CS-C3100 Computer Graphics

# 12.1 Ray Tracing: Intersections

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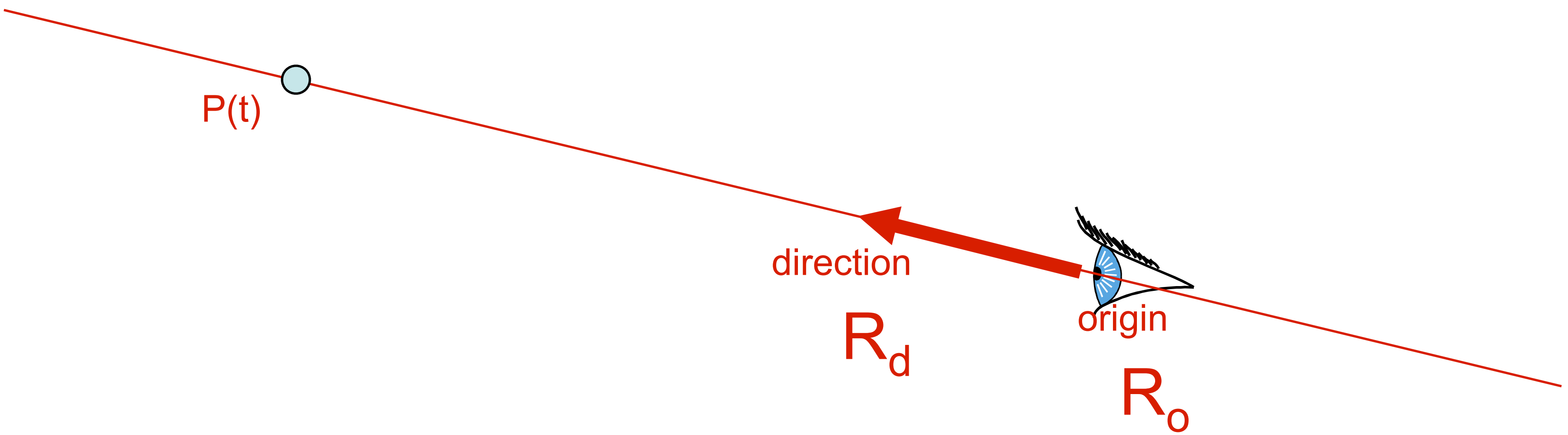
with lots of material from Frédo Durand

# In This Video

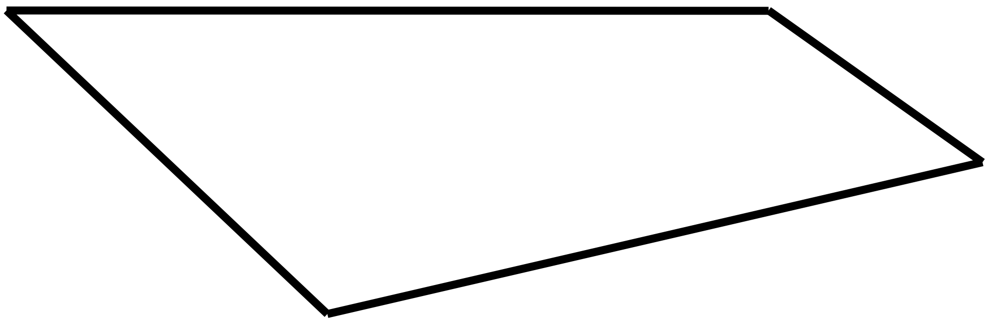
- Intersections
  - **Ray-plane**
    - implicit plane representation
  - Ray-sphere
    - simple, by analogy
  - Ray-triangle
    - barycentric coordinates (“painopistekoordinaatit”)
    - “Möller-Trumbore” intersection test
    - Interpolation over triangles

# Recall: Ray Representation

- Parametric line
- $P(t) = R_o + t * R_d$
- Explicit representation

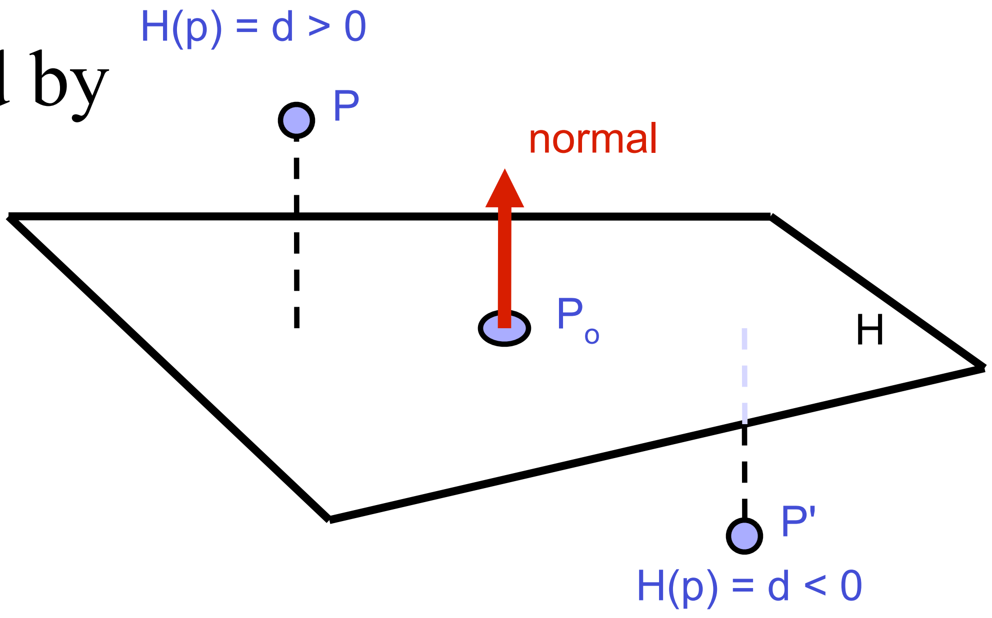


# 3D Plane Representation?



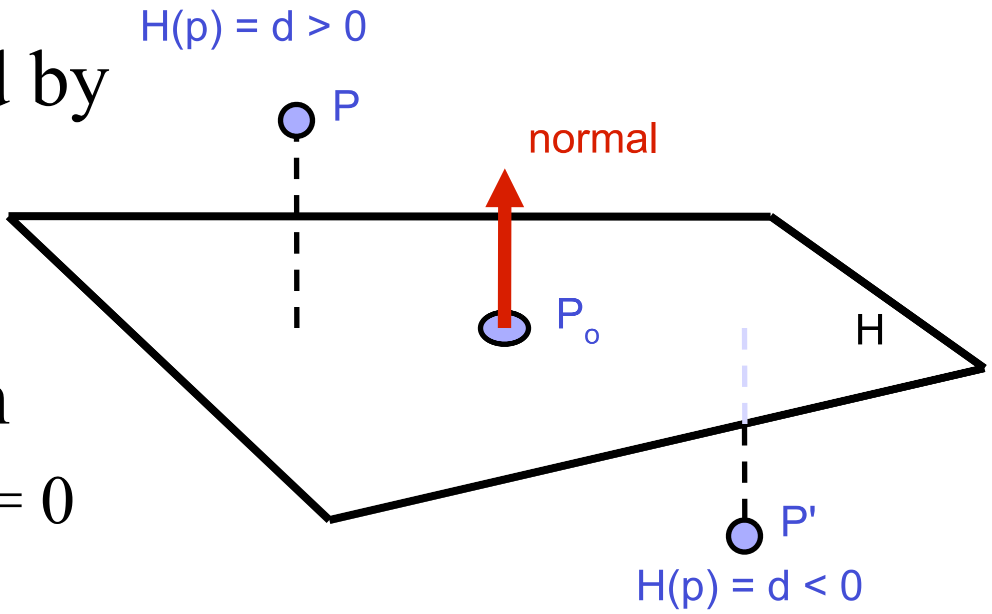
# 3D Plane Representation?

- (Infinite) plane defined by
  - $P_o = (x_0, y_0, z_0)$
  - $n = (A,B,C)$



# 3D Plane Representation?

- (Infinite) plane defined by
  - $P_o = (x_0, y_0, z_0)$
  - $n = (A,B,C)$
- Implicit plane equation
  - $H(P) = Ax+By+Cz+D = 0$
    $= n \cdot P + D = 0$



# 3D Plane Representation?

- (Infinite) plane defined by
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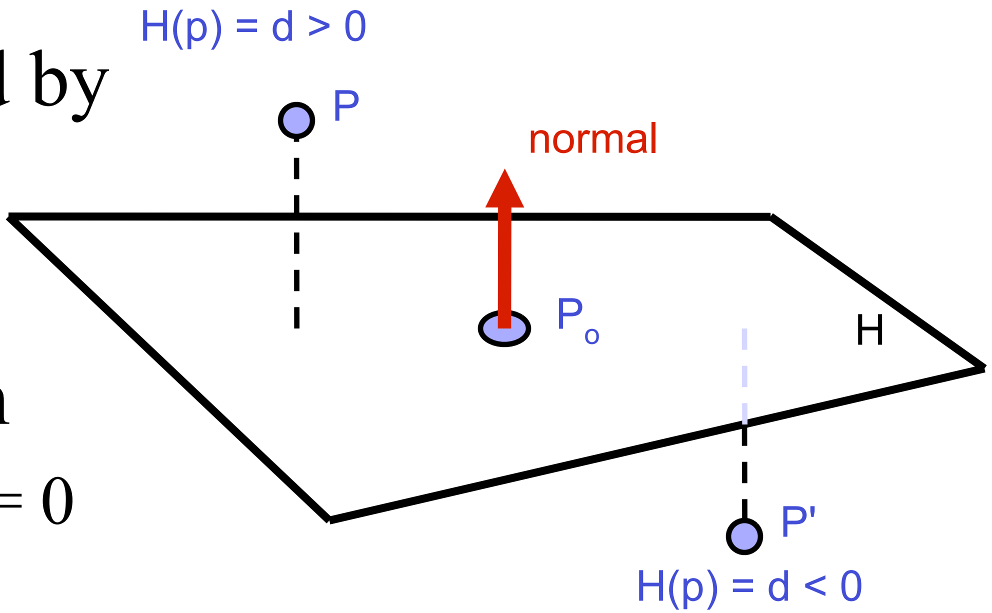
- Implicit plane equation
  - $H(P) = Ax+By+Cz+D = 0$
    $= n \cdot P + D = 0$
  - What is D?

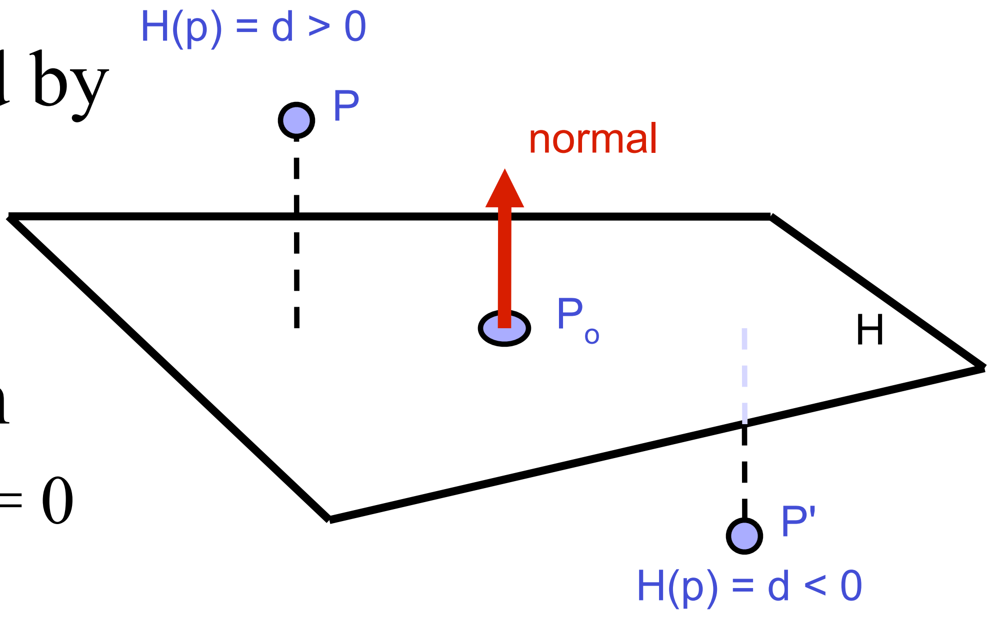



$$Ax_0 + By_0 + Cz_0 + D = 0$$ (Point $P_0$ must lie on plane)

$$\Rightarrow D = -Ax_0 - By_0 - Cz_0$$

# 3D Plane Representation?

- (Infinite) plane defined by
  - $P_o = (x_0, y_0, z_0)$
  - $n = (A,B,C)$
- Implicit plane equation
  - $H(P) = Ax+By+Cz+D = 0$
    $= n \cdot P + D = 0$
- Point-Plane distance?
  - If n is normalized,
    distance to plane, $d = H(P)$
  - d is the *signed distance*!



