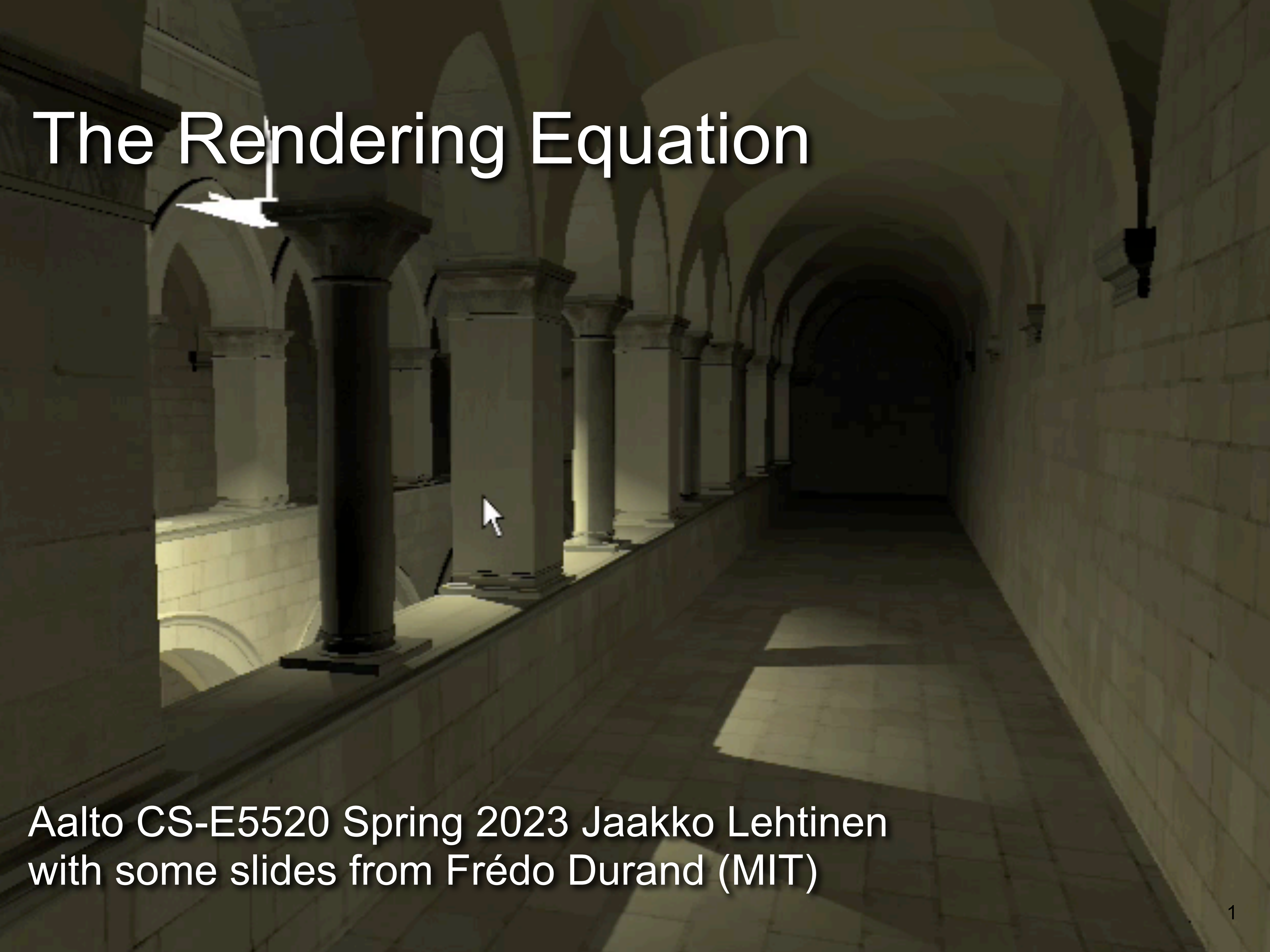


# The Rendering Equation

Aalto CS-E5520 Spring 2023 Jaakko Lehtinen
with some slides from Frédo Durand (MIT)

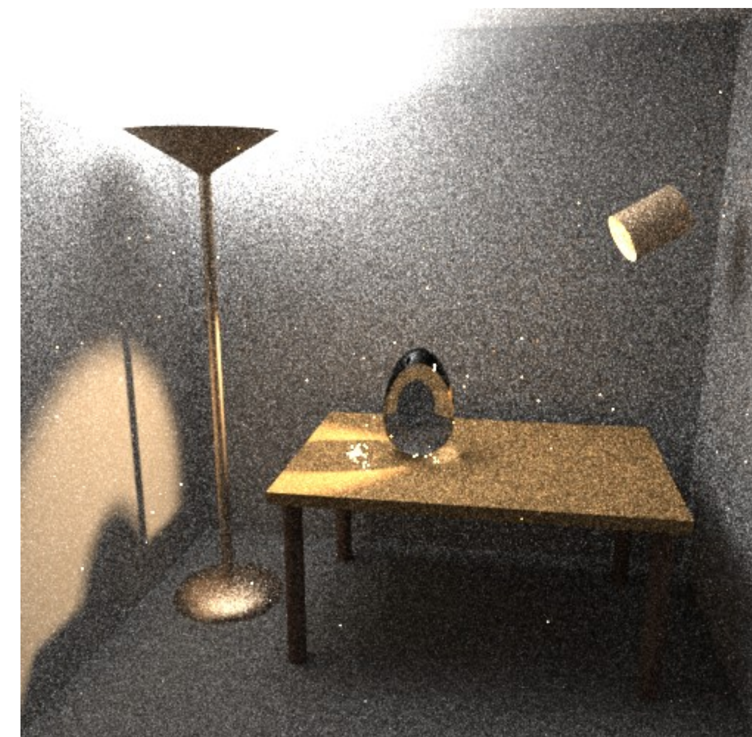
# Rendering <=>
# what is the radiance hitting my sensor?

Rendering <=>
what is the radiance hitting my sensor?

**What is the radiance hitting my sensor? <=>**
**Solution of the "rendering equation"**

# Today

- Global Illumination
  - Rendering Equation
  - Gets us indirect lighting
- Next time
  - Monte Carlo integration
  - Better sampling
    - importance
    - stratification



# Recap: Reflectance Equation

$$L(x \to \mathbf{v}) =$$

**outgoing radiance**

$$\int_\Omega f_r(x, \mathbf{l} \to \mathbf{v}) \, L(x \leftarrow \mathbf{l}) \, \cos\theta \, \mathrm{d}\mathbf{l}$$

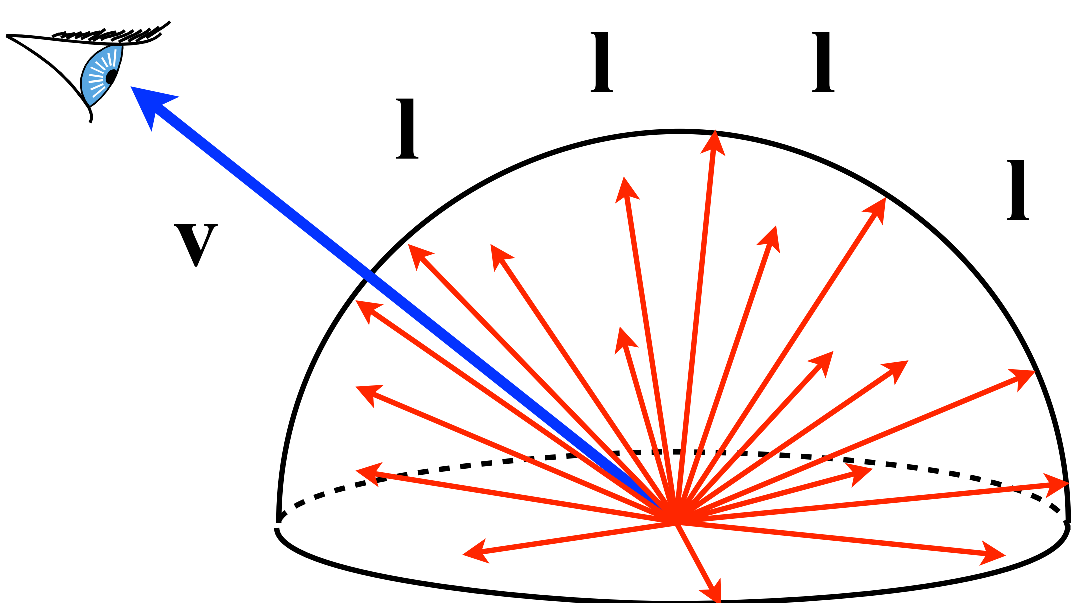
**L_in * cos = incident differential irradiance**

