \leq \sigma_1$,*

$$\mathcal{R}\left((\Lambda_0 - \Lambda_1)^{1/2}\right) \subseteq \mathcal{R}\left((\Lambda_0 - \Lambda_2)^{1/2}\right).$$

Conversely, if $\sigma_2 \leq \sigma_0$, it holds that

$$\mathcal{R}\left((\Lambda_2 - \Lambda_0)^{1/2}\right) \subseteq \mathcal{R}\left((\Lambda_1 - \Lambda_0)^{1/2}\right).$$

Proof. First of all, the above square roots are well-defined because the corresponding operators are positive semi-definite, self-adjoint and compact (see the previous lemma).

A functional analytic lemma that is frequently used for the factorization method is that for any continuous linear operator $A : H_1 \to H_2$, between Hilbert spaces H_1 and H_2,

$$y \in R(A) \quad \text{iff} \quad \exists C > 0 : \ \langle y, x \rangle_{H_2} \leq C \, ||A^* x||_{H_1} \quad \forall x \in H_2.$$

An immediate consequence for self-adjoint operators $A, B : H_1 \to H_1$ is that the existence of a constant $C > 0$ satisfying

$$||Ax|| \leq C\,||Bx|| \qquad \text{for all } x \in H_1$$

implies that $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

Let $\sigma_0 \leq \sigma_1 \leq \sigma_2$. The previous lemma tells us that for any $f \in L^2_\diamond(\partial D)$

$$\begin{aligned} \langle f, (\Lambda_0 - \Lambda_1) f \rangle &= \langle f, (\Lambda_0 - \Lambda_2) f \rangle - \langle f, (\Lambda_1 - \Lambda_2) f \rangle \\ &\leq \langle f, (\Lambda_0 - \Lambda_2) f \rangle \,, \end{aligned}$$

meaning that $\left|\left|(\Lambda_0 - \Lambda_1)^{1/2} f\right|\right| \leq \left|\left|(\Lambda_0 - \Lambda_2)^{1/2} f\right|\right|$ and, thus,

$$\mathcal{R}\left((\Lambda_0 - \Lambda_1)^{1/2}\right) \subseteq \mathcal{R}\left((\Lambda_0 - \Lambda_2)^{1/2}\right).$$

Since the second part of the assertion follows from the same line of reasoning, the proof is complete. $\square$

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Second and third lectures, June 29, 2010.

2. Factorization method

2.1 The factorization

A simple setting

In this subsection, it is assumed that the conductivity σ is of the form

$$\sigma = \begin{cases} 1+\kappa & \text{in } \Omega, \\ 1 & \text{in } D \setminus \overline{\Omega}, \end{cases}$$

where $\kappa > 0$ is a real number and Ω is a nonempty, simply connected and smooth domain such that $\overline{\Omega} \subset D$.

Take note that the results presented below are, actually, valid for less regular domains Ω and for variable and irregular κ — as well as for less regular D. However, our aim is to combine results for this simple framework with the monotonicity arguments of Section 1.1 to obtain an even stronger final theorem.

Three auxiliary operators

Let us introduce three auxiliary operators.

(i) To begin with, consider the boundary value problem

$$\Delta v = 0 \quad \text{in } D \setminus \overline{\Omega}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial D \qquad \frac{\partial v}{\partial \nu} = \phi \quad \text{on } \partial \Omega,$$

where the unit normals point out of $D \setminus \overline{\Omega}$. For $\phi \in H_\diamond^{-1/2}(\partial\Omega)$, this problem has a unique solution $v \in H^1(D \setminus \overline{\Omega})/\mathbb{R}$, and thus it follows from the trace theorem that the operator

$$L : \phi \mapsto v|_{\partial D}, \quad H_\diamond^{-1/2}(\partial\Omega) \to L_\diamond^2(\partial D)$$

is well-defined and bounded — actually, L is infinitely smoothening and, in particular, compact.

(ii) The dual operator of L is given by

$$L^* : \phi^* \mapsto v^*|_{\partial\Omega}, \quad L^2_\diamond(\partial D) \to H^{1/2}_\diamond(\partial\Omega),$$

where $v^* \in H^1(D \setminus \overline{\Omega})/\mathbb{R}$ is the unique solution of

$$\Delta v^* = 0 \quad \text{in } D \setminus \overline{\Omega}, \qquad \frac{\partial v^*}{\partial \nu} = \phi^* \quad \text{on } \partial D, \qquad \frac{\partial v^*}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

for $\phi^* \in L^2_\diamond(\partial D)$.

Indeed, by Green's formula,

$$\begin{aligned}
\langle \phi^*, L\phi \rangle_{\partial D} &= \langle \frac{\partial v^*}{\partial \nu}, v \rangle_{\partial D} \\
&= \int_{D \setminus \overline{\Omega}} \nabla v^* \cdot \nabla v \, dx - \langle \frac{\partial v^*}{\partial \nu}, v \rangle_{\partial\Omega} \\
&= \langle \frac{\partial v}{\partial \nu}, v^* \rangle_{\partial\Omega} + \langle \frac{\partial v}{\partial \nu}, v^* \rangle_{\partial D} = \langle \phi, L^* \phi^* \rangle_{\partial\Omega},
\end{aligned}$$

which proves the claim.

