Conditional Lower Bounds for Failed Literals and Related Techniques

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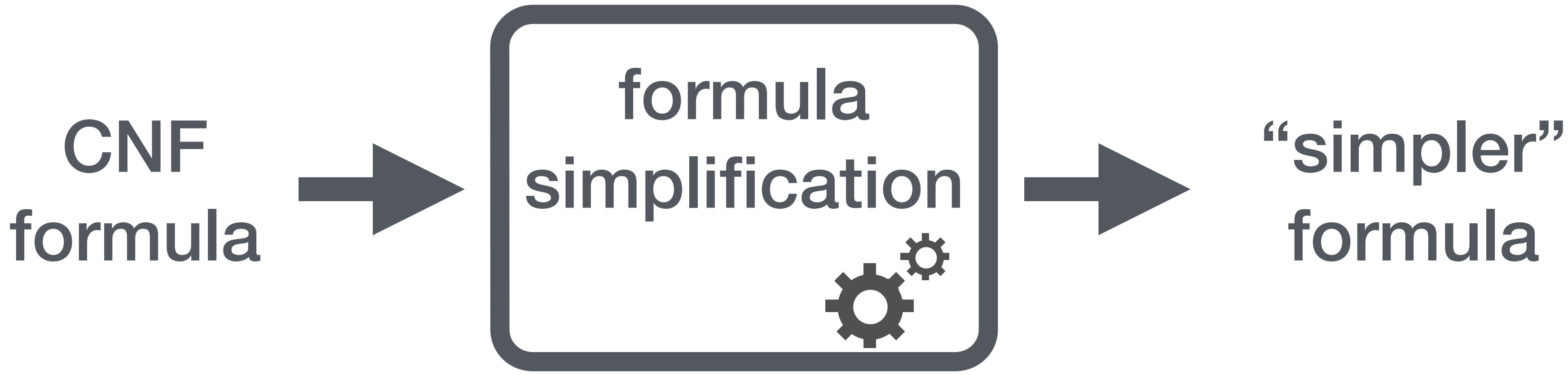
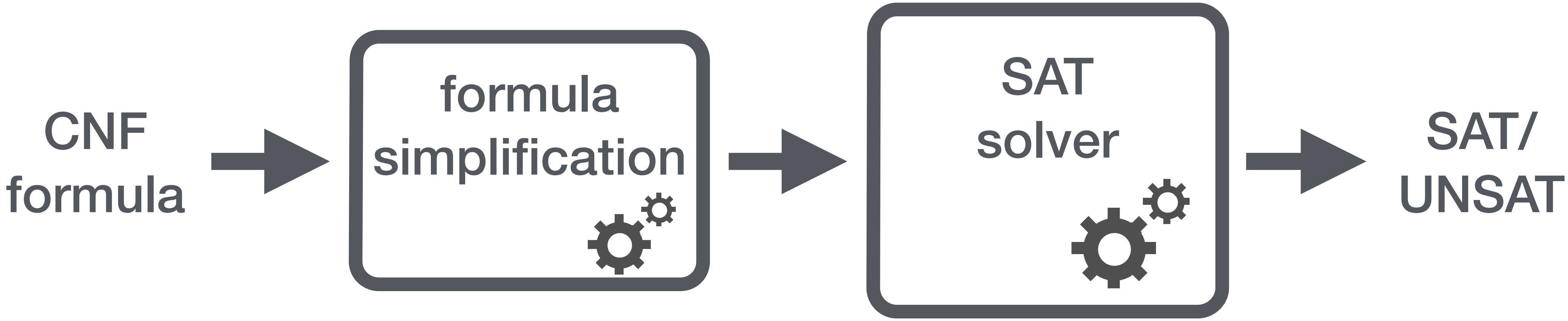
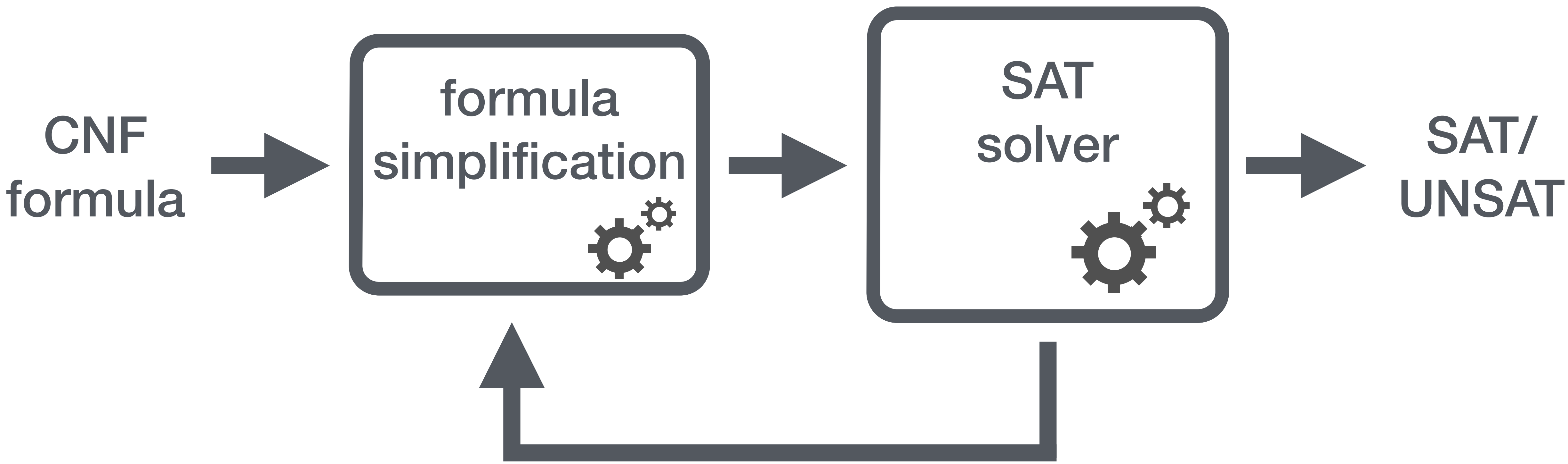
1.