**integral over hemisphere**

**BRDF**

**incoming radiance**

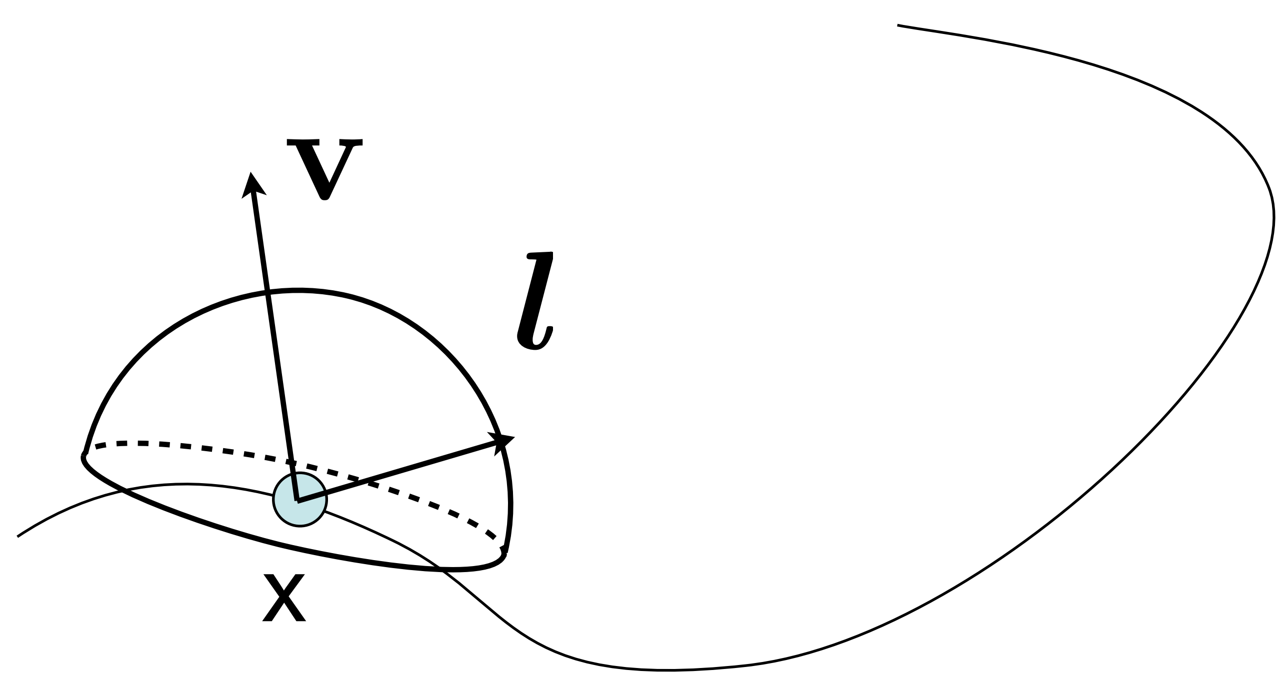
**cosine of incident angle**



# The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \,\mathrm{d}\mathbf{l}$$

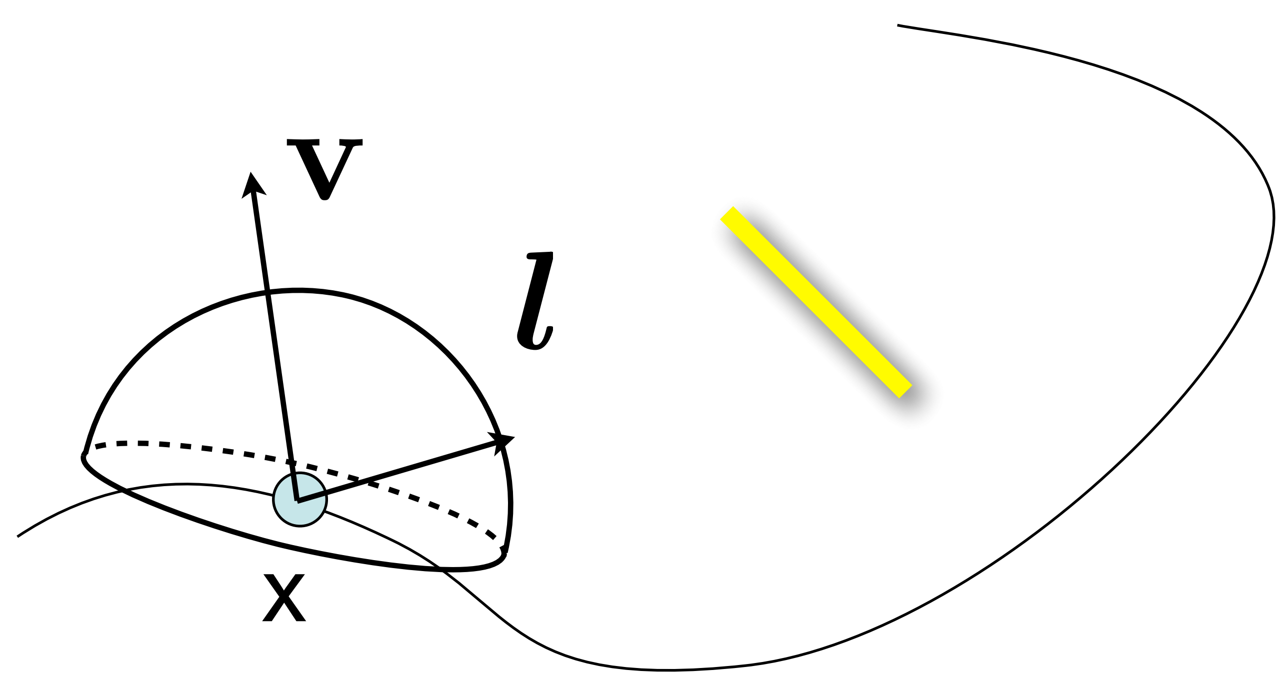
- Where does incident radiance L come from?



# The Way To Global Illumination

$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

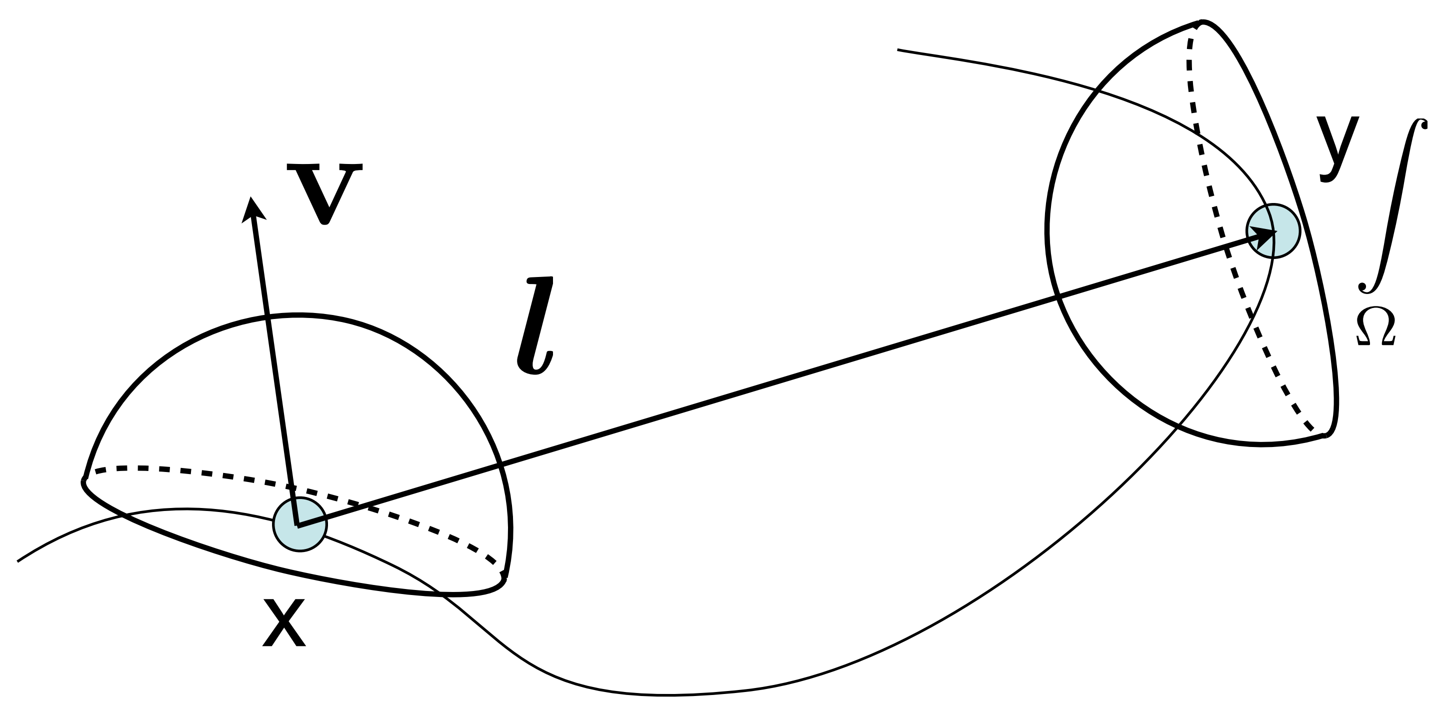
- Where does incident radiance L come from?
- Familiar case: From a light source, for certain **l**
  - But what about other incident directions?