(iii) Finally, let $\psi \in H^{1/2}_{\diamond}(\partial\Omega)$ and consider the problem

$$\Delta w = 0 \quad \text{in } D \setminus \partial\Omega, \qquad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial D,$$
$$\kappa \frac{\partial w}{\partial \nu}^{+} - \frac{\partial w}{\partial \nu}^{-} = 0, \quad w^{+} - w^{-} = \psi \quad \text{on } \partial\Omega,$$

where the superscripts $+$ and $-$ denote traces taken from within Ω and $D \setminus \overline{\Omega}$, respectively. Such a transmission problem has a unique solution w in $(H^1(\Omega) \oplus H^1(D \setminus \overline{\Omega}))/\mathbb{R}$.

The third auxiliary operator is defined via

$$F : \psi \mapsto \left. \frac{\partial(w_0 - w)}{\partial \nu} \right|^{-}_{\partial\Omega}, \quad H^{1/2}_{\diamond}(\partial\Omega) \to H^{-1/2}_{\diamond}(\partial\Omega),$$

where $w_0 \in (H^1(\Omega) \oplus H^1(D \setminus \overline{\Omega}))/\mathbb{R}$ is the solution of the above transmission problem when κ is replaced by 1.

Lemma. *The operator* $F : H_\diamond^{1/2}(\partial\Omega) \to H_\diamond^{-1/2}(\partial\Omega)$ *is an isomorphism. Furthermore,* F *is positive definite and allows a decomposition*

$$F = GG^*,$$

where $G : L_\diamond^2(\partial\Omega) \to H_\diamond^{-1/2}(\partial\Omega)$ *is also an isomorphism.*

Proof. Brühl 2001, Lemma 3.3 and Section 3.2. $\square$

The factorization of the factorization method

Theorem. *The relative Neumann to Dirichlet map* $\Lambda_0 - \Lambda$ *can be factored as*

$$\Lambda_0 - \Lambda \;=\; LFL^* = LGG^*L^* = LG(LG)^*.$$

Proof. Brühl 2001, Lemma 3.2. $\square$

During the rest of the considerations on the factorization method, it is important to bear in mind

- the general form of the above factorization,
- the definition of the operator L, and
- the fact that $G : L^2_\diamond(\partial\Omega) \to H_\diamond^{-1/2}(\partial\Omega)$ is an isomorphism.

2.2 A range test

A range identity

The following theorem composes the core of the factorization method.

Theorem: *Assume that the conductivity is as in the previous section. Then, it holds that*

$$\mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right) = \mathcal{R}(L).$$

Proof. First of all, the square root $(\Lambda_0 - \Lambda)^{1/2} : L^2_\diamond(\partial D) \to L^2_\diamond(\partial D)$ is well-defined because the original operator is positive definite, self-adjoint and compact. Furthermore, it follows easily — e.g., by using the singular value decomposition of LG — that

$$\mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right) = \mathcal{R}\left((LG(LG)^*)^{1/2}\right) = \mathcal{R}(LG) = \mathcal{R}(L)$$

because G is surjective.

The remarkable feature of this theorem is that the range of $(\Lambda_0 - \Lambda)^{1/2}$ *is independent of the (constant) conductivity inside the inclusion* Ω*!*

This property can be utilized constructively, e.g., by introducing a family of dipole potentials $\{\Phi_y\}_{y \in D}$ satisfying

$$\Delta_x \Phi_y(x) = \alpha \cdot \nabla_x \delta(x - y), \quad x \in D, \qquad \frac{\partial \Phi_y}{\partial \nu} = 0 \quad \text{on } \partial D,$$

where the dipole moment $0 \neq \alpha \in \mathbb{R}^n$ and the location of the electromagnetic dipole, $y \in D$, are free parameters. We denote the Dirichlet trace of Φ_y on ∂D by ϕ_y.

Notice that Φ_y is smooth away from y and has a singularity of strength

$$\frac{1}{|x - y|^{n-1}}$$

at $y \in D$.

The range test

Theorem. *Assume that σ is of the simple piecewise constant form introduced in Section 2.1. Then, the inclusion Ω has the following characterization:*

$$y \in \Omega \quad \Longleftrightarrow \quad \phi_y \in \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right).$$

Proof. Assume first that $y \in \Omega$. Then, it is easy to see that

$$\phi_y \;=\; L\left(\frac{\partial \Phi_y}{\partial \nu}|_{\partial\Omega}\right),$$

which means that $\phi_y \in \mathcal{R}(L) = \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right)$.