Background:

Formula Simplification


CNF
formula
formula
simplification
"simpler"
formula


CNF
formula
formula
simplification
SAT
solver
SAT/
UNSAT


CNF
formula
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SAT/
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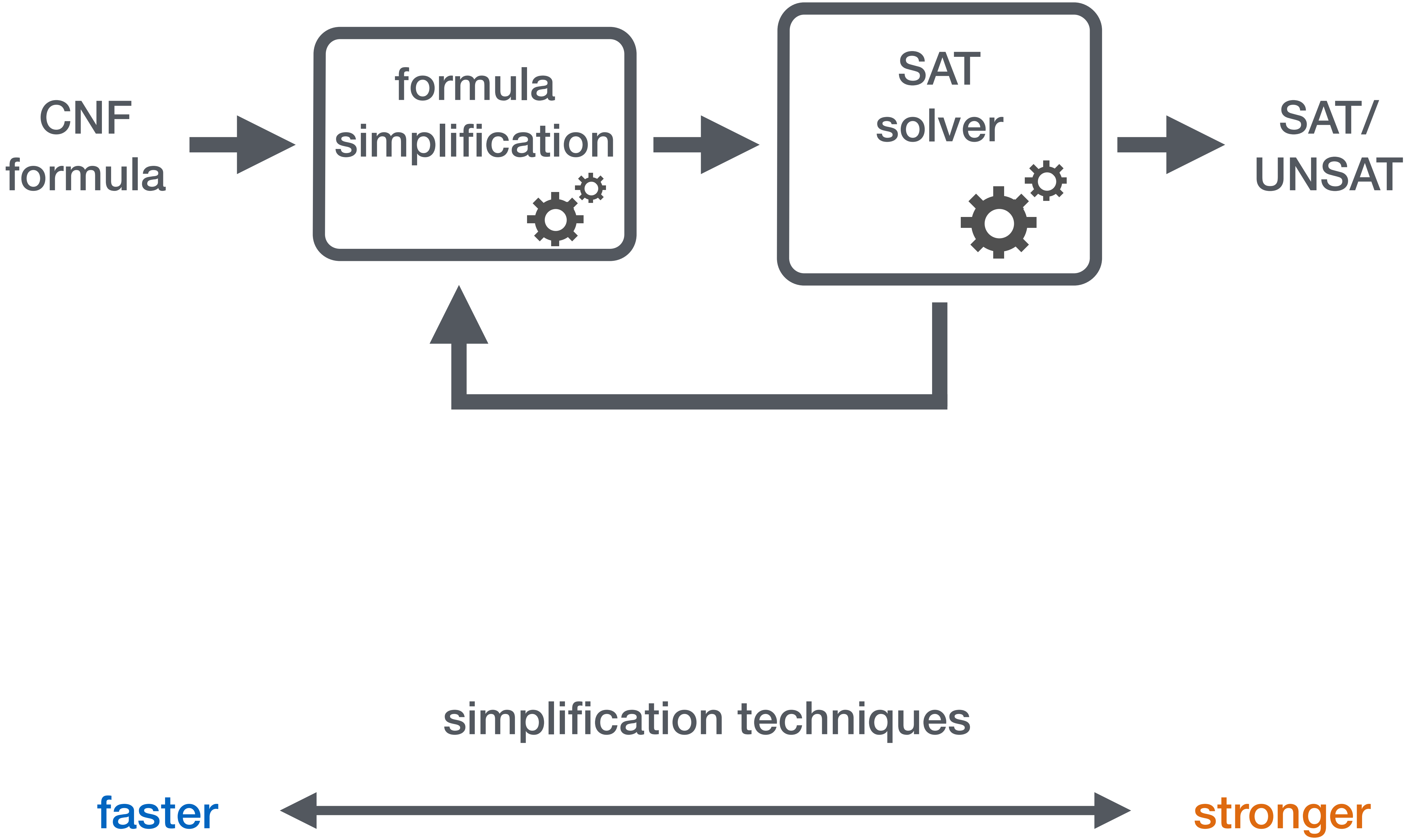
CNF
formula
formula
simplification
SAT
solver
SAT/
UNSAT
simplification techniques
faster
stronger