# The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

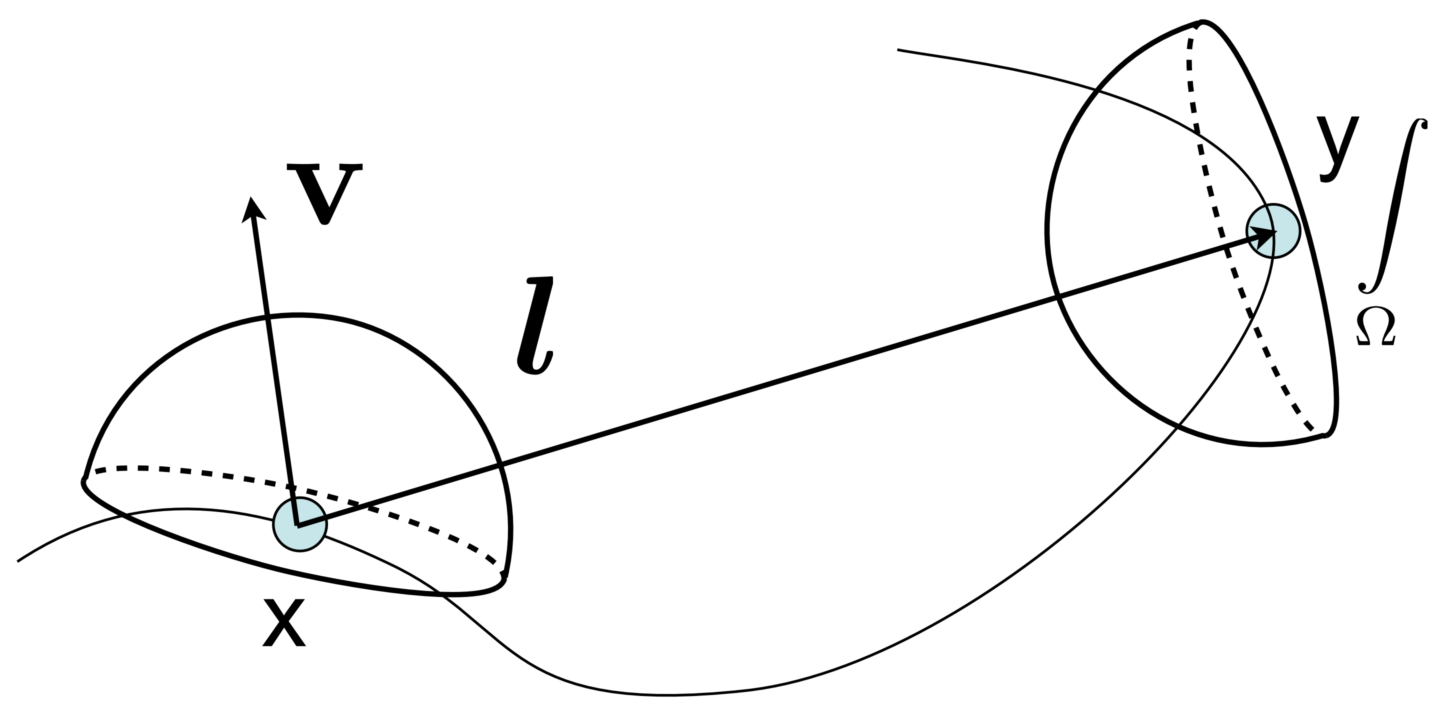
- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l***



# The Way To Global Illumination

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \,\mathrm{d}\mathbf{l}$$

- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l*** ==> must compute similar integral for every ***l***!
    - Recursive!

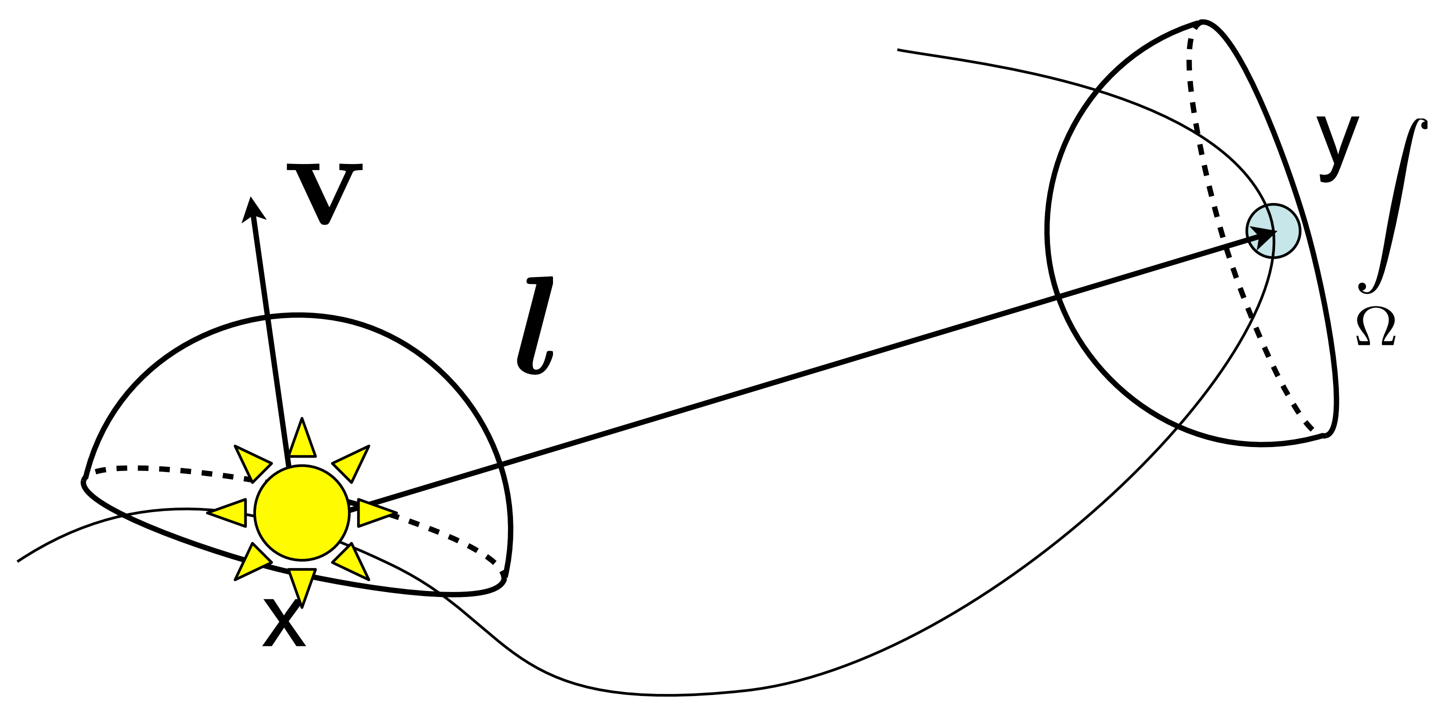


# Rendering Equation

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta\, \mathrm{d}\mathbf{l}$$

$$+ \; E(x \to \mathbf{v})$$

- Where does incident radiance L come from?
  - It is the light reflected towards *x* from the surface point *y* in direction ***l*** ==> must compute similar integral for every ***l***!
    - Recursive!
- ...and if *x* happens to be on a light source, we add its emitted contribution *E*