Suppose next that $\phi_y \in \mathcal{R}(L) = \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right)$ for some $y \in D \setminus \Omega$. Then, according to the definition of L, there exists $v \in H^1(D \setminus \overline{\Omega})$ satisfying

$$\Delta v = 0 \quad \text{in } D \setminus \overline{\Omega}, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial D, \qquad v = \phi_y \quad \text{on } \partial D.$$

In particular, such v has the same Cauchy data as Φ_y on ∂D, and it thus follows from the principle of unique continuation for the Laplacian that

$$v \;=\; \Phi_y \quad \text{in } (D \setminus \overline{\Omega}) \setminus \{y\},$$

which is a contradiction since Φ_y has a relatively strong singularity at y.
$\square$

2.3 Generalized range test

A less simple setting

In this subsection, it is still assumed that the conductivity σ is of the form

$$\sigma = \begin{cases} 1+\kappa & \text{in } \Omega, \\ 1 & \text{in } D \setminus \overline{\Omega}. \end{cases}$$

However, now we only require that Ω is open, $\overline{\Omega} \subset D$ and $D \setminus \overline{\Omega}$ is connected, and that for each $y \in \Omega$ there exist scalar constants $\epsilon_y, r_y > 0$ such that $\kappa \in L^\infty(\Omega)$ satisfies

$$\kappa > \epsilon_y \qquad \text{almost everywhere in } B(y, r_y) \subset D, \tag{2}$$

where $B(y, r_y)$ denotes the open ball of radius r_y centered at y.

Generalized range test

Theorem. *Assume that σ is of the form introduced above. Then, the inclusion Ω has the following characterization (modulo $\partial\Omega$):*

$$y \in \Omega \quad \Longrightarrow \quad \phi_y \in \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right)$$

and

$$y \in D \setminus \overline{\Omega} \quad \Longrightarrow \quad \phi_y \notin \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right).$$

Proof. Let $y \in \Omega$ and note that by assumption there exist scalars $\epsilon_y, r_y > 0$ such that $\kappa > \epsilon_y$ almost everywhere in $B(y, r_y) \subset \Omega$. We define an auxiliary conductivity by

$$\sigma_y = \begin{cases} 1 + \epsilon_y & \text{in } B(y, r_y), \\ 1 & \text{in } D \setminus \overline{B(y, r_y)}, \end{cases}$$

and denote the associated Neumann-to-Dirichlet map by Λ_y. The range test in the "simple setting" indicates that $\phi_y \in \mathcal{R}\{(\Lambda_0 - \Lambda_y)^{1/2}\}$. Furthermore, since $\sigma_y < \sigma$, it follows from the second monotonicity lemma of Section 2.1 that also

$$\phi_y \in \mathcal{R}\left\{(\Lambda_0 - \Lambda)^{1/2}\right\}.$$

Next, let $y \in D \setminus \overline{\Omega}$. Since $D \setminus \overline{\Omega}$ is open and connected, there exists a simply connected open set Ω_y such that $y \notin \Omega_y$, $\Omega \subset \Omega_y$, $D \setminus \overline{\Omega}_y$ is connected and $\partial\Omega_y$ is smooth. We redefine the auxiliary conductivity by

$$\sigma_y = \begin{cases} 1 + k & \text{in } \Omega_y, \\ 1 & \text{in } D \setminus \overline{\Omega}_y, \end{cases}$$

where the scalar constant $k > 0$ is chosen so that $\sigma_y > \sigma$ almost everywhere in Ω. Now, it follows from the range test in the simple setting and the monotonicity relation for the ranges that

$$\phi_y \notin \mathcal{R}\left\{(\Lambda_0 - \Lambda_y)^{1/2}\right\} \supseteq \mathcal{R}\left\{(\Lambda_0 - \Lambda)^{1/2}\right\},$$

where Λ_y is again the Neumann-to-Dirichlet map corresponding to σ_y. $\square$

2.4 Algorithmic implementation

Picard criterion

There are several ways to numerically implement the range test introduced above. However, the most successful algorithms are arguably based on the so-called Picard criterion:

Assume the setting of Section 2.3. Since $\Lambda_0 - \Lambda : L^2_\diamond(\partial D) \to L^2_\diamond(\partial D)$ is a compact, self-adjoint and positive definite operator, it has an orthonormal basis of eigenfunctions $\{\varphi_k\}_{k=1}^\infty \subset L^2_\diamond(\partial D)$ and corresponding positive eigenvalues $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{R}_+$ (arranged in descending order) such that

$$(\Lambda_0 - \Lambda)\varphi_k \;=\; \lambda_k \varphi_k, \qquad k = 1, 2, \dots \; .$$

The square root $(\Lambda_0 - \Lambda)^{1/2}$ has the same eigenfunctions as the original operator and the eigenvalues $\{\sqrt{\lambda_k}\}_{k=1}^\infty$.

According to the Picard criterion, the following equivalence holds:

$$\phi_y \in \mathcal{R}\left((\Lambda_0 - \Lambda)^{1/2}\right) \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} \frac{\langle \phi_y, \varphi_k \rangle^2_{L^2(\partial D)}}{\lambda_k} < \infty. \qquad (3)$$

This can be proved in a straightforward manner: Just solve the equation

$$(\Lambda_0 - \Lambda_y)^{1/2} f \;=\; \phi_y$$

formally using the above introduced eigensystem, and then note that the squared norm of the obtained formal solution equals the series on the right-hand side of (3).

Practical issues

Naturally, real-life measurements — or even numerical simulations — do not provide enough information to carry out the test on the right-hand side of (3) exactly:

In practice, one is forced to work with some kind of a finite-dimensional and noisy approximation of $\Lambda_0 - \Lambda$, which can be assumed to be presented as a symmetric matrix $A \in \mathbb{R}^{m\times m}$, $m \in \mathbb{N}$, with respect to some suitable orthonormal (incomplete) basis of $L^2_\diamond(\partial D)$.