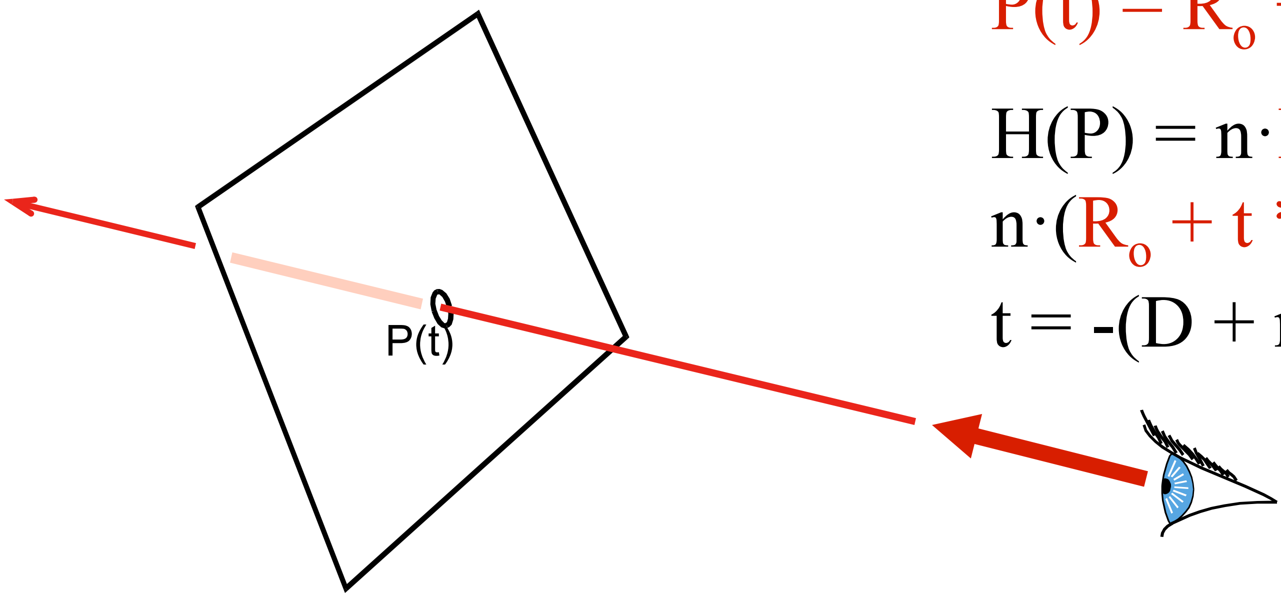
# Explicit vs. Implicit?

- Ray equation is explicit $P(t) = R_o + t * R_d$
  - Parametric
  - Generates points
  - Hard to verify that a point is on the ray
- Plane equation is implicit $H(P) = n \cdot P + D = 0$
  - Solution of an equation
  - Does not generate points
  - Verifies that a point is on the plane

- Exercise: Explicit plane and implicit ray?

# Ray-Plane Intersection

- Intersection means both equations are satisfied
- So, insert explicit equation of ray into implicit equation of plane & solve for *t*




$P(t) = R_o + t * R_d$

$H(P) = n \cdot P + D = 0$

$n \cdot (R_o + t * R_d) + D = 0$

$t = -(D + n \cdot R_o) / n \cdot R_d$

**Done!**

# Ray-Plane Intersection

- Intersection means both equations are satisfied
- So, insert explicit equation of ray into implicit equation of plane & solve for *t*

$$P(t) = R_o + t * R_d$$

$$H(P) = n \cdot P + D = 0$$

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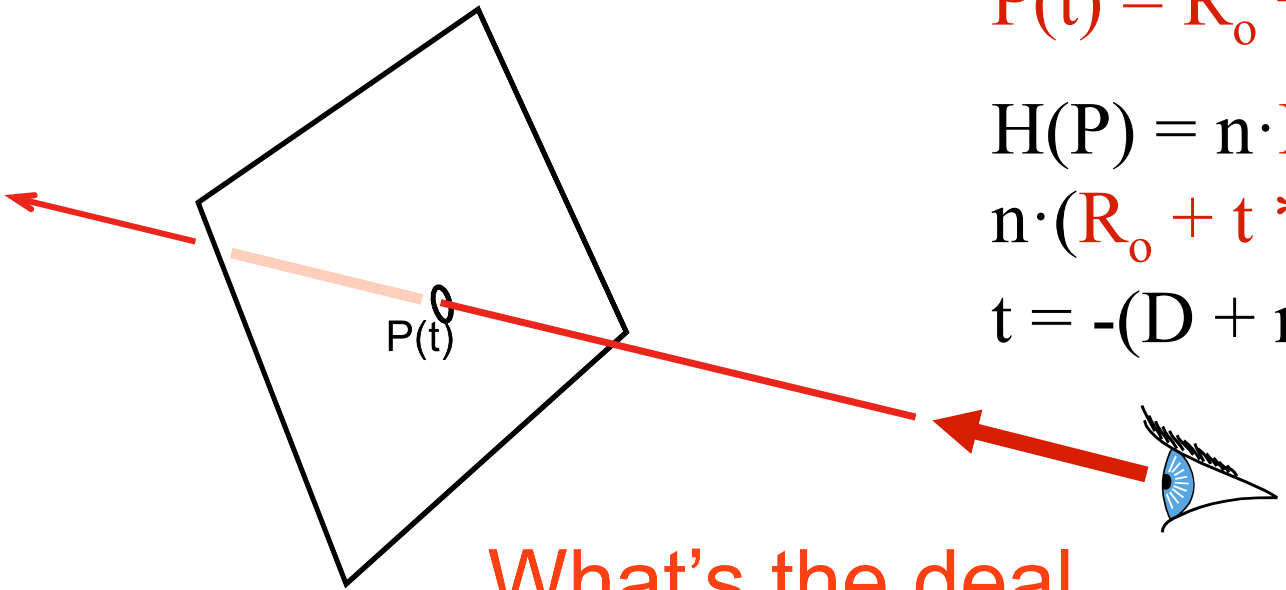



**Done!**

What's the deal when $n \cdot R_d = 0$?

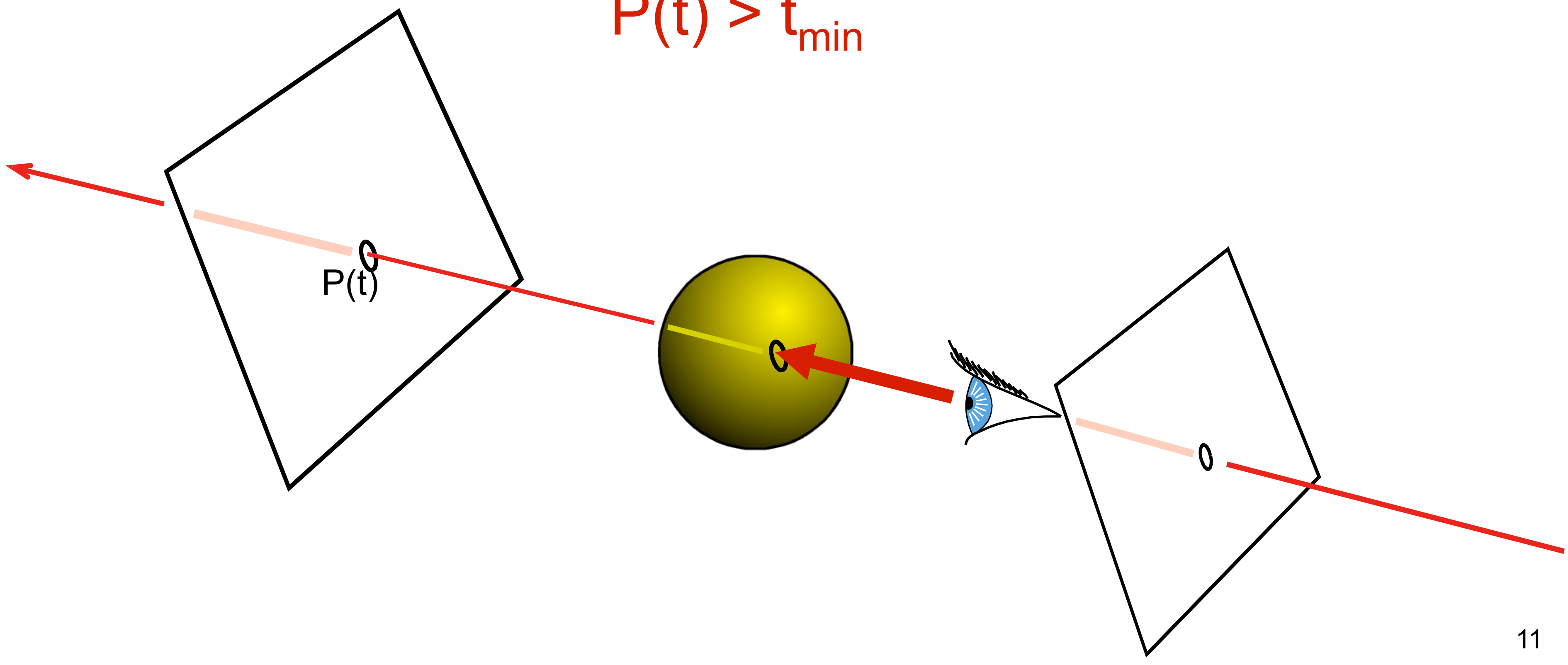
# Additional Housekeeping

- Verify that intersection is closer than previous

$$P(t) < t_{current}$$

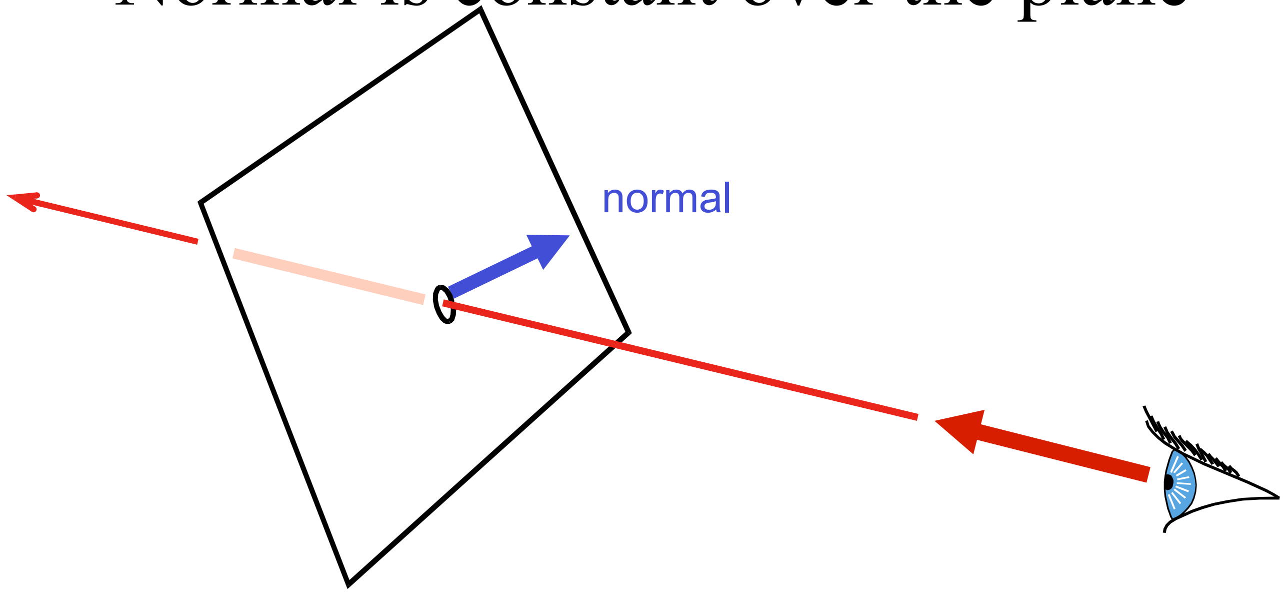
- Verify that it is not out of range (behind eye)

$$P(t) > t_{min}$$



# Normal

- Also need surface normal for shading
  - (Diffuse: dot product between light direction and normal, clamp to zero)
- Normal is constant over the plane