# The Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_{\Omega} L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

$$+ E(x \rightarrow \mathbf{v})$$

- Let's bask in its glory for a moment

# The Rendering Equation

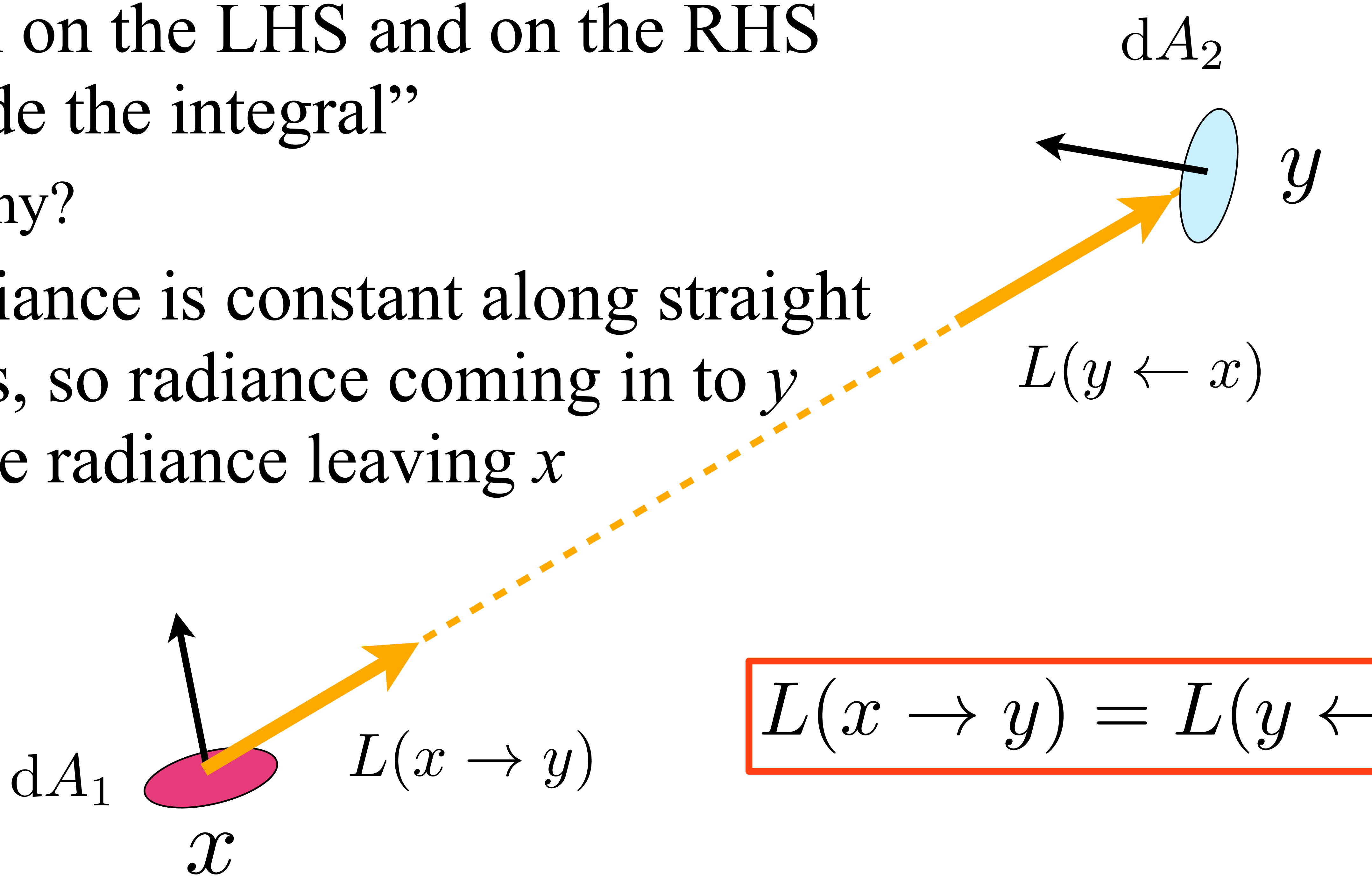
$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

$$+ \; E(x \to \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS (left-hand side) and on the RHS inside the integral

# Hmmh..

- “the unknown solution function L is both on the LHS and on the RHS inside the integral”
  - Why?
- Radiance is constant along straight lines, so radiance coming in to *y* is the radiance leaving *x*




$$L(x \rightarrow y) = L(y \leftarrow x)$$

# The Rendering Equation

$$L(x \to \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \to \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$
$$+ \; E(x \to \mathbf{v})$$

- The rendering equation describes the appearance of the scene, including direct and indirect illumination
  - An "integral equation", the unknown solution function L is both on the LHS and on the RHS inside the integral
    - More precisely: a "Fredholm equation of the 2nd kind"
  - Originally described by Kajiya and Immel et al. in 1986
  - **Take a class in Functional Analysis to learn more!**

# The Rendering Equation

- The unknown in this equation is the *function* $L(x \to \mathbf{v})$ defined for all points $x$ and all directions $\mathbf{v}$
- Analytic (exact) solution is impossible in all cases of practical interest
- Lots of ways to solve approximately
  - Monte Carlo techniques use random samples
  - Finite element methods (FEM) discretize using *basis functions*
    - Radiosity, wavelets, precomputed radiance transfer, etc.
  - Topic of next lecture!

# Questions?


Stack Studios, Rendered using Maxwell

# Next: Let’s Distill Things Down

- We’ll abstract the integrals away to see what is really going on: a linear system of equations
  - Infinitely many of them, though
  - Result: the *operator form* of the rendering equation

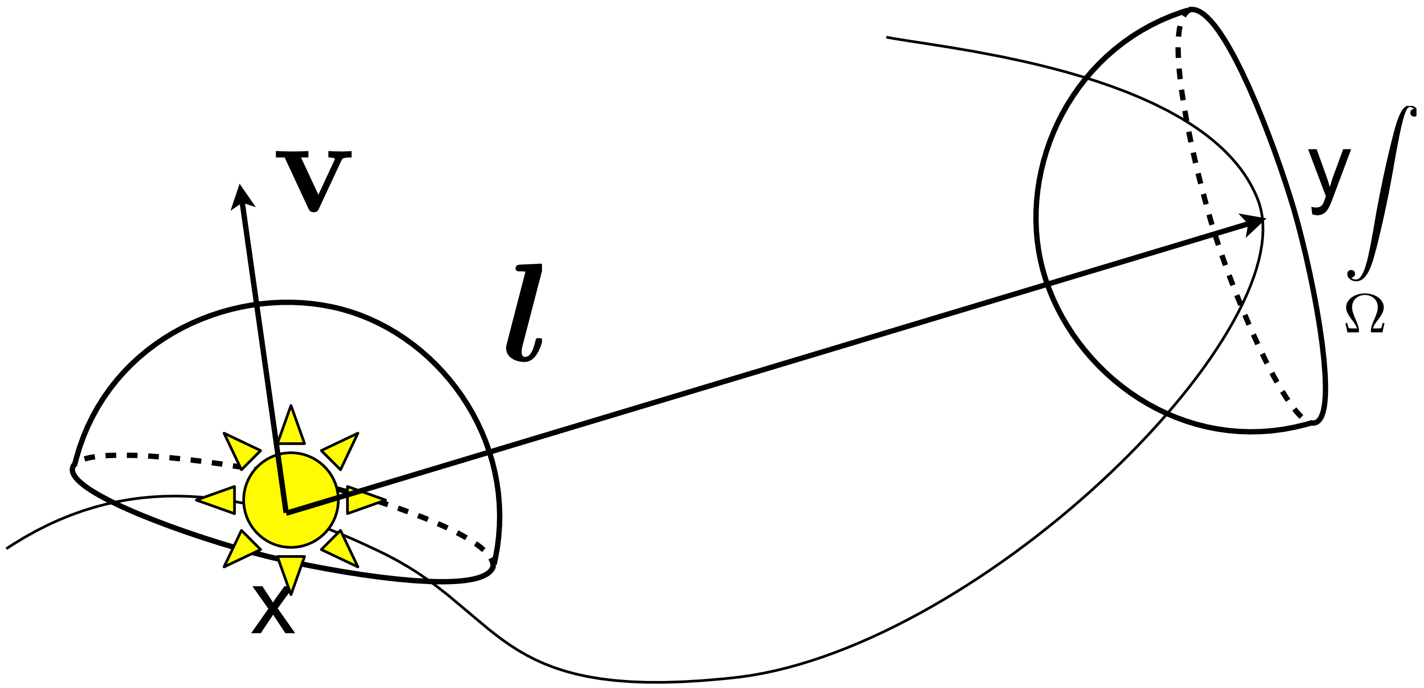
- This is helpful for..
  - understanding the nature of the solutions
  - thinking of numerical methods for solution

# The Rendering Equation

$$L(x \rightarrow \mathbf{v}) = \int_\Omega L(x \leftarrow \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta \, \mathrm{d}\mathbf{l}$$