Moreover, the boundary potentials $\{\phi_y\}_{y\in D}$ can be given explicitly only in some simple geometries, and thus one is typically forced to work with inaccurate test dipoles. (The computational cost of approximating ϕ_y also depends heavily on the geometry.)

Numerical implementation

Let $\{v_k\}_{k=1}^m \subset \mathbb{R}^m$ and $\{\mu_k\}_{k=1}^m \subset \mathbb{R}$, respectively, be the eigenvectors and eigenvalues (in descending order) of the finite-dimensional matrix approximation $A \in \mathbb{R}^{m\times m}$, and assume that $\{h_y\}_{y\in Z} \subset \mathbb{R}^m$ are the available approximations of $\{\phi_y\}_{y\in Z}$ in the same basis with respect to which A is given. Here, $Z \subset D$ is some finite grid of test points .

Instead of the infinite series in (3), we are forced to consider the finite-dimensional analogue

$$I(y) = \sum_{k=1}^{m_0} \frac{(h_y \cdot v_k)^2}{\mu_k}, \quad y \in Z. \tag{4}$$

Notice that, in general, it is not reasonable to choose the upper limit m_0 to be the dimension of the matrix A, i.e., m. Indeed, because the eigenvalues of $\Lambda_0 - \Lambda$ converge to zero, for 'large' k the values $1/\lambda_k$ and $1/\mu_k$ can differ arbitrarily much even without any measurement noise.

The choice of an appropriate cut-off index m_0 is a subtle issue and will not be considered here more thoroughly.

After choosing m_0, i.e., the number of 'reliable' eigenvectors and eigenvalues of A, one can, e.g., plot the function

$$\mathrm{Ind}(y) \;=\; \frac{1}{I(y)}.$$

Intuitively, Ind should attain 'large' values inside the inhomogeneity Ω and 'small' values in its exterior.

Another successful technique is to use the 'reliable' eigenvectors and eigenvalues of A to apply a logarithmic regression model to both the numerators and denominators of the terms in the series (4), i.e.,

$$2\log|h_y \cdot v_k| \approx ak + b, \qquad \log \mu_k \approx ck + d, \qquad a, b, c, d \in \mathbb{R}.$$

Under the assumption that these approximations are feasible, one can postulate that the original test series converges if and only if $a < c$.

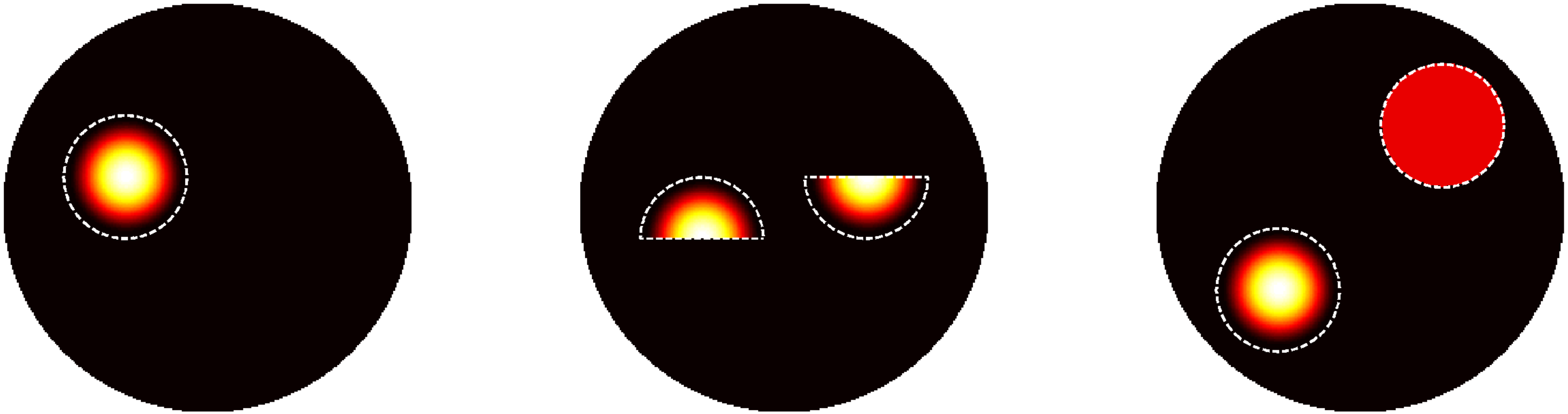

Figure 1: Exact conductivities for the three test cases.