# Math Digression

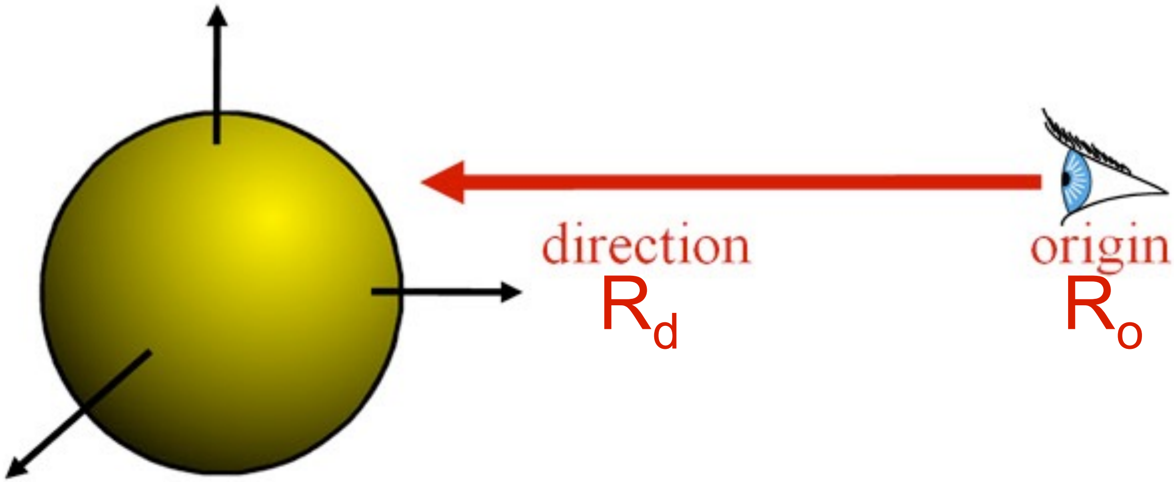
- Duality: points and planes are "dual" when you use homogeneous coordinates
- Point (x, y, z, 1)
- Plane (A, B, C, D)
- Plane equation → dot product
- You can map planes to points and points to planes in a dual space.
- Lots of cool equivalences
  - intersection of 3 planes define a point
  - 3 points define a plane!

# In This Video

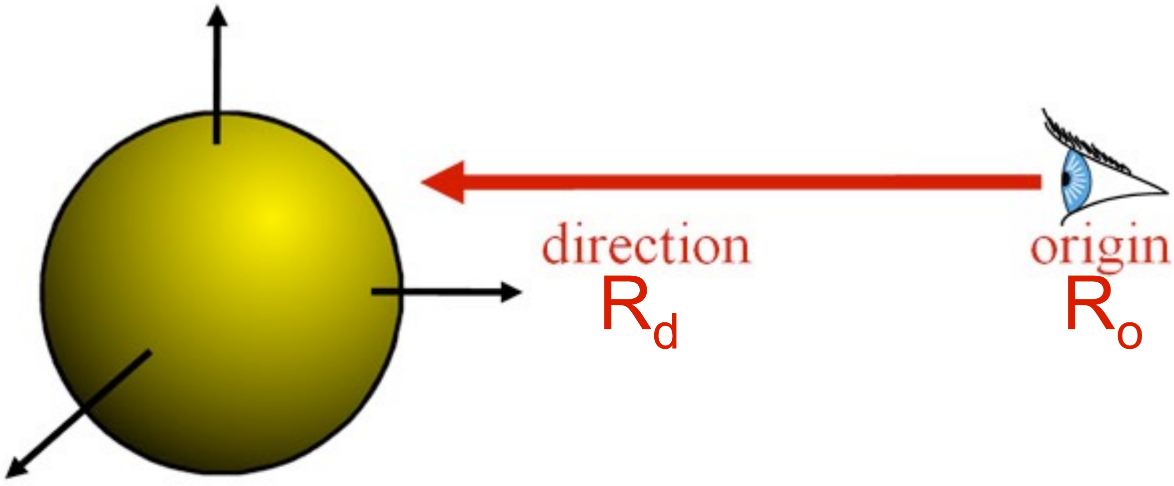
- Intersections
  - Ray-plane
  - **Ray-sphere**
  - Ray-triangle
    - barycentric coordinates (“painopistekoordinaatit”)
    - “Möller-Trumbore” intersection test
    - Interpolation over triangles

# Sphere Representation?

- Implicit sphere equation
  - Assume centered at origin (easy to translate)
  - $H(P) = ||P||^2 - r^2 = P \cdot P - r^2 = 0$



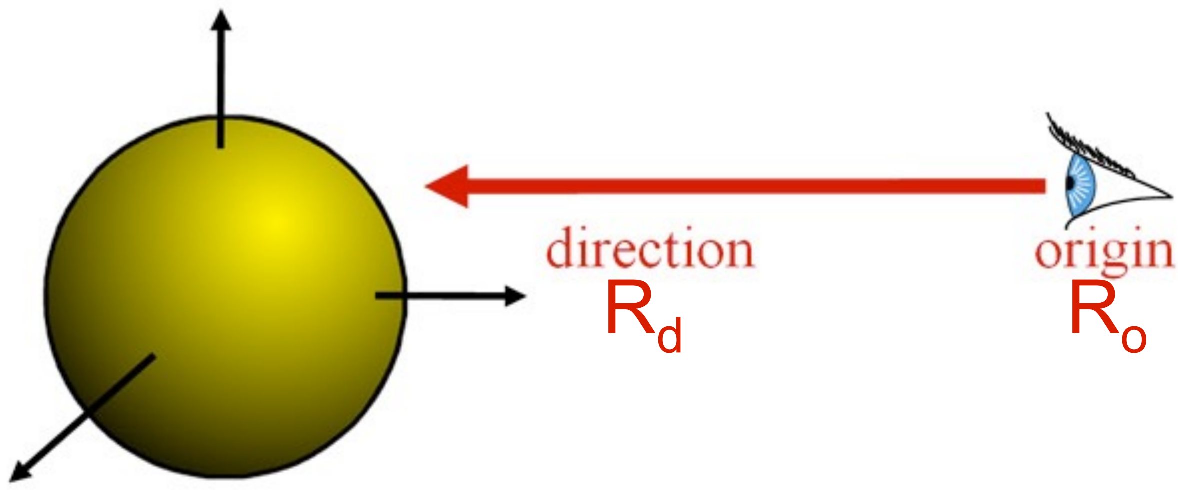
# Ray-Sphere Intersection



# Ray-Sphere Intersection

- Insert explicit equation of ray into implicit equation of sphere & solve for t

$P(t) = R_o + t{*}R_d \quad ; \quad H(P) = P \cdot P - r^2 = 0$



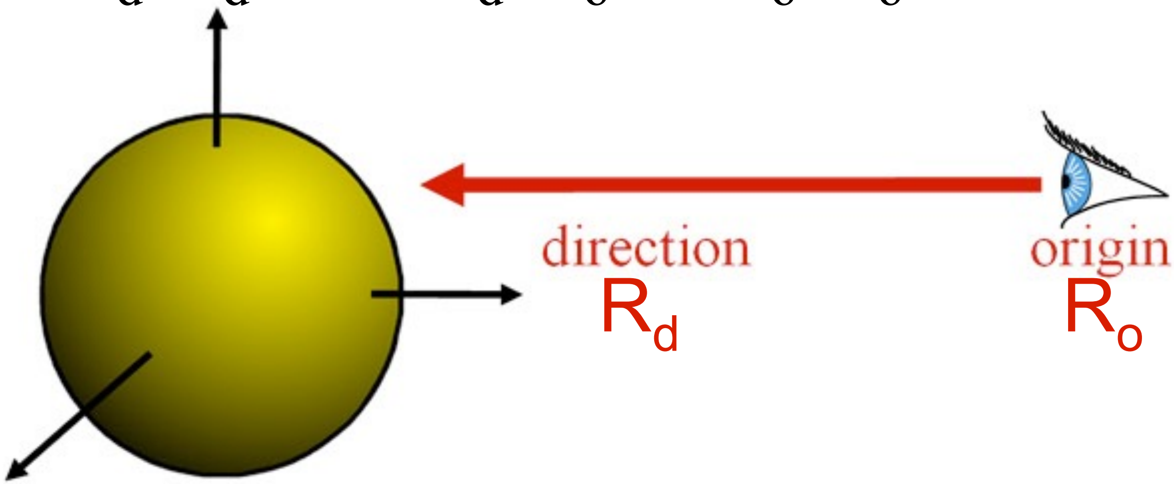
# Ray-Sphere Intersection

- Insert explicit equation of ray into implicit equation of sphere & solve for t

$$P(t) = R_o + t{*}R_d \quad ; \quad H(P) = P \cdot P - r^2 = 0$$

$$(R_o + tR_d) \cdot (R_o + tR_d) - r^2 = 0$$

$$R_d \cdot R_d t^2 + 2R_d \cdot R_o t + R_o \cdot R_o - r^2 = 0$$



# Ray-Sphere Intersection

- Quadratic: $at^2 + bt + c = 0$
  - $a = R_d \cdot R_d$
  - $b = 2R_d \cdot R_o$
  - $c = R_o \cdot R_o - r^2$

- with discriminant

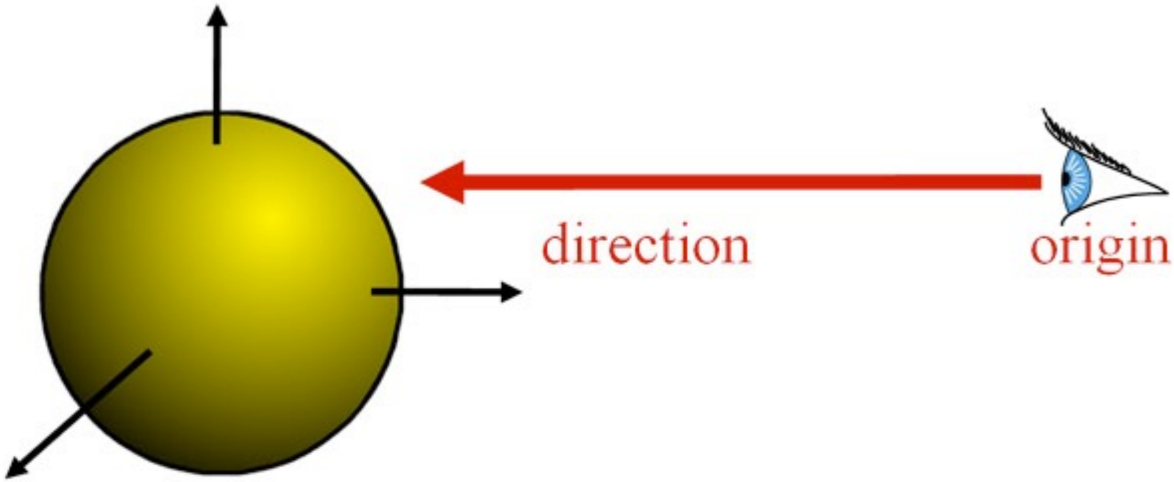
$$d = \sqrt{b^2 - 4ac}$$

- and solutions

$$t_{\pm} = \frac{-b \pm d}{2a}$$

# Ray-Sphere Intersection

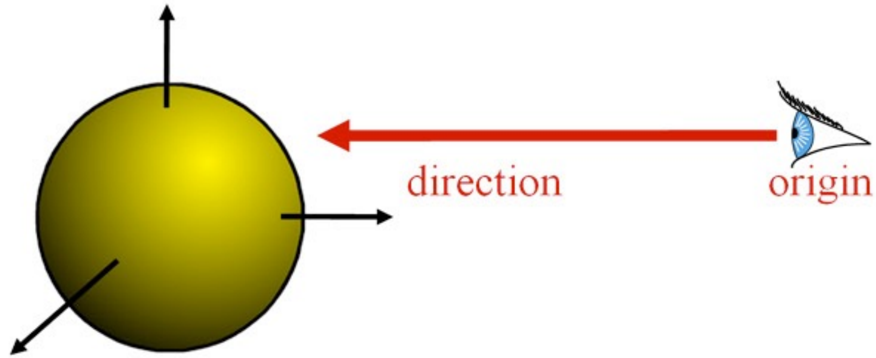
- 3 cases, depending on the sign of $b^2 - 4ac$
- What do these cases correspond to?
- Which root (t+ or t-) should you choose?
  - Closest positive!