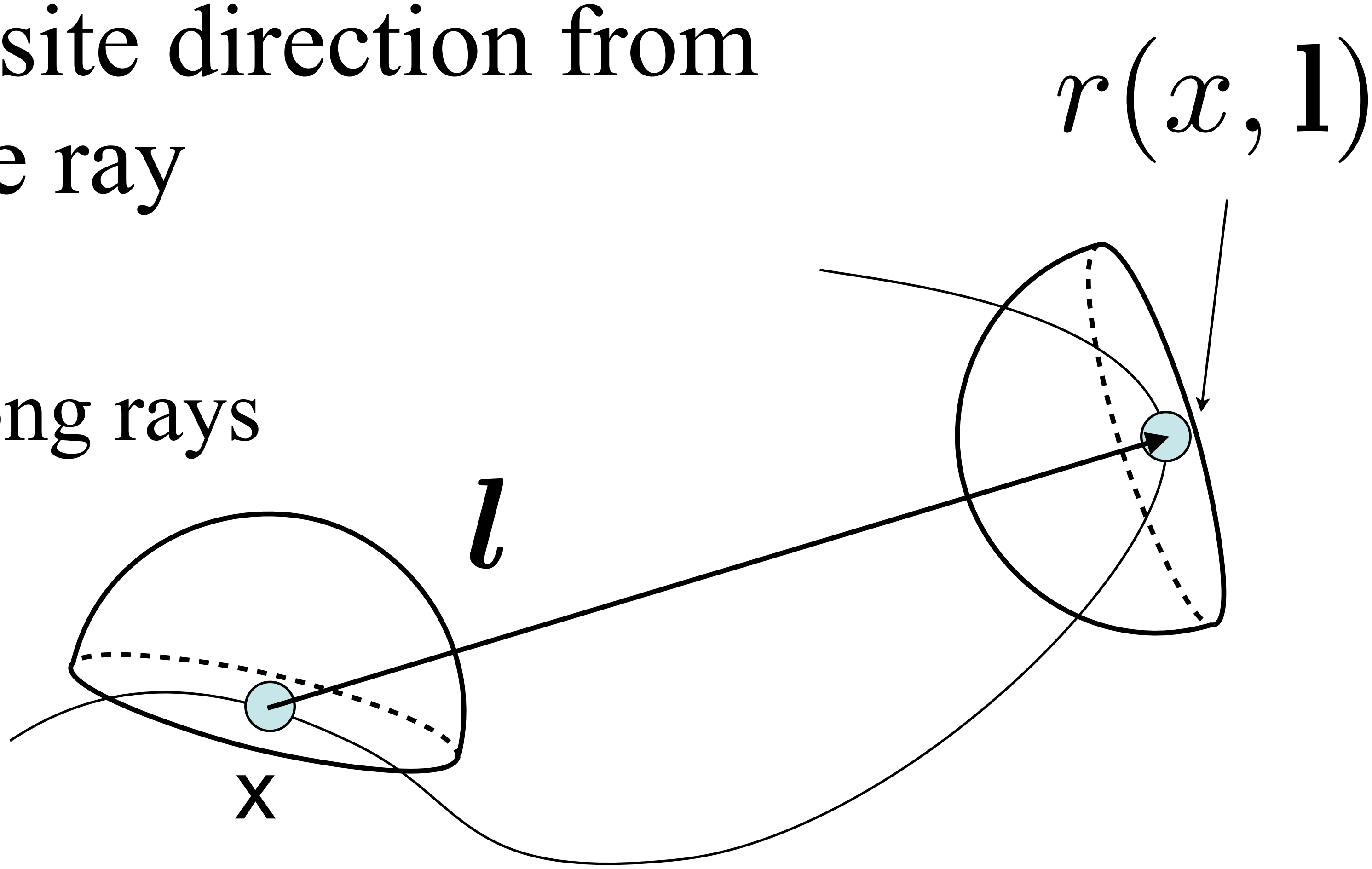
$$+ \; E(x \rightarrow \mathbf{v})$$

- Recursive!
  - To know incident radiance at *x*, must know outgoing radiance at *all points y seen by x*



# Operator Formulation 1

- "The lighting incident to $x$ from $\mathbf{l}$ is the light exiting to the opposite direction from the point r($x$,$\mathbf{l}$) where the ray from $x$ towards $\mathbf{l}$ hits"
  - Constancy of radiance along rays
  - "Ray-cast function" $r(x,\mathbf{l})$ returns point hit by ray from $x$ towards $\mathbf{l}$

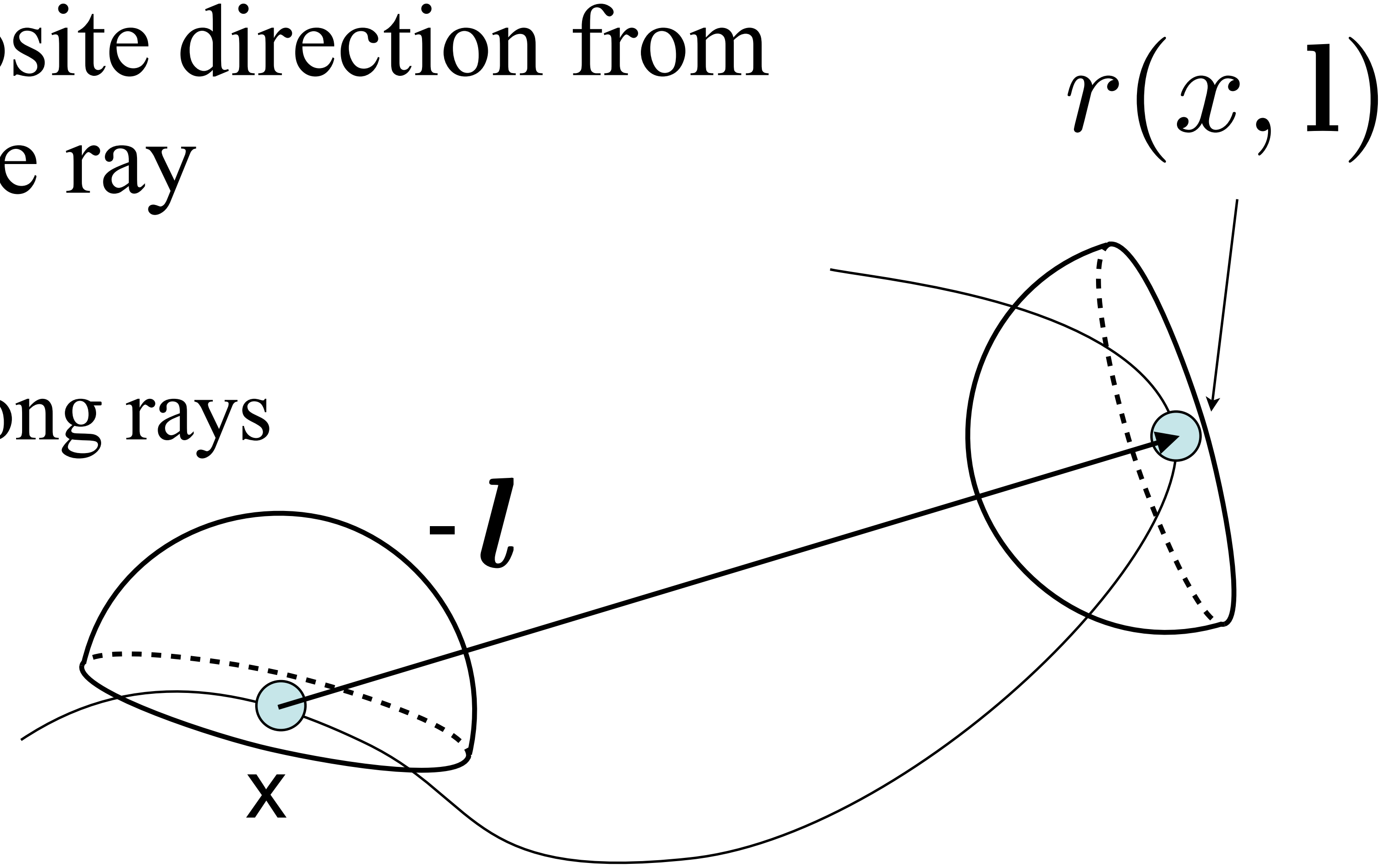


# Operator Formulation 1

- Let's define the *propagation operator* G

$$L_{\text{in}}(x, \mathbf{l}) = (\mathcal{G}L_{\text{out}}) \stackrel{\text{def}}{=} L_{\text{out}}(r(x, \mathbf{l}) \to -\mathbf{l})$$