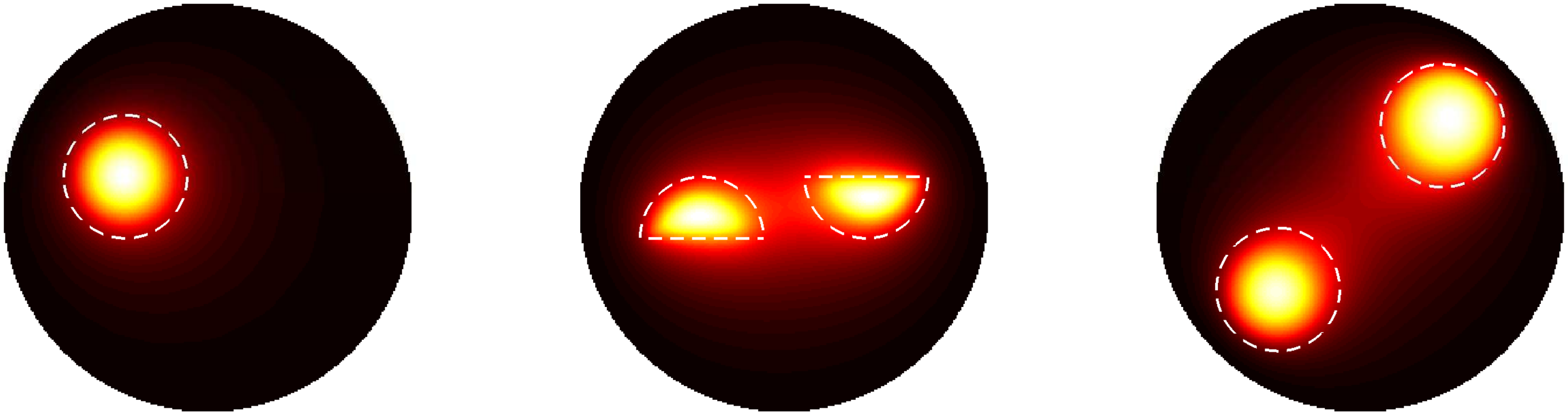

Figure 2: Numerical reconstructions for exact simulated data.

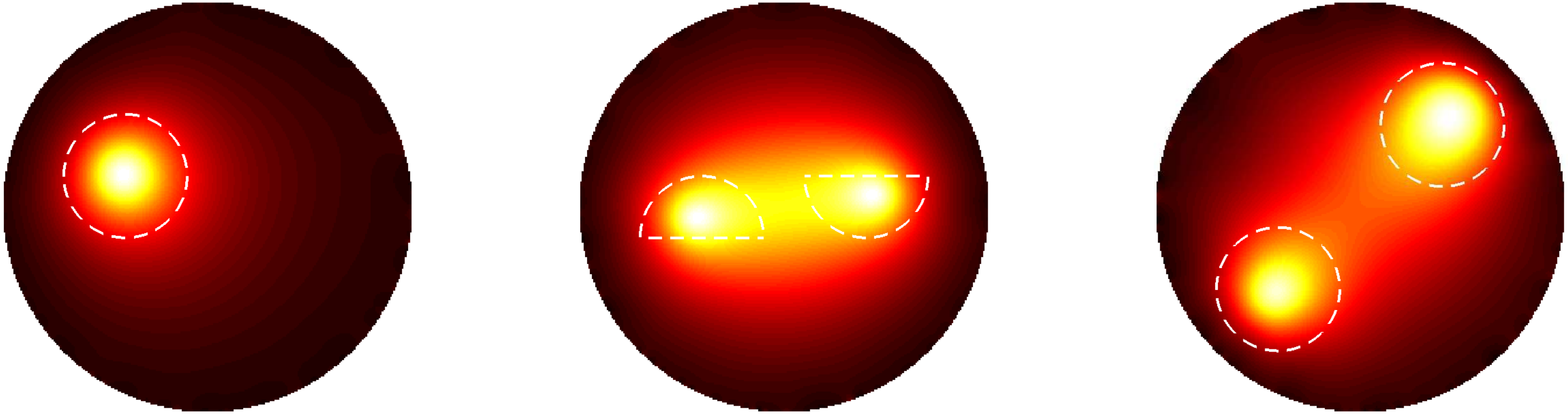

Figure 3: Numerical reconstructions for noisy simulated data.

Factorization and source support methods for electrical impedance tomography

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2.5 Concluding remarks on the factorization method

Auxiliary remarks

- The above considerations can be generalized, e.g., to the cases of
  - nonconstant but known background conductivity,
  - inclusions that are less conductive than the background,
  - inclusion that have 'holes' in them, and
  - measurements only on some subset of the boundary (Brühl and Hanke 2003).
- The choice of the upper limit in the Picard criterion and the regularization properties of the factorization method have been considered rigorously by Lechleiter (2006).
- The factorization algorithm has been implemented in the framework of — more or less — realistic electrode models of EIT by Brühl, Hakula, Hanke, H, Lechleiter etc..

- In my opinion, the factorization method has two obvious weaknesses:
  - All inclusions must — at least according to the current theory — be either more or less conductive than the background.
  - In principle, the method requires the knowledge of the full Neumann-to-Dirichlet map (at least, on some open and nonempty part of the boundary).

3 Convex source support algorithm

Motivation and basic ideas

- The *convex source support* (CSS) algorithm is a noniterative method for localizing sources in electrostatics.
- It can also be used for inclusion detection in EIT with only one pair of current density and boundary potential as the measurement data.
- The background theory for the CSS algorithm is independent of the spatial dimension $n \geq 2$, but the corresponding reconstruction algorithm is (currently) based on tools of complex analysis and is thus inherently two-dimensional.