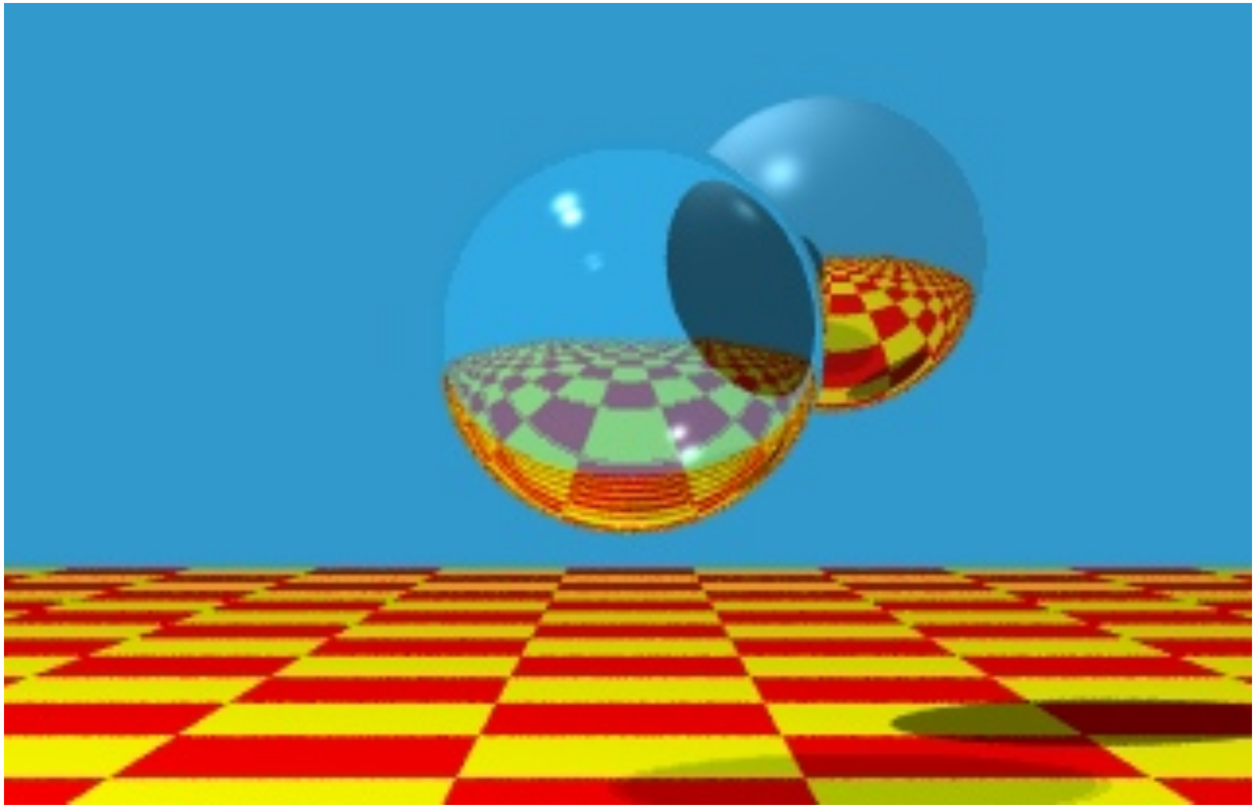


# Ray-Sphere Intersection

- It's so easy that all ray-tracing images have spheres!

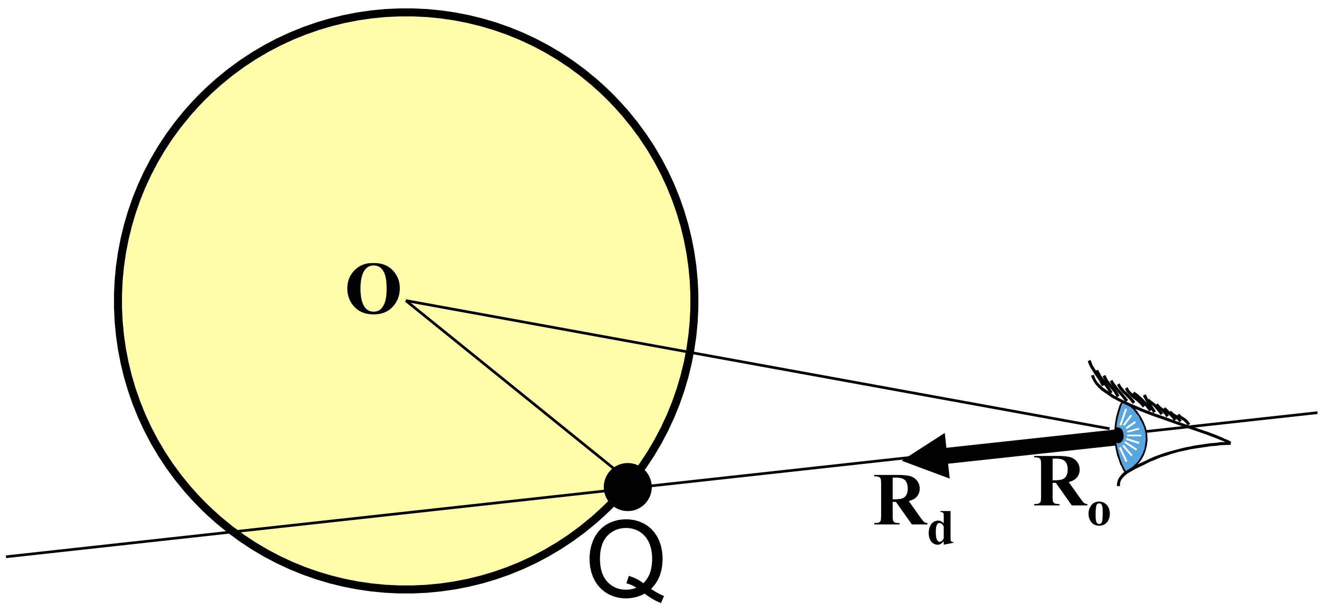
:-)



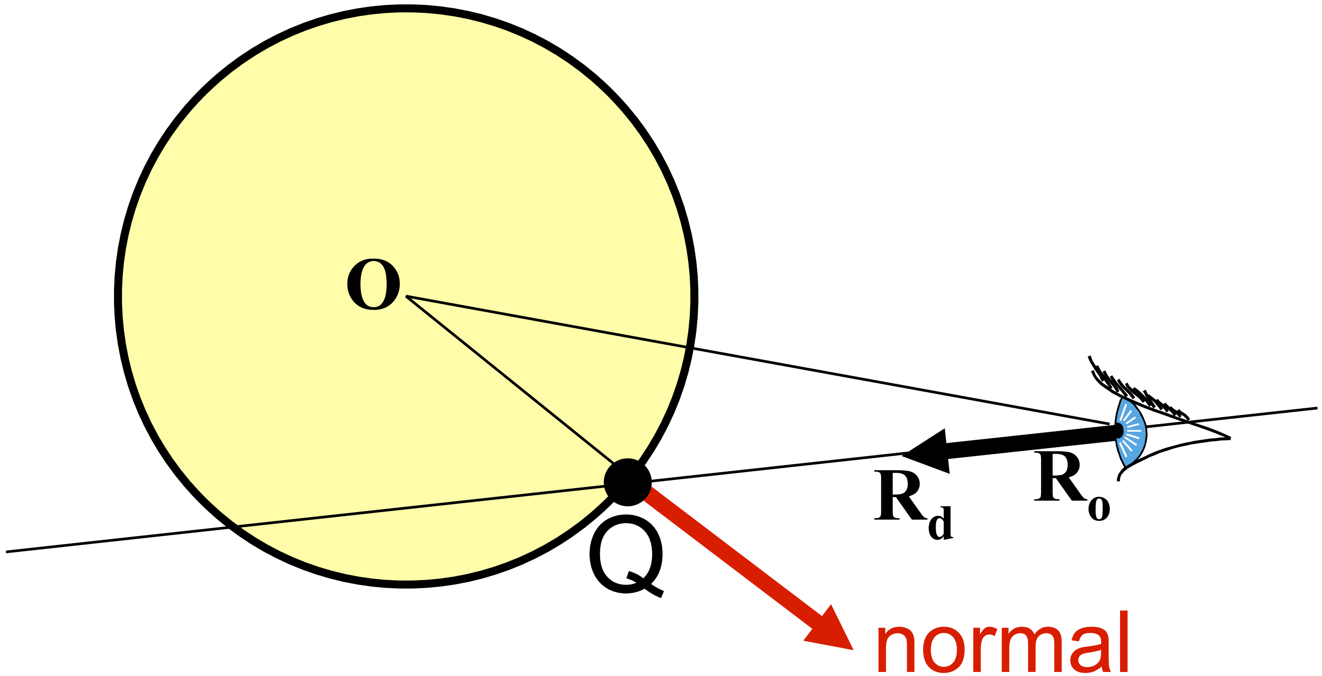
Turner Whitted

# Sphere Normal



# Sphere Normal

- Simply Q/||Q||
  - Q = P(t), intersection point
  - (for spheres centered at origin)



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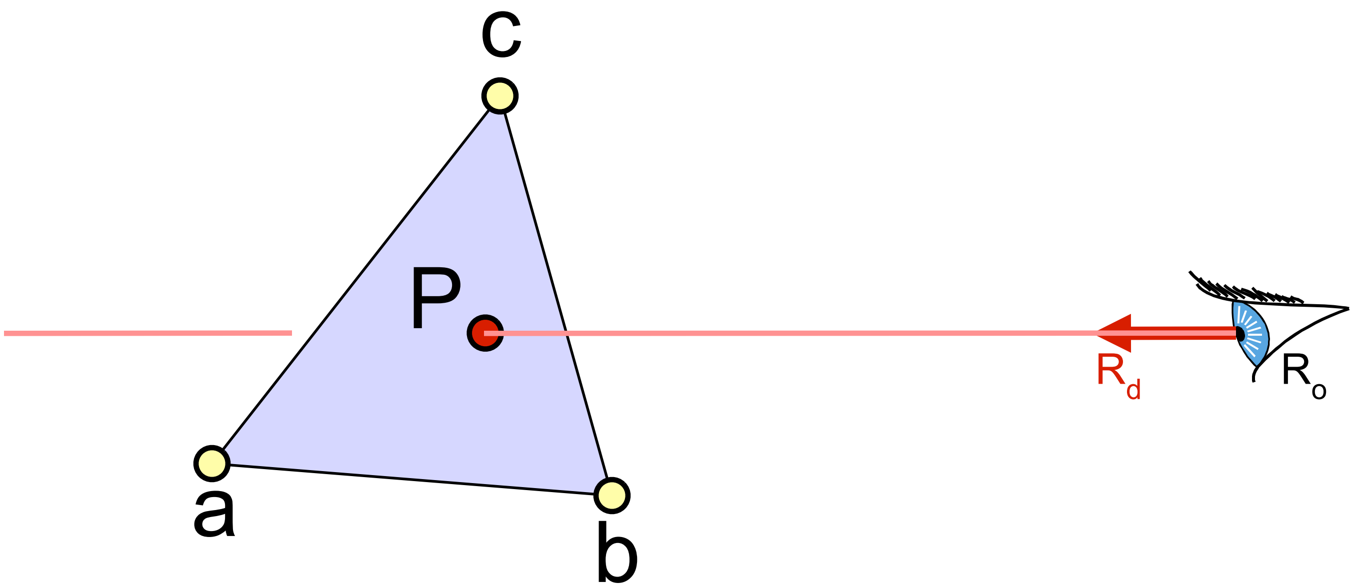
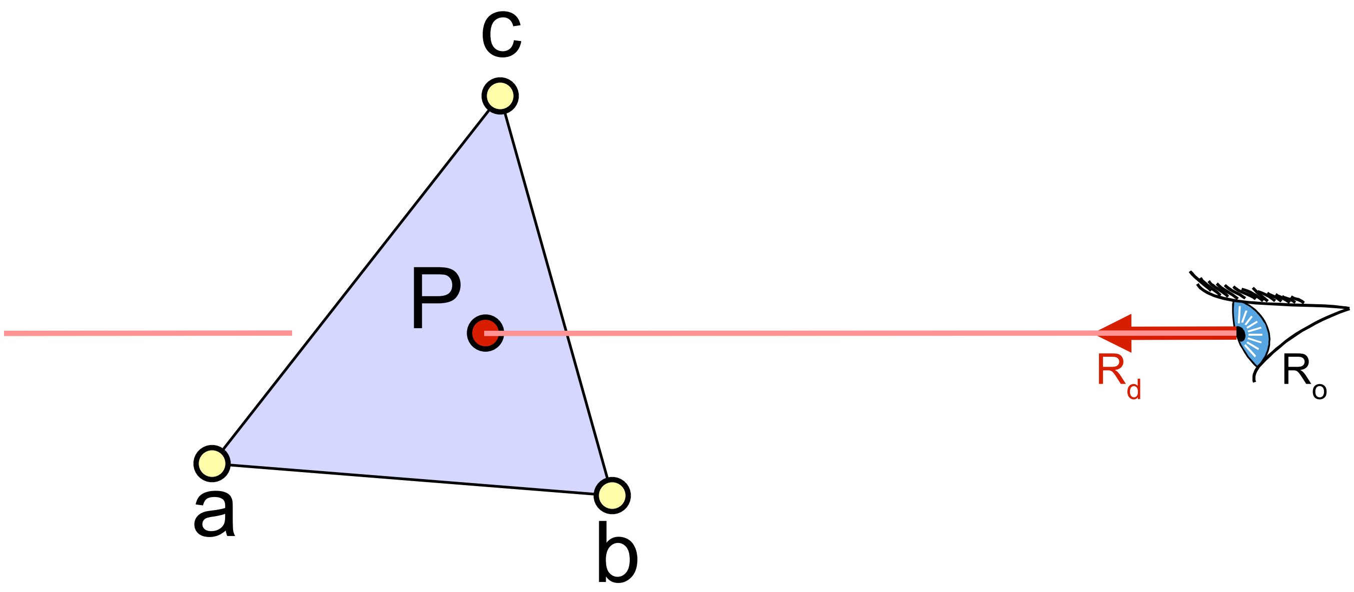
# Ray-Triangle Intersection