- "The lighting incident to *x* from **l** is the light exiting to the opposite direction from the point r(*x*,**l**) where the ray from *x* towards **l** hits"
  - Constancy of radiance along rays
  - "Ray-cast function" *r*(*x*,**l**) returns point hit by ray from *x* towards **l**

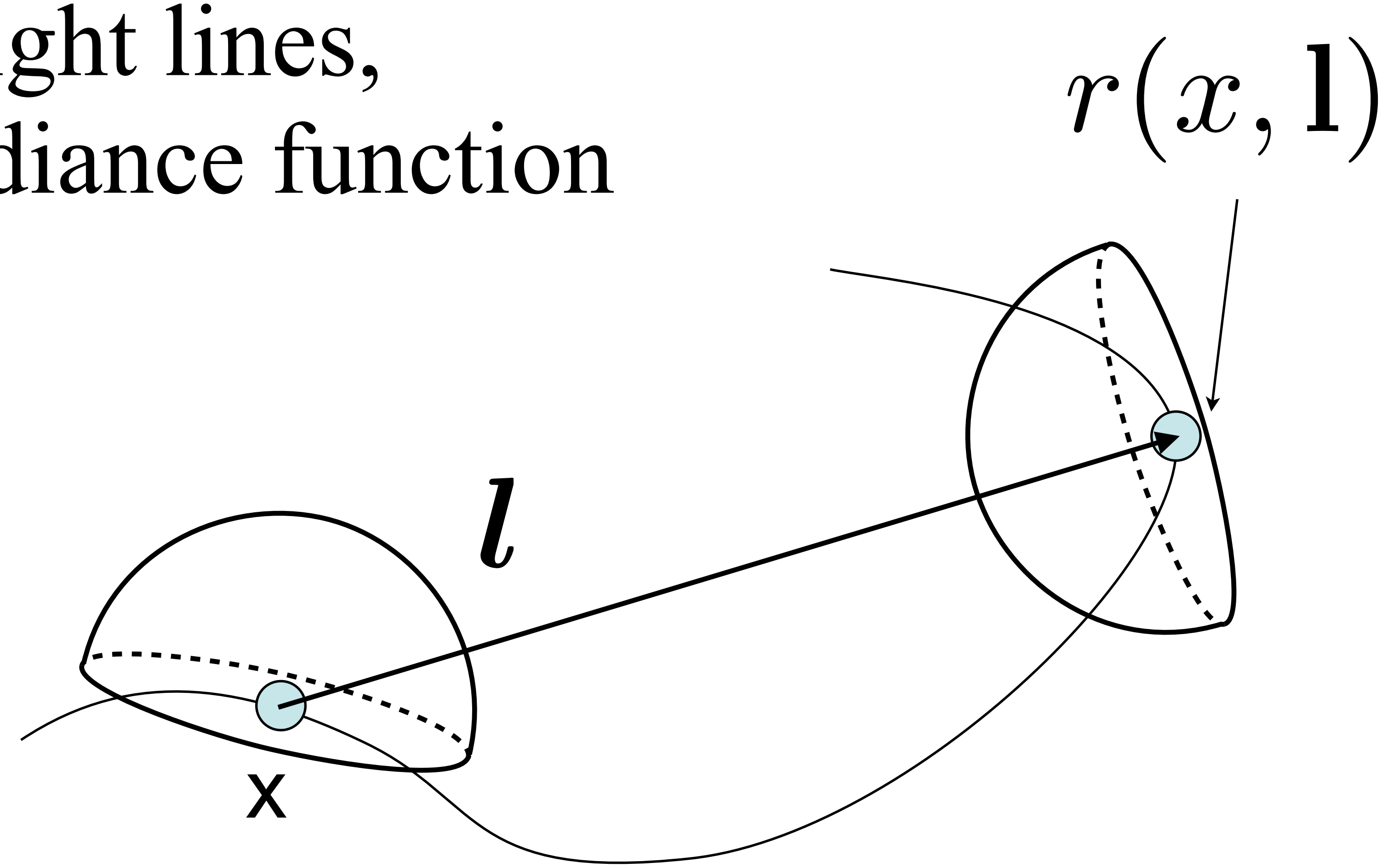


# Operator Formulation 1

- Let's define the *propagation operator* G

$$L_{\text{in}}(x, \mathbf{l}) = (\mathcal{G}L_{\text{out}}) \stackrel{\text{def}}{=} L_{\text{out}}(r(x, \mathbf{l}) \to -\mathbf{l})$$

- G takes an outgoing radiance function, propagates it along straight lines, produces an incident radiance function



# Operator Formulation cont'd

- ..and the *local reflection operator* R

$$L_{\text{out}}(x, \mathbf{v}) = (\mathcal{R} L_{\text{in}}) =$$

$$\int_\Omega L_{\text{in}}(x, \mathbf{l}) \, f_r(x, \mathbf{l} \rightarrow \mathbf{v}) \, \cos\theta \, \mathrm{d}\mathbf{l}$$

- Takes incident radiance function (defined for all points and directions), produces outgoing radiance function (defined for all points and directions)
- This is just another way of writing the reflectance integral you saw already

# These operators are not complicated

take in one function

do something to it

return another function

runes.nu, rendered using Maxwell

# Operator Form of Rendering Eq.

$$L_{\text{out}}(x, \mathbf{v}) = \int_\Omega L_{\text{in}}(x, \mathbf{l})\, f_r(x, \mathbf{l} \rightarrow \mathbf{v})\, \cos\theta_{\text{in}}\, \mathrm{d}\mathbf{l} + E_{\text{out}}(x, \mathbf{v})$$

- Propagation + reflectance operators

$$L_{\text{out}} = \mathcal{R} L_{\text{in}}$$

$$L_{\text{in}} = \mathcal{G} L_{\text{out}}$$

# Operator Form of Rendering Eq.

$$L_{\text{out}}(x,\mathbf{v}) = \int_\Omega L_{\text{in}}(x,\mathbf{l})\, f_r(x,\mathbf{l}\rightarrow\mathbf{v})\, \cos\theta_{\text{in}}\, \mathrm{d}\mathbf{l} + E_{\text{out}}(x,\mathbf{v})$$

- Propagation + reflectance operators

$$L_{\text{out}} = \mathcal{R} L_{\text{in}}$$

$$L_{\text{in}} = \mathcal{G} L_{\text{out}}$$

- Let's put them together and add emission $E$:

$$\boxed{L_{\text{out}} = \mathcal{R}\mathcal{G} L_{\text{out}} + E}$$

# Operator Form of Rendering Eq.

$$L_{\text{out}} = \mathcal{R}\mathcal{G}L_{\text{out}} + E$$

- Let's call propagation followed by reflection, the *transport operator T*: $\mathcal{T} = \mathcal{R}\mathcal{G}$
- Looks a lot like a linear system Ax=b, doesn't it?
  - Well, it *is* a linear system.
    Just with functions instead of vectors.
  - Easy to verify linearity: T(aX+bY) = aTX + bTY for any functions X,Y and scalars a,b

$$(\mathcal{I} - \mathcal{T})L_{\text{out}} = E$$

# Consequence of Linearity

$$(\mathcal{I} - \mathcal{T})L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1}\, E$$

- This is kind of a deep result, although simple: *the lighting solution is linear w.r.t. the emission.*
  – I.e., solution is a linear function of the emission.

# Consequence of Linearity, Pt 2

$$(\mathcal{I} - \mathcal{T}) L_{\text{out}} = E$$

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} E$$

- Light is additive, i.e., we can break emission into parts

$$L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} (E_1 + E_2)$$

$$= \boxed{(\mathcal{I} - \mathcal{T})^{-1} E_1} + \boxed{(\mathcal{I} - \mathcal{T})^{-1} E_2}$$

# “Neumann Series”

See link by clicking title!

$$\Leftrightarrow L_{\text{out}} = (\mathcal{I} - \mathcal{T})^{-1} E$$

- The Neumann series says

$$(\mathcal{I} - \mathcal{T})^{-1} = \sum_{i=0}^{\infty} \mathcal{T}^i$$

$$= \mathcal{I} + \mathcal{T} + \mathcal{T}^2 + \mathcal{T}^3 + \ldots$$

- I.e. $$L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots$$

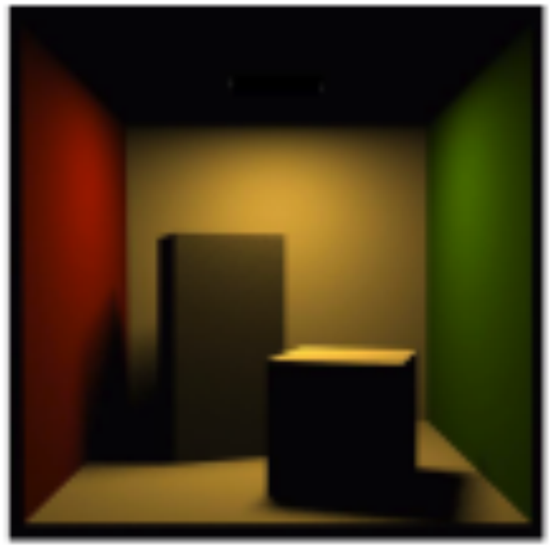
# Neumann Series

$$L_{\text{out}} = E + \mathcal{T}E + \mathcal{T}\mathcal{T}E + \ldots$$

- The lighting solution is the sum of
  - emitted light E,
  - light reflected once TE,
  - light reflected twice TTE, etc.