3.1 Convex source support in electrostatics

An inverse source problem

Let $D \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth enough boundary and consider the Poisson problem

$$\Delta v = F \quad \text{in } D, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial D,$$

where $F \in \mathcal{E}'_\diamond(D) := \{g \in \mathcal{E}'(D) \mid \langle g, 1\rangle = 0\}$ is a compactly supported mean-free distribution. Such a source problem has a unique solution $v \in \cup_s H^s(D)/\mathbb{R}$, which is smooth away from the source F.

We define a linear 'measurement map' by

$$L : F \mapsto v|_{\partial D}, \quad \mathcal{E}'_\diamond(D) \to L^2_\diamond(\partial D).$$

Our inverse problem: Let $F_0 \in \mathcal{E}'_\diamond(D)$ be an unknown source. Extract information on $\overline{\Omega} := \operatorname{supp} F_0$ from the measurement $g := LF_0$.

Convex source support

For $g \in \mathcal{R}(L)$, the CSS is defined to be (Hanke, H, Lehn, and Reusswig 2008)

$$\mathcal{C}g := \bigcap_{LF=g} \mathrm{supp}_c F,$$

where $\mathrm{supp}_c F$ denotes the convex hull of the support of F.

The idea of the CSS originates from the works of Kusiak, Sylvester and their co-authors in the framework of inverse scattering.

Main property of the convex source support

Theorem. *Let $g \in \mathcal{R}(L)$. Then, given any $\epsilon > 0$, there exists a source $F_\epsilon \in \mathcal{E}'_\diamond(D)$ such that $LF_\epsilon = g$ and*

$$\mathcal{C}g \subset \mathrm{supp}_c F_\epsilon \subset \overline{N_\epsilon(\mathcal{C}g)}.$$

Moreover, $\mathcal{C}g = \emptyset$ if and only if $g = 0$.

Proof. For simplicity, let us assume that D is convex.

Suppose that $\mathcal{C}g \neq \emptyset$. Then, if we fix an arbitrary $\epsilon > 0$ such that $N_\epsilon(\mathcal{C}g) \subset D$, a simple compactness argument shows that we can find a finite number $F_1, \ldots, F_m$ of sources satisfying $LF_1, \ldots, LF_m = g$ and

$$C := \bigcap_{k=1,\ldots,m} \mathrm{supp}_c F_k \subset N_\epsilon(\mathcal{C}g).$$

For each $k = 1, \ldots, m$, there exists a harmonic function v_k that solves

$$\Delta v_k = 0 \quad \text{in } D \setminus \operatorname{supp}_c F_k, \qquad v_k = g \quad \text{and} \quad \frac{\partial v_k}{\partial \nu} = 0 \quad \text{on } \partial D.$$

Since $\operatorname{supp}_c F_k$, $k = 1, \ldots, n$, are convex sets, the principle of unique continuation shows that any two of the functions v_k coincide in the subset of D where both are harmonic, and all can be extended to the same (harmonic) function v that solves the above Cauchy problem with $D \setminus \operatorname{supp}_c F_k$ replaced by $D \setminus C \supset D \setminus N_\epsilon(\mathcal{C}g)$.

Thus, the source $F = \Delta v_\epsilon \in \mathcal{E}'_\diamond(D) \cap H^{-2}(D)$, with

$$v_\epsilon = \begin{cases} v & \text{in } D \setminus N_\epsilon(\mathcal{C}g), \\ 0 & \text{in } N_\epsilon(\mathcal{C}g), \end{cases}$$

satisfies $LF_\epsilon = g$ and $\operatorname{supp}_c F_\epsilon \subset \overline{N_\epsilon(\mathcal{C}g)}$.

The case when $\mathcal{C}g = \emptyset$ can be handled in a similar way. $\square$

Remark. *If* $\operatorname{supp}_{\mathrm{c}} F$ *is replaced by* $\operatorname{supp} F$ *in the definition of the CSS, the resulting intersection is empty. This may hold even if the holes in* $\operatorname{supp} F$ *are covered before the intersection is taken (Hanke, H, Lehn and Reusswig 2008).*

Extension to a disk

Let $B_\rho \subset \mathbb{R}^2$ be an open disk of radius $\rho > 0$ centered at the origin and enclosing $\overline{D}$. Consider the auxiliary Poisson problem

$$\Delta v_\rho = F \quad \text{in } B_\rho, \qquad \frac{\partial v_\rho}{\partial \nu} = 0 \quad \text{on } \partial B_\rho,$$

where $F \in \mathcal{E}'_\diamond(B_\rho)$ is a compactly supported mean-free distribution. Analogously to the original setting, we define a (virtual) measurement map by

$$L_\rho : F \mapsto v_\rho|_{\partial B_\rho}, \quad \mathcal{E}'_\diamond(B_\rho) \to L^2_\diamond(\partial B_\rho).$$

We set $g_\rho := L_\rho F_0$ and note that such ‘propagated data’ can be computed stably using the actual measurement g and a double layer potential:

$$g_\rho(x) = 2 \int_{\partial D} \frac{\partial \Phi(x-y)}{\partial \nu(y)} g(y)\, dS(y), \qquad x \in \partial B_\rho.$$

The convex source support corresponding to g_ρ is defined in the natural way:

$$\mathcal{C}g_\rho := \bigcap_{L_\rho F = g_\rho} \mathrm{supp}_c F.$$

Reconstructing $\mathcal{C}g_\rho$ is almost equivalent to reconstructing $\mathcal{C}g$:

- If D is convex, the sets $\mathcal{C}g$ and $\mathcal{C}g_\rho$ coincide. If not, $\mathcal{C}g_\rho$ may be a proper subset of $\mathcal{C}g$, but is still a nonempty subset of $\mathrm{supp}_\mathrm{c} F_0$ for a nonzero g. (In particular, $\mathcal{C}g_\rho$ is — a little counter intuitively — independent of ρ.)