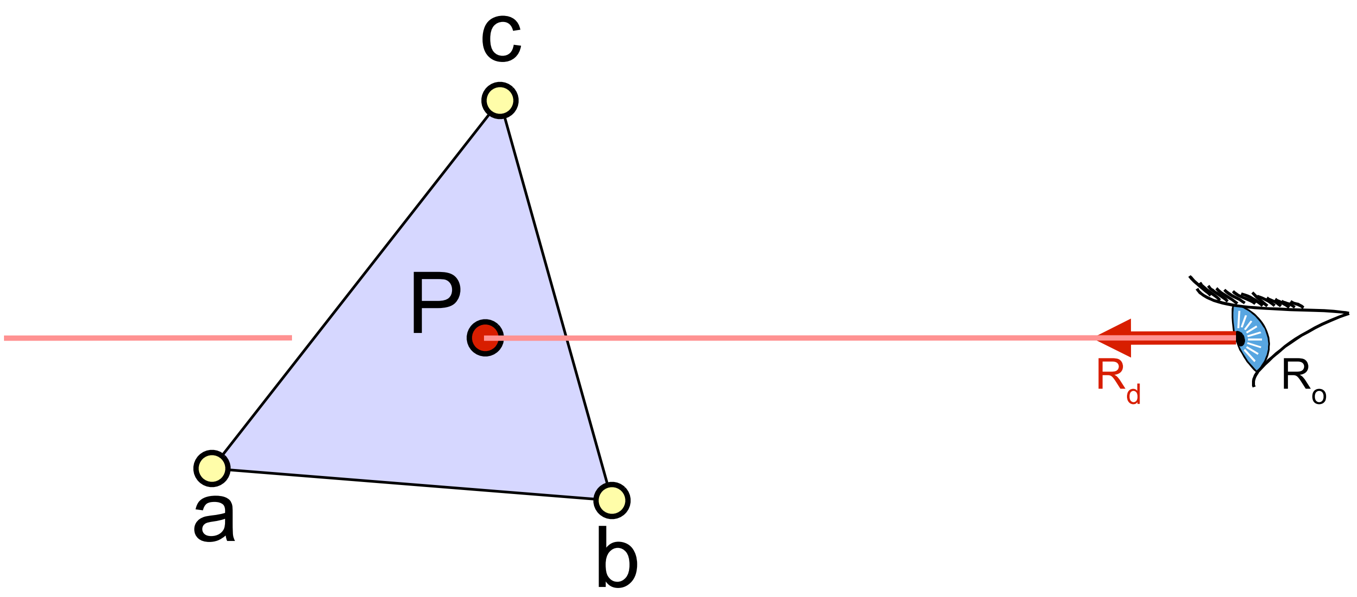
# Ray-Triangle Intersection

- Use ray-plane intersection followed by in-triangle test



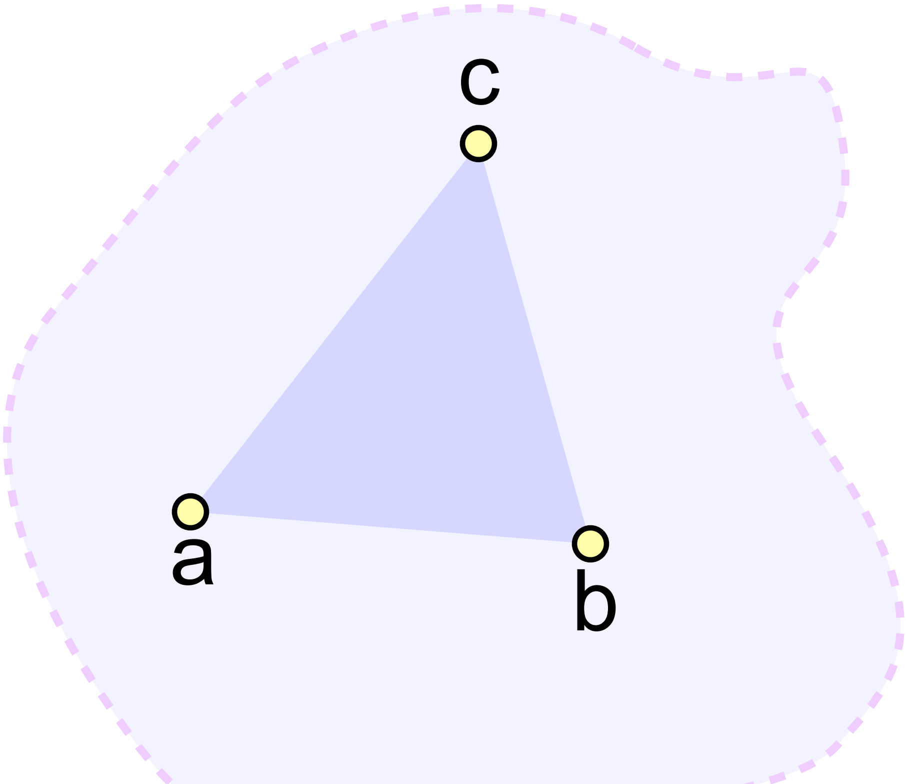
# Ray-Triangle Intersection

- Use ray-plane intersection followed by in-triangle test
- Or try to be smarter
  - Use barycentric coordinates



# Barycentric Definition of a Plane

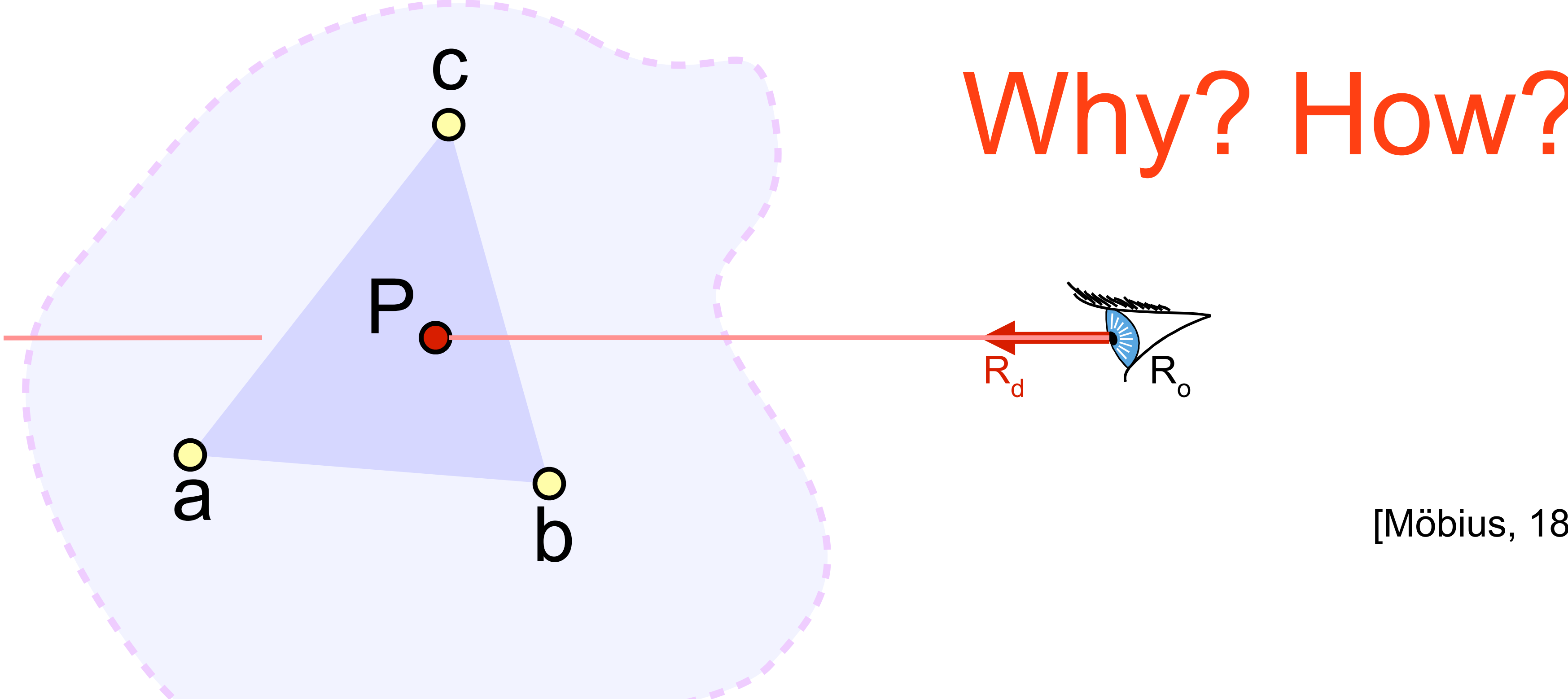
- A non-degenerate (?) triangle (**a**,**b**,**c**) defines a plane




[Möbius, 1827]

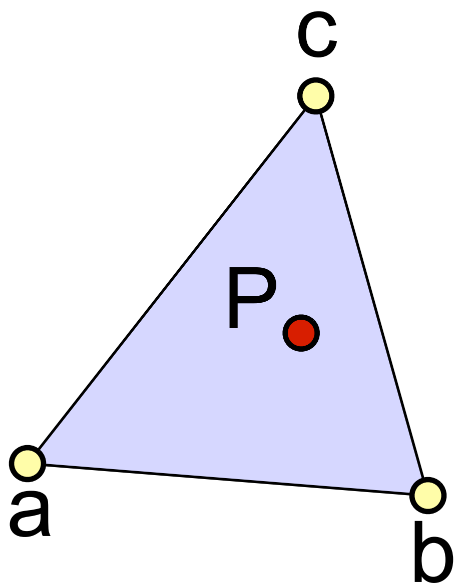
# Barycentric Definition of a Plane

- A non-degenerate triangle (**a**,**b**,**c**) defines a plane
- Any point **P** on this plane can be written as
  $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$,
  with $\alpha + \beta + \gamma = 1$




[Möbius, 1827]