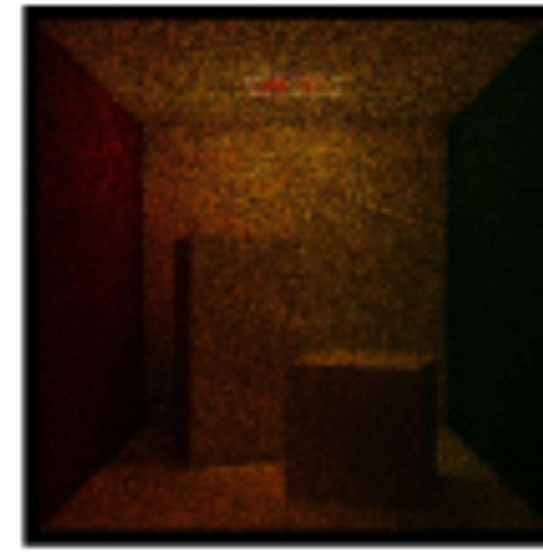
- Monte Carlo methods compute these integrals probabilistically

$E$

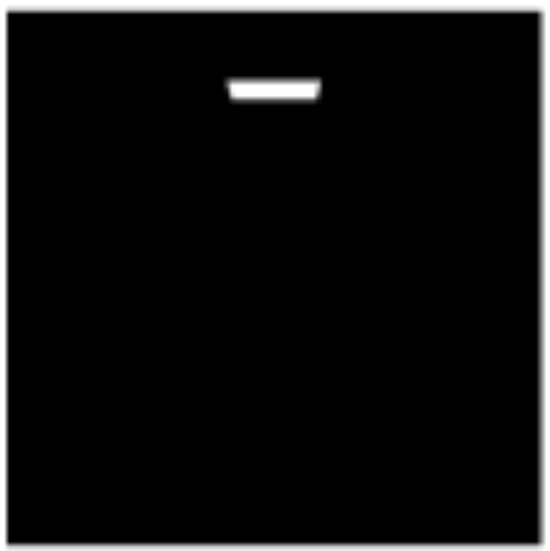


$\mathcal{T}E$

$\mathcal{T}\mathcal{T}E$

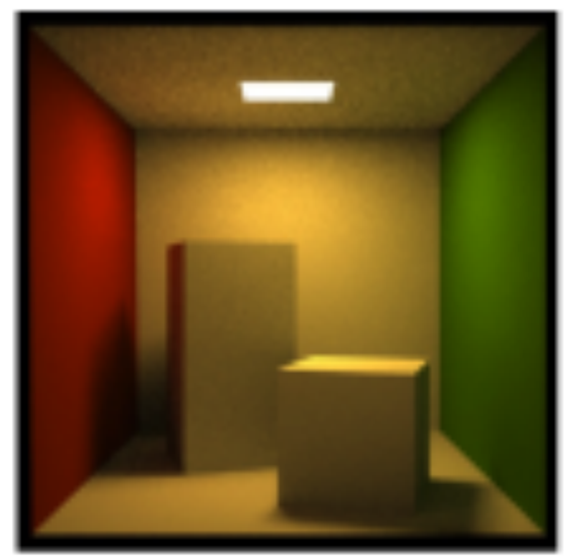


$\mathcal{T}\mathcal{T}\mathcal{T}E$

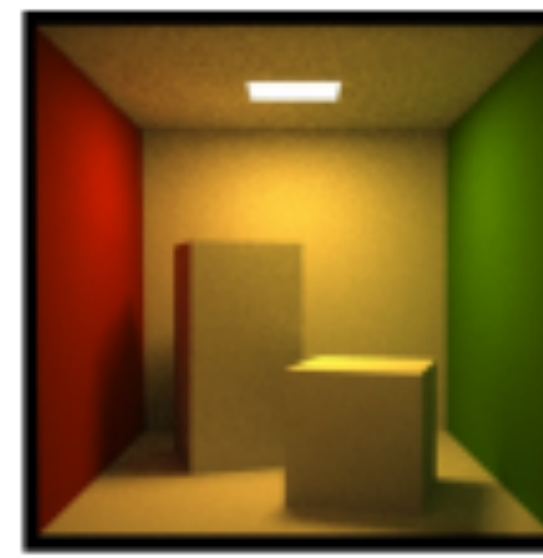


$E$

$E + \mathcal{T}E$



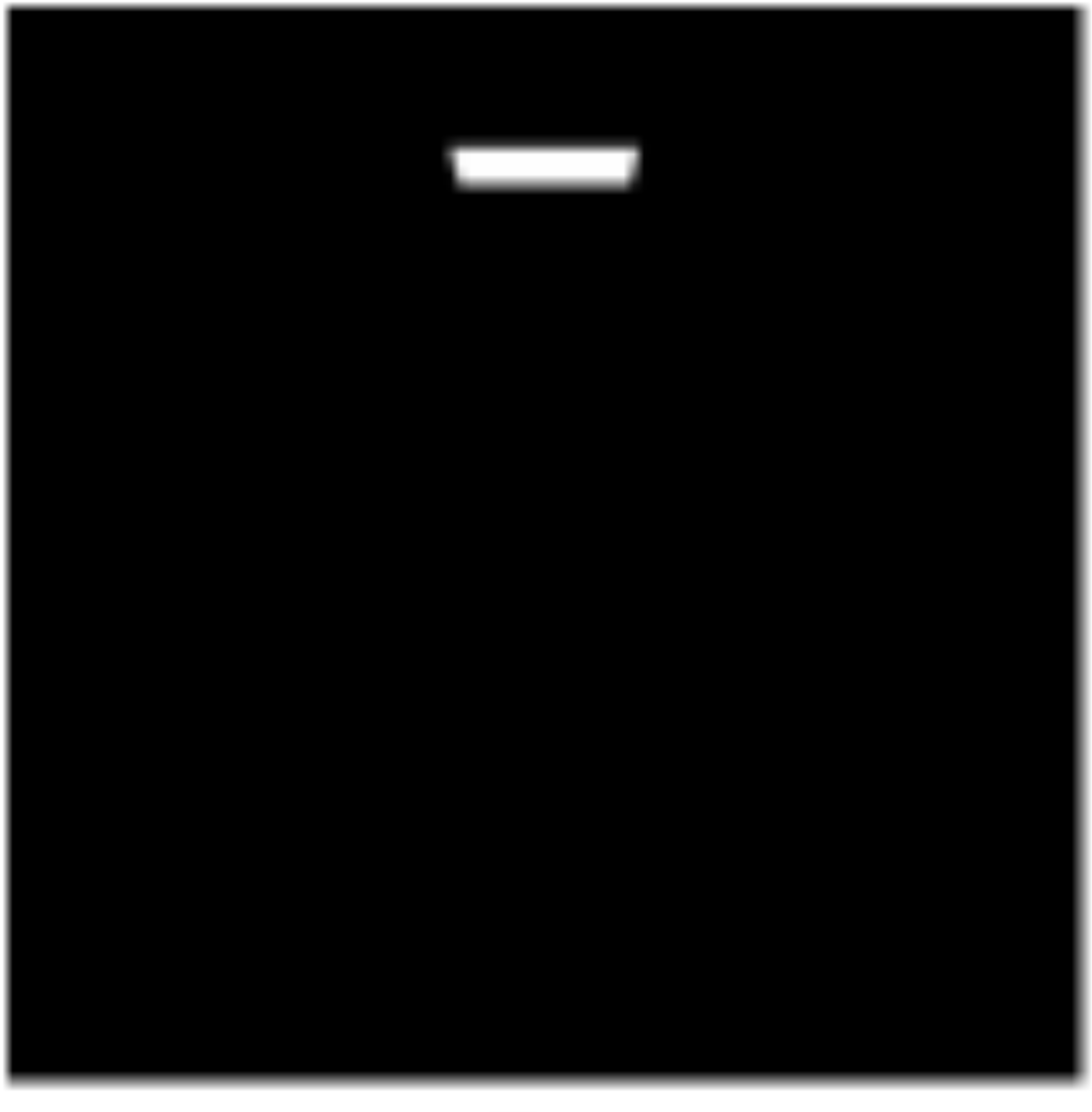