Remark. *After g_ρ is computed for one ρ, it can be obtained practically for free for any larger ρ with the help of the fast Fourier transformation.*

2.2 Constructive identification of the CSS

The concentric case

Fix ρ, assume that we have access to the propagated data g_ρ and denote the (complex) Fourier coefficients of g_ρ with respect to the polar angle by $\{\alpha_j\}_{j=-\infty}^{\infty}$. Let us consider another open disk B_R of radius $0 < R < \rho$ centered at the origin.

If the Cauchy problem

$$\Delta w = 0 \quad \text{in } B_\rho \setminus \overline{B}_R, \qquad w = g_\rho \quad \text{and} \quad \frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial B_\rho,$$

has a solution, it can be written using the Fourier coefficients of g_ρ as

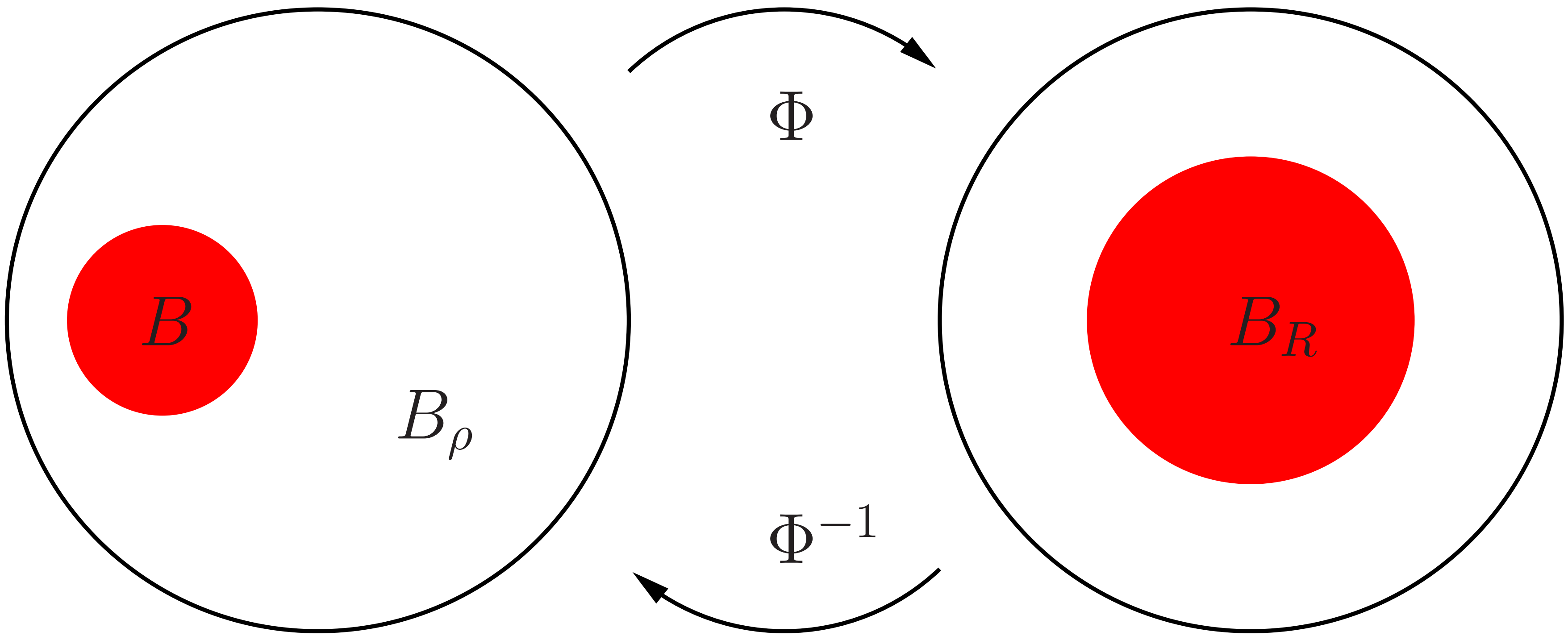
$$w(r,\theta) = \sum_{j=-\infty}^{\infty} \frac{\alpha_j}{2} \left(\left(\frac{r}{\rho}\right)^j + \left(\frac{\rho}{r}\right)^j \right) e^{\mathrm{i}j\theta}, \qquad (r,\theta) \in (R,\rho) \times (-\pi,\pi].$$

By checking when the above series representation of w converges, one sees relatively easily that $\mathcal{C}g_\rho \subset \overline{B}_R$ if and only if

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j|^2}{(R/\rho+\epsilon)^{2|j|}} < \infty$$

for all $\epsilon > 0$.