# α, β, γ = Barycentric Coordinates



# α, β, γ = Barycentric Coordinates

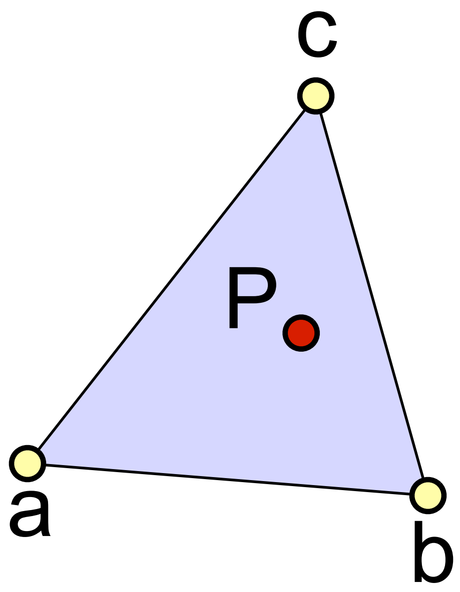
- Since $\alpha + \beta + \gamma = 1$, we can write $\alpha = 1 - \beta - \gamma$

$$\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

rewrite

$$\mathbf{P}(\beta, \gamma) = (1-\beta-\gamma)\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$$

$$= \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$$



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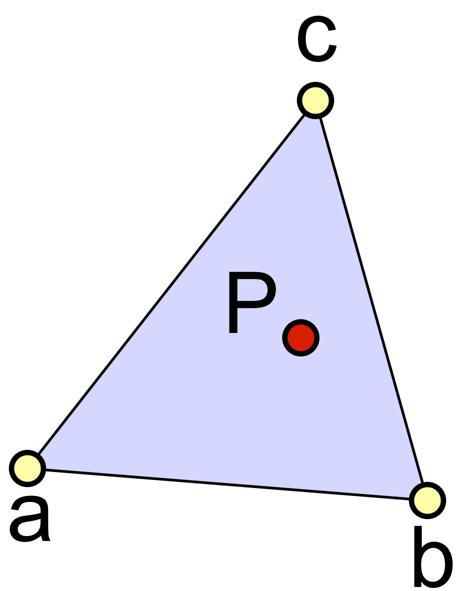
$$= \mathbf{a} + \beta(\underbrace{\mathbf{b}-\mathbf{a}}) + \gamma(\underbrace{\mathbf{c}-\mathbf{a}})$$

**Vectors that lie on the triangle plane**



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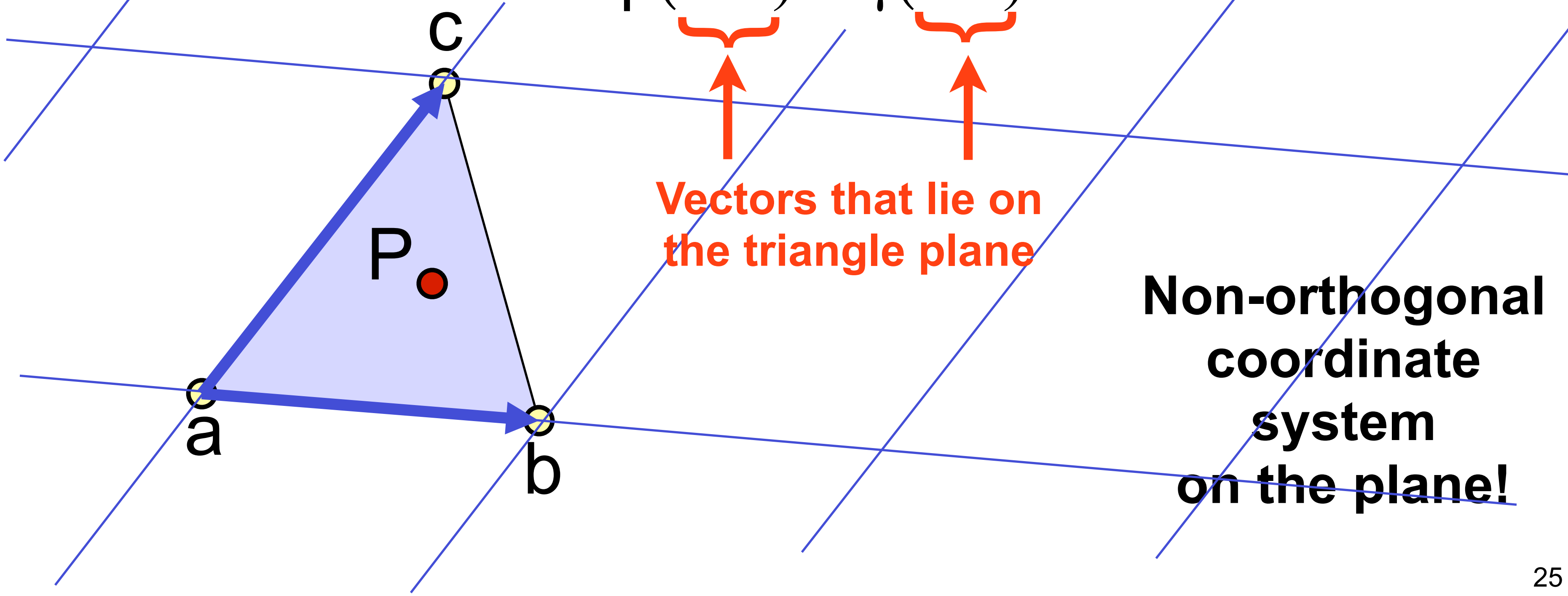
**Vectors that lie on the triangle plane**

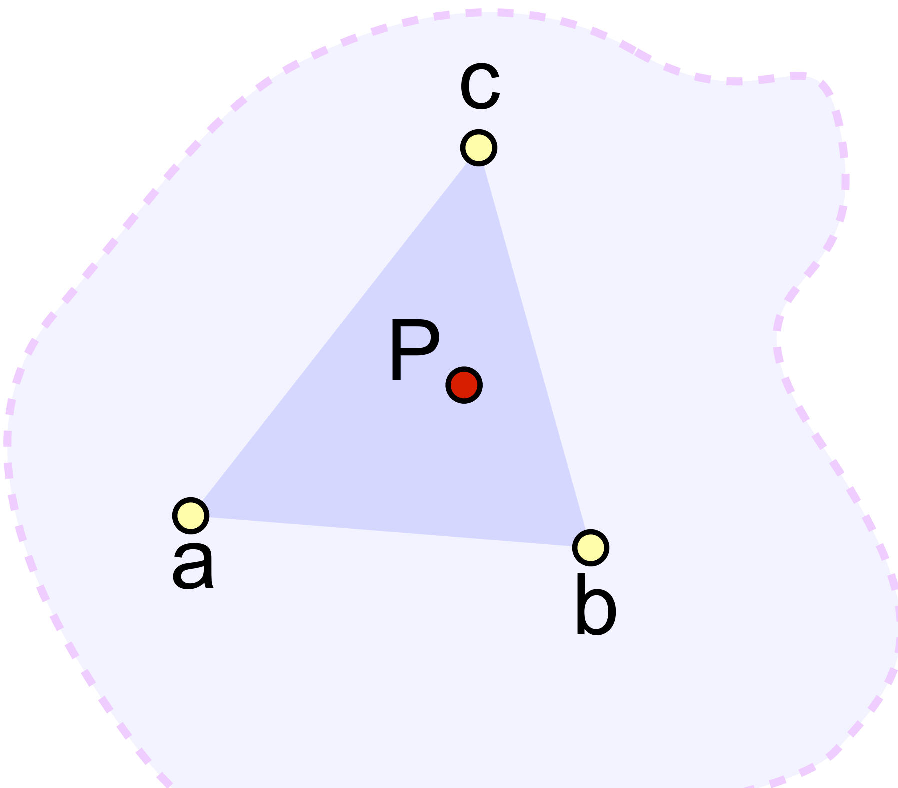



**Non-orthogonal coordinate system on the plane!**

# Barycentric Definition of a Plane

[Möbius, 1827]

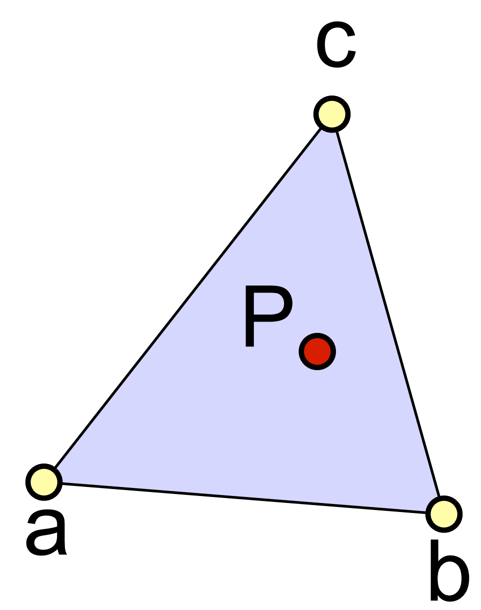
- $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$
  with $\alpha + \beta + \gamma = 1$
- Is it explicit or implicit?




**Fun to know:**
**P** is the **barycenter**, the single point upon which the triangle would balance if weights of size $\alpha$, $\beta$, & $\gamma$ are placed on points **a**, **b** & **c**.

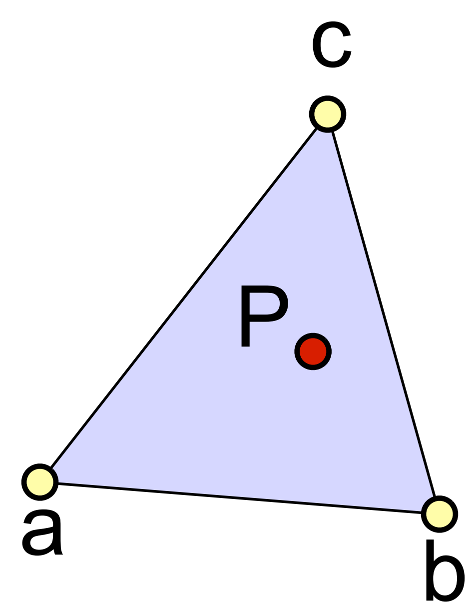
# Barycentric Definition of a Triangle

- $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$
  with $\alpha + \beta + \gamma = 1$ parametrizes the entire plane



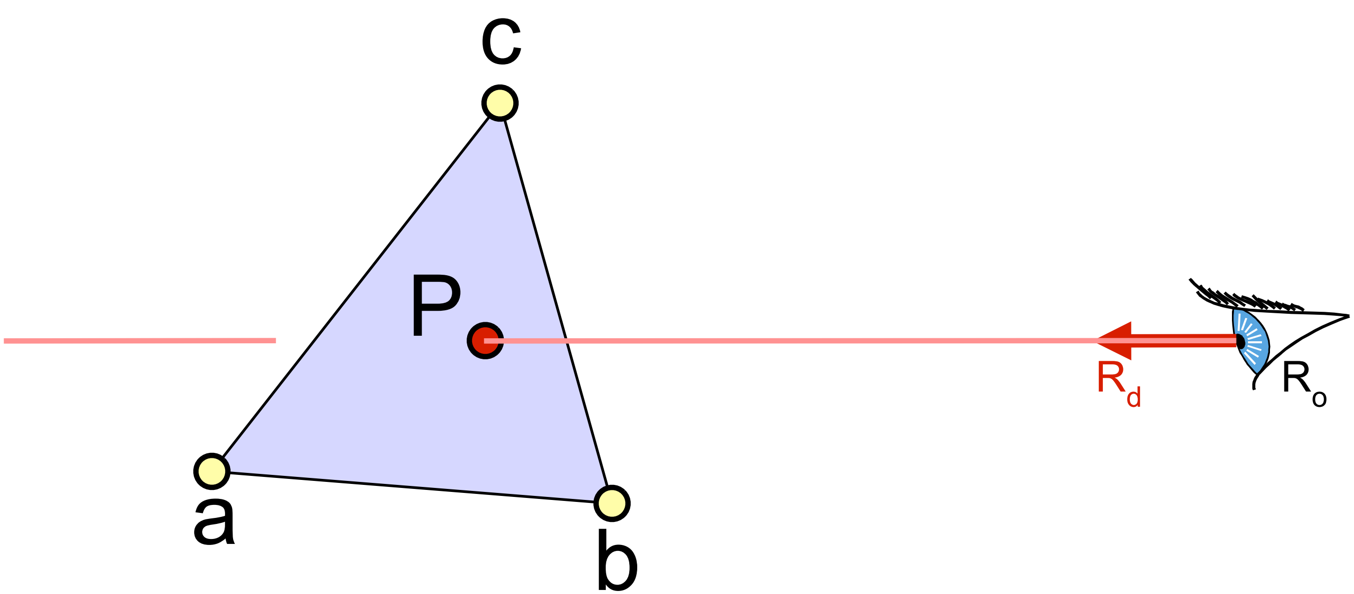
# Barycentric Definition of a Triangle

- $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$
  with $\alpha + \beta + \gamma = 1$ parametrizes the entire plane
- If we require in addition that
  $\alpha, \beta, \gamma >= 0$, we get just the triangle!
  – Note that with $\alpha + \beta + \gamma = 1$ this implies
  $0 < \alpha < 1 \quad \& \quad 0 < \beta < 1 \quad \& \quad 0 < \gamma < 1$
  – Verify:
    - $\alpha=0 \;\; => \;\; \mathbf{P}$ lies on line **b-c**
    - $\alpha,\beta=0 \;\; => \;\; \mathbf{P} = \mathbf{c}$
    - etc.