$E + \mathcal{T}E + \mathcal{T}^2E$



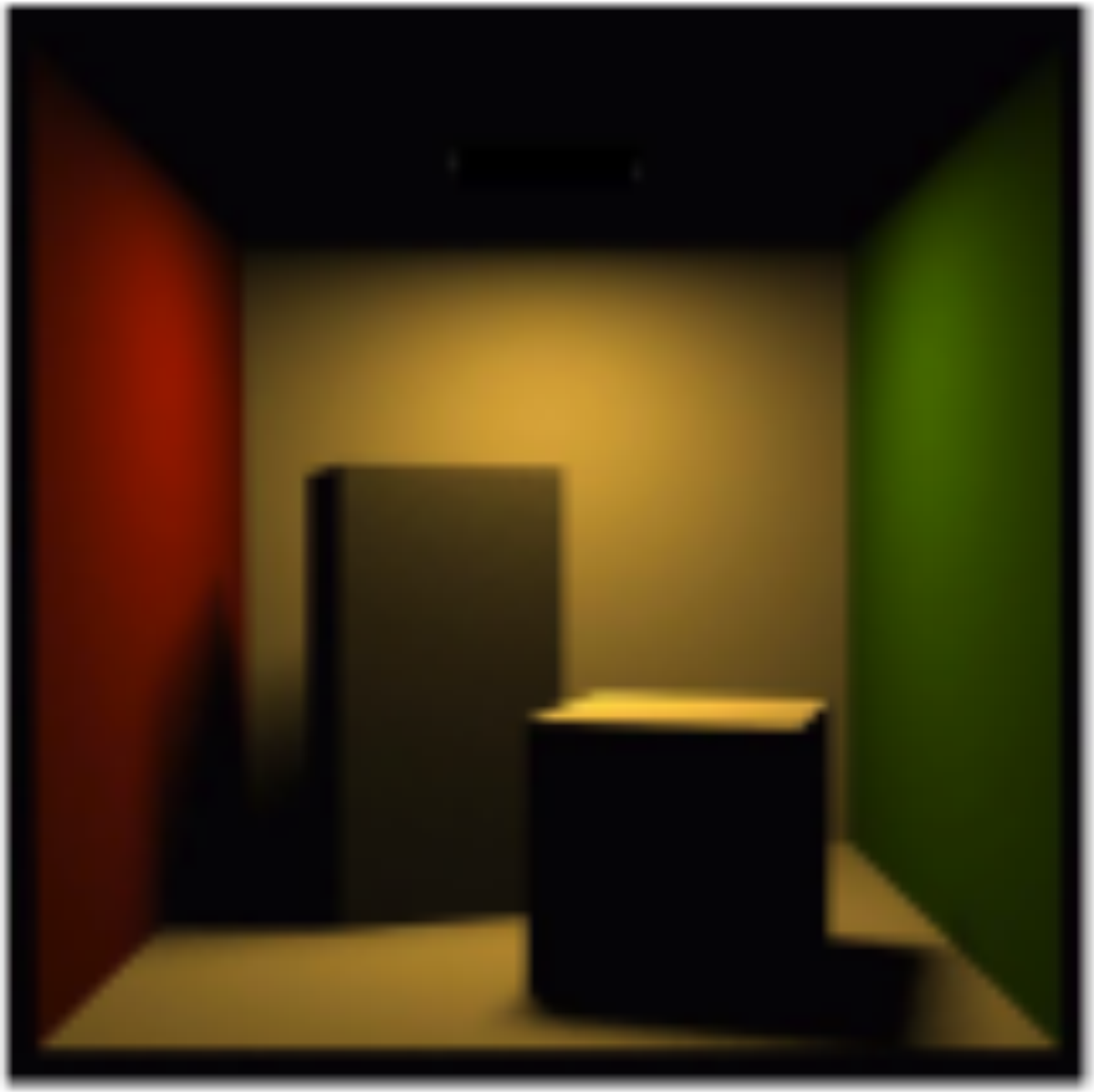
$E + \mathcal{T}E + \mathcal{T}^2E + \mathcal{T}^3E$

adapted from **Pat Hanrahan, Spring 2010**

# E = Emitted Radiance (Light sources)

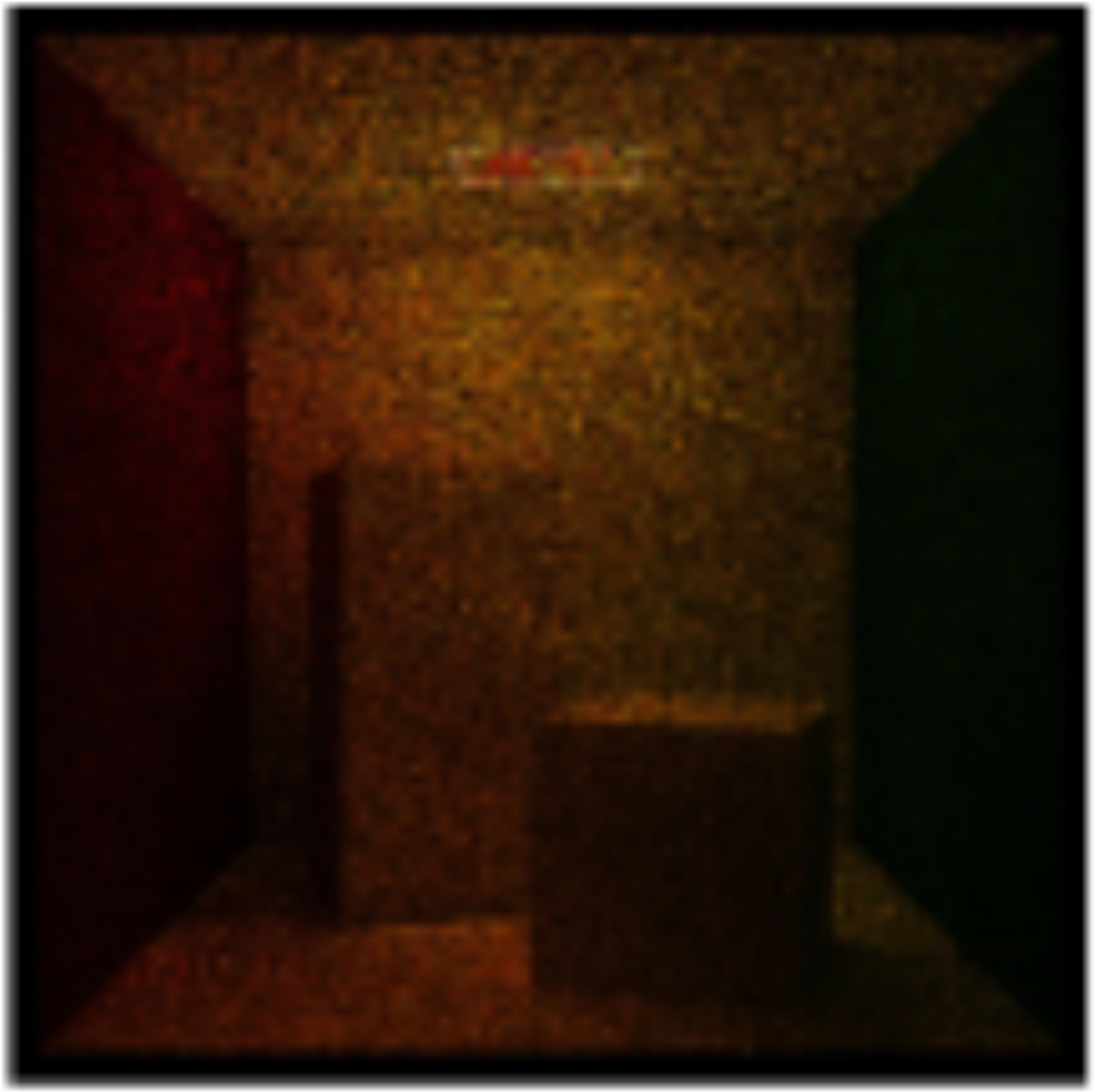


# TE = Direct Lighting



# TTE = First Indirect Bounce

# TTTE = Second Indirect Bounce



# Questions?