A useful Möbius transformation



The nonconcentric case

For any closed disk $B \subset B_\rho$, there exists a Möbius transformation Φ that maps $\overline{B}_\rho$ onto itself and B onto some disk $\overline{B}_R \subset B_\rho$ around the origin, with $R = R(B, \rho)$ uniquely determined by B and ρ. We denote by $\{\alpha_j(\Phi)\}_{j=-\infty}^{\infty}$ the Fourier coefficients of $g_\rho \circ \Phi^{-1}$ with respect to the polar angle.

Using the above concentric result and the fact that conformal maps interplay well with harmonic functions and homogeneous Neumann boundary conditions, we obtain the following characterization (Hanke, H, Lehn, Reusswig 2008): It holds that $\mathcal{C}g_\rho \subset B$ if and only if

$$\sum_{j=-\infty}^{\infty} \frac{|\alpha_j(\Phi)|^2}{(R/\rho + \epsilon)^{2|j|}} < \infty, \qquad (5)$$

for $R = R(B, \rho)$ and every $\epsilon > 0$.

Remark. *The above convergence test gives a means to construct* $\mathcal{C}g_\rho$ *because a convex set is uniquely determined by the disks enclosing it.*

Factorization and source support methods for electrical impedance tomography

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2.3 The CSS reconstruction algorithm

Algorithmic implementation (the *short* version)

The Fourier coefficients $\{\alpha_j(\Phi)\}$ show typically a geometric decay in $|j|$. Hence, we approximate

$$\log|\alpha_j(\Phi)| \approx a|j| + b, \qquad |j| \geq 1, \tag{6}$$

in the sense of least squares and postulate that the test series (5) converges if and only if

$$R \geq \rho e^a.$$

We carry out this test for a certain family of closed disks and approximate the CSS $\mathcal{C}g$ by the intersection of those disks that were found to enclose it. (Note that in practice the number of Fourier coefficients considered reliable for (6) depends on both Φ and the measurement noise level.)

Let us be a bit more precise:

Up to rotations of the image space, all Möbius transformations of $\overline{B}_\rho$ onto itself can be given as

$$\Phi_\zeta(z) \;=\; \rho^2 \frac{z-\zeta}{\rho^2 - \overline{\zeta} z},$$

where $\zeta \in B_\rho \subset \mathbb{C}$ is a free parameter that is mapped to the origin. Note that we have here identified the real plane $\mathbb{R}^2$ with its complex counterpart $\mathbb{C}$.

Let $R_\zeta = \rho e^{a_\zeta}$, where a_ζ is the slope obtained from the logarithmic regression model (6) in the case that $\Phi = \Phi_\zeta$. Under the courtesy of the assumption that the regression model is exact, it is easy to see that the closed disk $\Phi_\zeta^{-1}(\overline{B}_{R_\zeta})$ contains $\mathcal{C}g$ but the same is not true if R_ζ is replaced by any smaller radius.

Hence, we choose a set of (complex) test points $Z \subset B_\rho$ and approximate

$$\mathcal{C}g \approx \bigcap_{\zeta \in Z} \Phi_\zeta^{-1}(\overline{B}_{R_\zeta}).$$

2.3 Application to EIT

Locating inclusions using one measurement of EIT

(Re)consider the conductivity equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } D, \qquad \frac{\partial u}{\partial \nu} = f \quad \text{on } \partial D,$$

and assume that we can apply one current pattern $f \in L^2_\diamond(\partial D)$ and measure the corresponding boundary potential $u|_{\partial D} \in L^2_\diamond(\partial D)$. We continue denoting $\Omega := \operatorname{supp}(\sigma - 1)$, which is still assumed to be a compact subset of D.

Our old/new inverse problem: Extract information about Ω from a *single* measurement pair $(u|_{\partial D}, f)$ of EIT.

Interpretation as a source problem

As always, let u_0 be the solution of the conductivity equation corresponding to the unit background conductivity and the same boundary current density f as above, and denote $g := (u - u_0)|_{\partial D}$. It is easy to see that

$$g = LF_0,$$

where $F_0 = F_0(f, \sigma) = \Delta(u - u_0) = \Delta u$ is supported in Ω.

Our inclusion detection algorithm is based on reconstructing the convex source support $\mathcal{C}g$ — or more precisely the 'extended' one $\mathcal{C}g_\rho$.

(Notice that it is possible that $g = 0$, in which case the corresponding CSS is empty.)

2.4 Numerical examples with Matlab

2.5 Concluding remarks on the CSS algorithm

Auxiliary remarks

- The above considerations can be generalized, e.g., to the cases of
  - electrostatics in half-plane geometry (Harhanen, H, 2010),
  - the so-called backscatter data of EIT (Hanke, H, Reusswig, 2010), and
  - (simulated) electrode measurements of EIT (Hakula, H, 2008).
- Since the CSS corresponding to each measurement pair of EIT is contained in the convex hull of the inhomogeneity, one may obtain reconstructions displaying more information by taking the union of CSSs corresponding to different boundary measurement pairs.