# Intersection with Barycentric Triangle

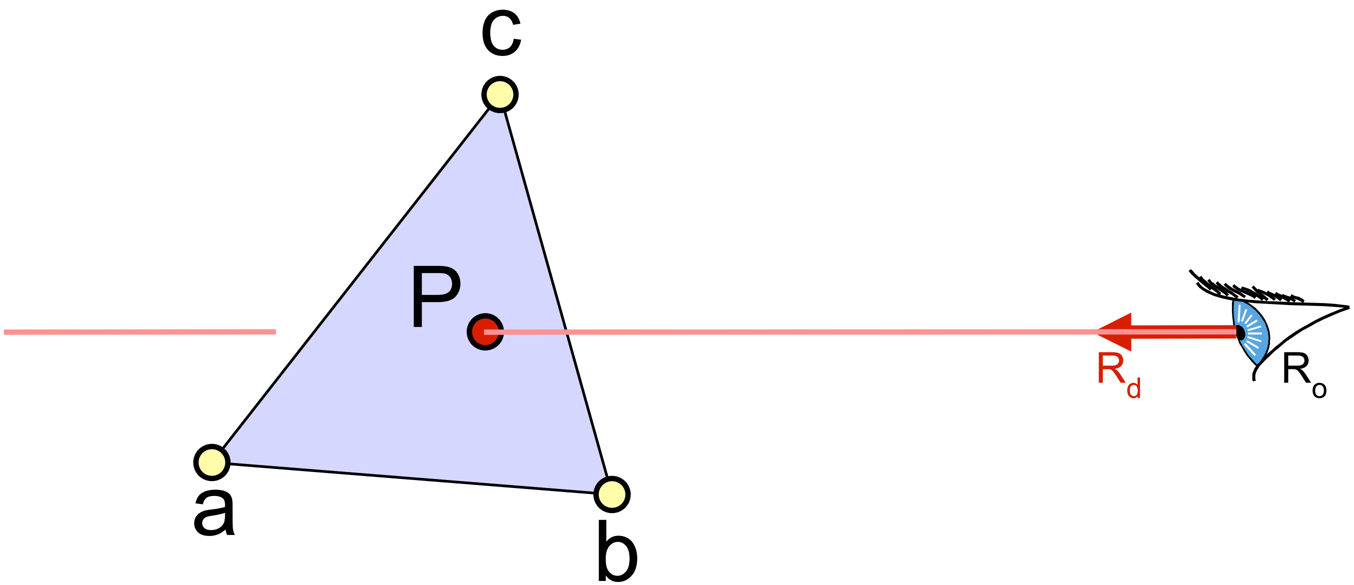
Known
Unknown



# Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

Known
Unknown

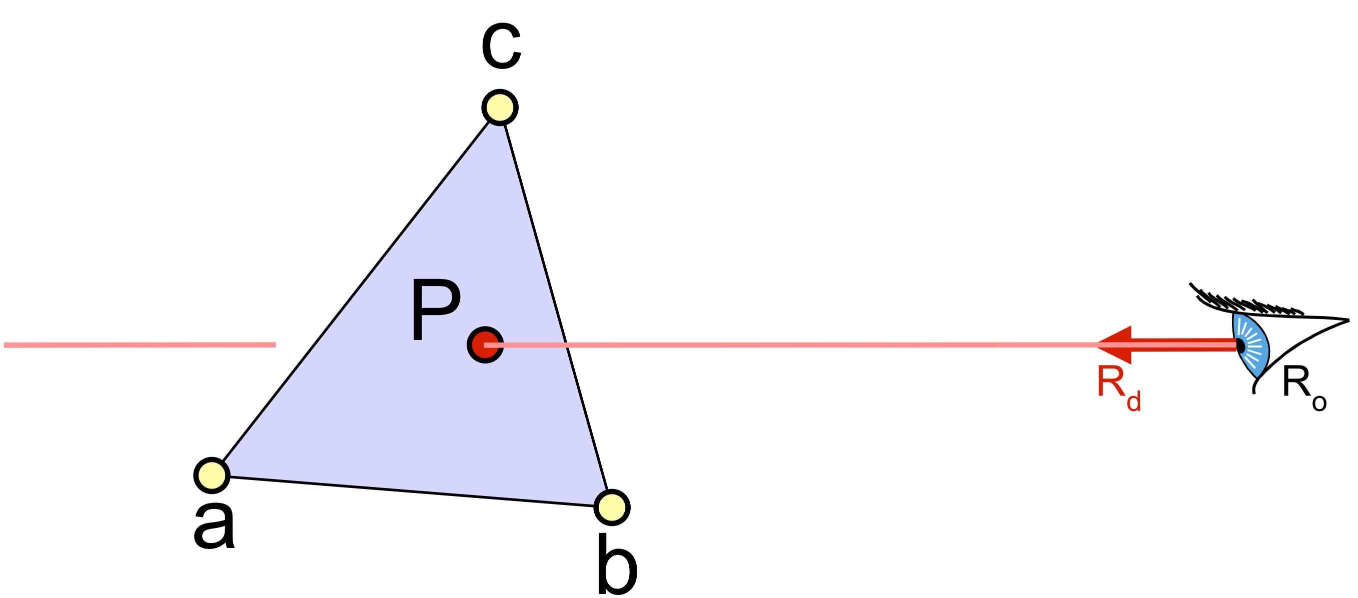


# Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

$$\mathbf{P}(t) = \mathbf{P}(\beta, \gamma)$$

Known
Unknown



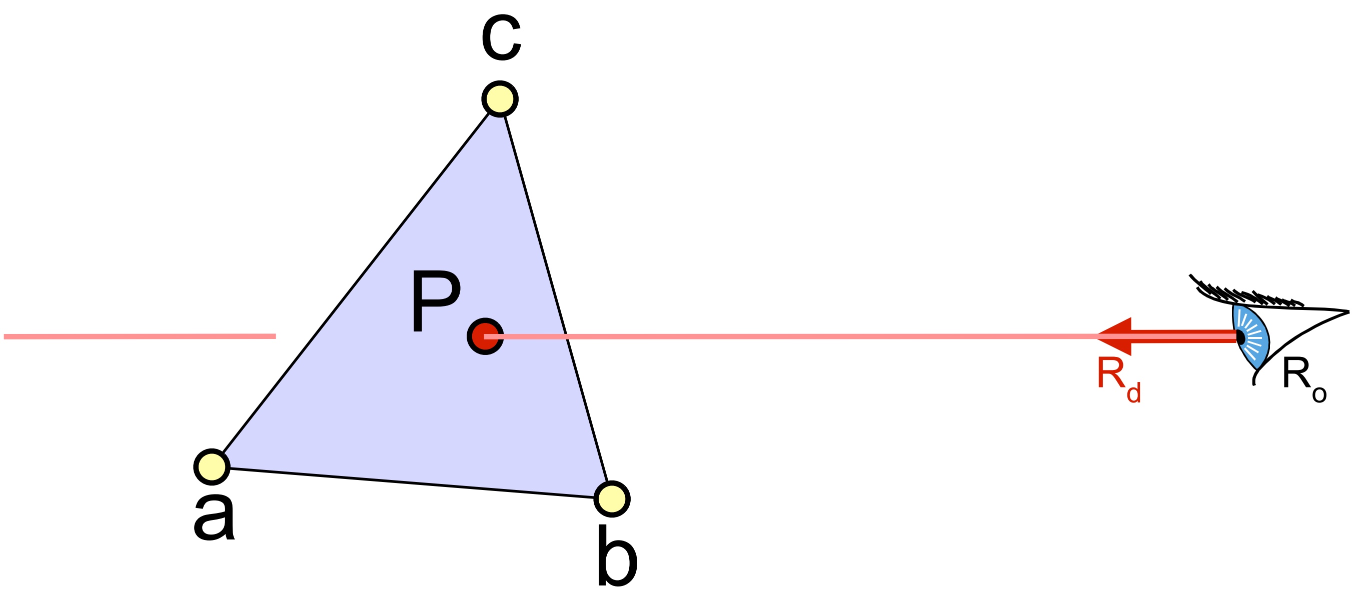
# Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

$$\mathbf{P}(t) = \mathbf{P}(\beta, \gamma)$$

$$\mathbf{R_o} + t * \mathbf{R_d} = \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$$

Known
Unknown



# Intersection with Barycentric Triangle

- Set ray equation equal to barycentric equation

$$\mathbf{P}(t) = \mathbf{P}(\beta, \gamma)$$

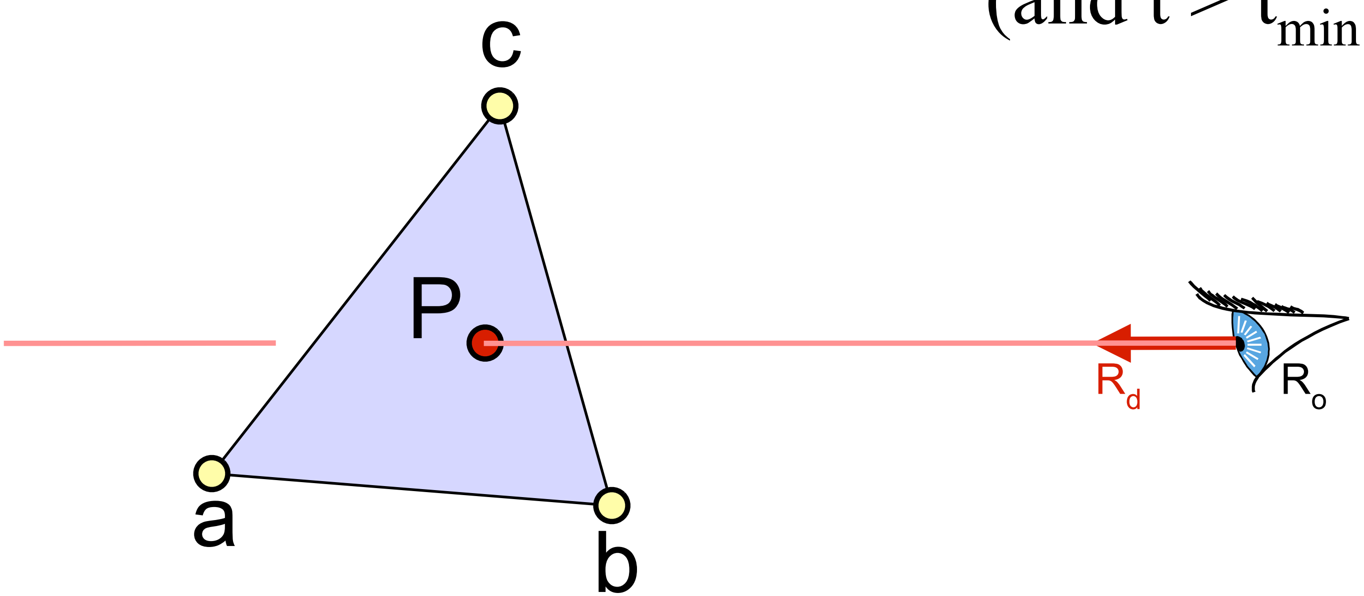
$$\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$$

Known

Unknown

- Intersection if $\beta + \gamma < 1$ & $\beta > 0$ & $\gamma > 0$

(and $t > t_{min}$ … )



# Intersection with Barycentric Triangle

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- $\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$

# Intersection with Barycentric Triangle

- $\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$

$$R_{ox} + tR_{dx} = a_x + \beta(b_x - a_x) + \gamma(c_x - a_x)$$

$$R_{oy} + tR_{dy} = a_y + \beta(b_y - a_y) + \gamma(c_y - a_y)$$

$$R_{oz} + tR_{dz} = a_z + \beta(b_z - a_z) + \gamma(c_z - a_z)$$

# Intersection with Barycentric Triangle

- $\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}-\mathbf{a}) + \gamma(\mathbf{c}-\mathbf{a})$

$$\left.\begin{array}{l} R_{ox} + tR_{dx} = a_x + \beta(b_x-a_x) + \gamma(c_x-a_x) \\ R_{oy} + tR_{dy} = a_y + \beta(b_y-a_y) + \gamma(c_y-a_y) \\ R_{oz} + tR_{dz} = a_z + \beta(b_z-a_z) + \gamma(c_z-a_z) \end{array}\right\} \text{3 equations, 3 unknowns}$$

# Intersection with Barycentric Triangle

- $\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}\text{-}\mathbf{a}) + \gamma(\mathbf{c}\text{-}\mathbf{a})$

$$\left.\begin{aligned} R_{ox} + tR_{dx} &= a_x + \beta(b_x\text{-}a_x) + \gamma(c_x\text{-}a_x) \\ R_{oy} + tR_{dy} &= a_y + \beta(b_y\text{-}a_y) + \gamma(c_y\text{-}a_y) \\ R_{oz} + tR_{dz} &= a_z + \beta(b_z\text{-}a_z) + \gamma(c_z\text{-}a_z) \end{aligned}\right\}$$

3 equations,
3 unknowns

- Regroup & write in matrix form $\mathbf{Ax}=\mathbf{b}$ ($\Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$)

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

# Intersection with Barycentric Triangle

- $\mathbf{R}_o + t * \mathbf{R}_d = \mathbf{a} + \beta(\mathbf{b}\text{-}\mathbf{a}) + \gamma(\mathbf{c}\text{-}\mathbf{a})$

$$\left.\begin{aligned} R_{ox} + tR_{dx} &= a_x + \beta(b_x - a_x) + \gamma(c_x - a_x) \\ R_{oy} + tR_{dy} &= a_y + \beta(b_y - a_y) + \gamma(c_y - a_y) \\ R_{oz} + tR_{dz} &= a_z + \beta(b_z - a_z) + \gamma(c_z - a_z) \end{aligned}\right\}$$

3 equations,
3 unknowns

- Regroup & write in matrix form $\mathbf{Ax}=\mathbf{b}$ ($\Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$)

$$\begin{bmatrix} a_x - b_x & a_x - c_x & R_{dx} \\ a_y - b_y & a_y - c_y & R_{dy} \\ a_z - b_z & a_z - c_z & R_{dz} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - R_{ox} \\ a_y - R_{oy} \\ a_z - R_{oz} \end{bmatrix}$$

# Solving the System

$$\begin{bmatrix} a_x - b_x & a_x - c_x & R_{dx} \\ a_y - b_y & a_y - c_y & R_{dy} \\ a_z - b_z & a_z - c_z & R_{dz} \end{bmatrix} \begin{bmatrix} \beta \\ \gamma \\ t \end{bmatrix} = \begin{bmatrix} a_x - R_{ox} \\ a_y - R_{oy} \\ a_z - R_{oz} \end{bmatrix}$$

- Just invert the matrix by brute force (OK for 3x3)
- Or use Cramer's rule (next slide)
- In the end, all triangle intersection algorithms have to perform these computations
  - Differences lie in what parts they precompute, and in which order they check for early-outs

# Solving Ax=b, Cramer's Rule

- Used to solve for one variable at a time in system of equations

$$\beta = \frac{\begin{vmatrix} a_x - R_{ox} & a_x - c_x & R_{dx} \\ a_y - R_{oy} & a_y - c_y & R_{dy} \\ a_z - R_{oz} & a_z - c_z & R_{dz} \end{vmatrix}}{|A|} \qquad \gamma = \frac{\begin{vmatrix} a_x - b_x & a_x - R_{ox} & R_{dx} \\ a_y - b_y & a_y - R_{oy} & R_{dy} \\ a_z - b_z & a_z - R_{oz} & R_{dz} \end{vmatrix}}{|A|}$$

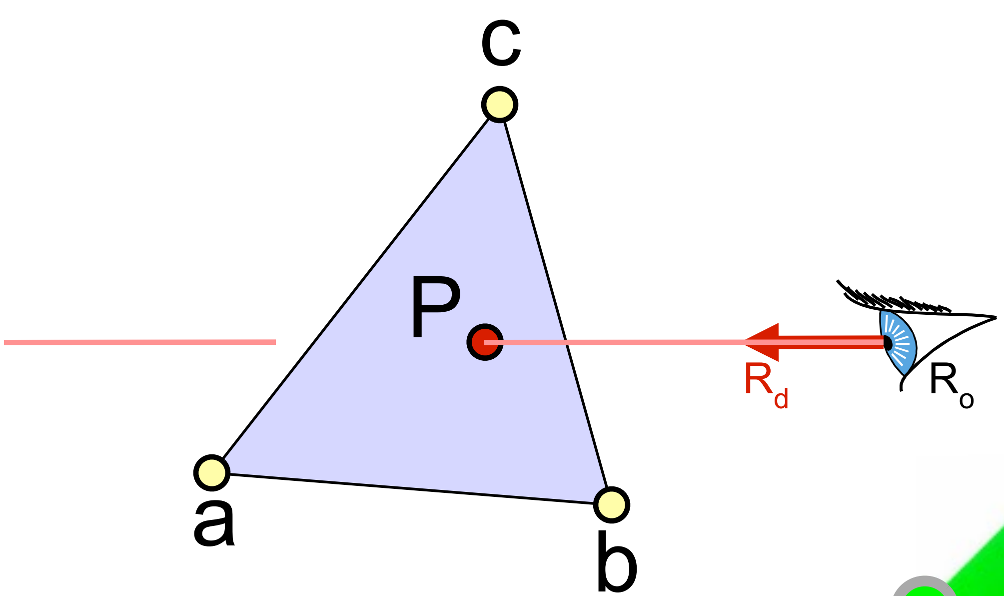
$$t = \frac{\begin{vmatrix} a_x - b_x & a_x - c_x & a_x - R_{ox} \\ a_y - b_y & a_y - c_y & a_y - R_{oy} \\ a_z - b_z & a_z - c_z & a_z - R_{oz} \end{vmatrix}}{|A|}$$

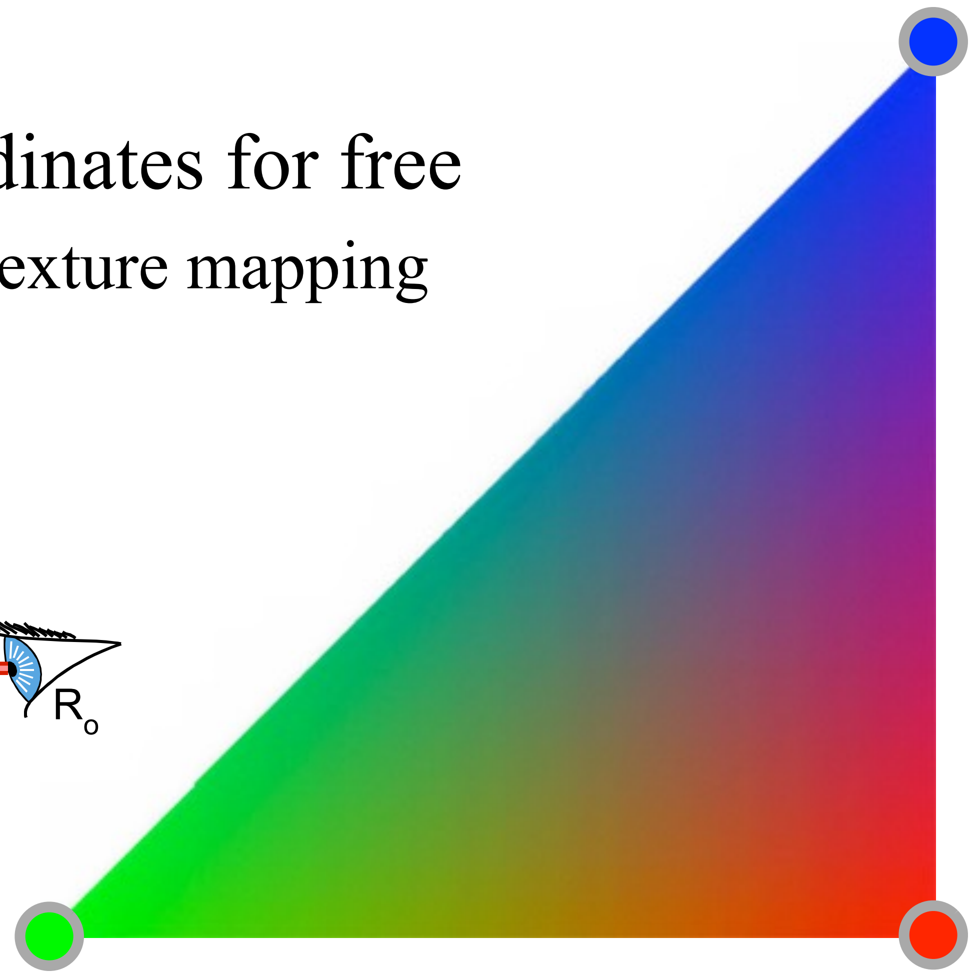
| | denotes the determinant

Can be copied mechanically into code

# Barycentric Intersection Pros

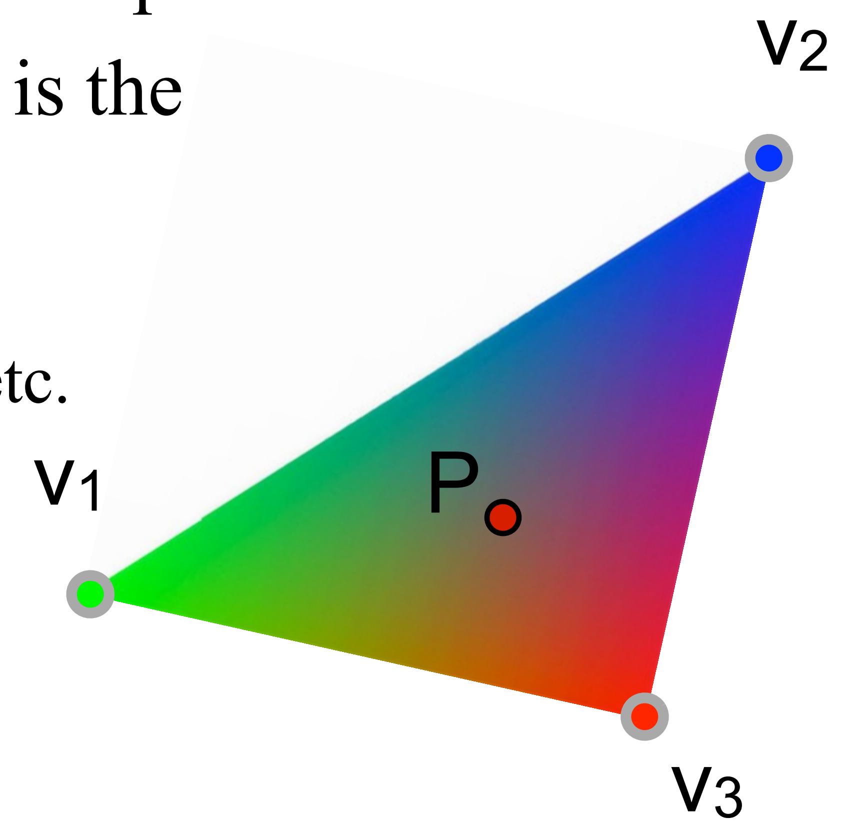
- Efficient
- Stores no plane equation
- Get the barycentric coordinates for free
  - Useful for interpolation, texture mapping



# Barycentric Interpolation

- Values $v_1$, $v_2$, $v_3$ defined at **a**, **b**, **c**
  - Colors, normal, texture coordinates, etc.
- $\mathbf{P}(\alpha, \beta, \gamma) = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ is the point...
- $v(\alpha, \beta, \gamma) = \alpha v_1 + \beta v_2 + \gamma v_3$ is the barycentric interpolation of $v_1, v_2, v_3$ at point **P**
  - Sanity check: $v(1,0,0) = v_1$, etc.
- I.e, once you know $\alpha$, $\beta$, $\gamma$, you can interpolate values using the same weights.
  - Convenient!



# That's It!

- Image computed using the RADIANCE system by Greg Ward