2.

Main Result:

Lower Bound for Failed Literal Existence

- **Unit propagation**

$$\left.\begin{array}{l}(\ell_1 \vee \ldots \vee \ell_k \vee \ell) \\ (\neg \ell_1), (\neg \ell_2), \ldots, (\neg \ell_k)\end{array}\right\} \rightarrow (\ell)$$

  - apply until fixpoint
  - we write $F \vdash_{\text{up}} (\ell)$ if (ℓ) can be derived from F by repeated application of unit resolution rule

- **Unit propagation**

$$\left.\begin{array}{l}(\ell_1 \vee \ldots \vee \ell_k \vee \ell) \\ (\neg\ell_1),\ (\neg\ell_2),\ \ldots,\ (\neg\ell_k)\end{array}\right\} \rightarrow (\ell)$$

  - apply until fixpoint
  - we write $F \vdash_{up} (\ell)$ if (ℓ) can be derived from F by repeated application of unit resolution rule

- **Failed literals**
  - a literal $\ell \in F$ is a **failed literal** if $F \wedge (\ell) \vdash_{up} (\ell'), (\neg\ell')$ for some $\ell' \in F$
  - replace F with $F \wedge (\neg\ell)$ if ℓ is a failed literal

Failed literal existence problem

Input: CNF formula F

Problem: Decide whether F has a failed literal

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Problem: Decide whether F has a failed literal

- **Upper bounds** (assuming bounded clause width)
  - unit propagation $O(n+m)$
  - failed literal existence $O(n(n+m))$
  - failed literal elimination fixpoint $O(n^2(n+m))$
  - **Can we do any better?**

Theorem. If failed literal existence can solved in

$$O((n+m)^{2-\varepsilon})$$

time on Horn-3-CNFs for some $\varepsilon > 0$, then CNF-SAT can be solved in time

$$2^{(1-\varepsilon/2)n}\,\mathrm{poly}(n,m)$$

on formulas of unrestricted clause length.

- Recall that CNF formula is **Horn** if each clause has at most one unnegated variable
- Horn-SAT is solvable in linear time

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- We do not know how to solve CNF-SAT in time $2^{(1-\varepsilon)n} \text{ poly}(n,m)$ for any $\varepsilon > 0$
- This would give **exponential** speed-up for CNF-SAT!

The Strong Exponential Time Hypothesis

$$\liminf_{n \to \infty} \{\delta : k\text{-SAT can be solved in time } O(2^{\delta n})\} = 1$$

$$\Downarrow$$

CNF-SAT with unrestricted clause length cannot be solved in time $2^{(1-\varepsilon)n}\,\mathrm{poly}(n,m)$ for any $\varepsilon > 0$

[Calabro, Impagliazzo, and Paturi 2009]

Corollary. Failed literal existence restricted to Horn-3-CNFs cannot be solved in time $O((n+m)^{2-\varepsilon})$ for any $\varepsilon > 0$ unless SETH fails.

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- **Compare with other similar results:** for any $\varepsilon > 0$ we cannot solve
  - k-dominating set for $k \geq 3$ in time $O((n+m)^{k-\varepsilon})$
  - 2-SAT with **$O(n)$ clauses** and **two unrestricted length clauses** in time $O(n^{2-\varepsilon})$

    [Pătraşcu and Williams 2010]
  - Local alignment of two binary strings in time $O(n^{2-\varepsilon})$

    [Abboud, Vassilevska Williams, and Weimann 2014]

3.

Proof of the Failed Literal Existence Lower Bound

3a.

Proof Overview

Theorem. If failed literal existence can solved in

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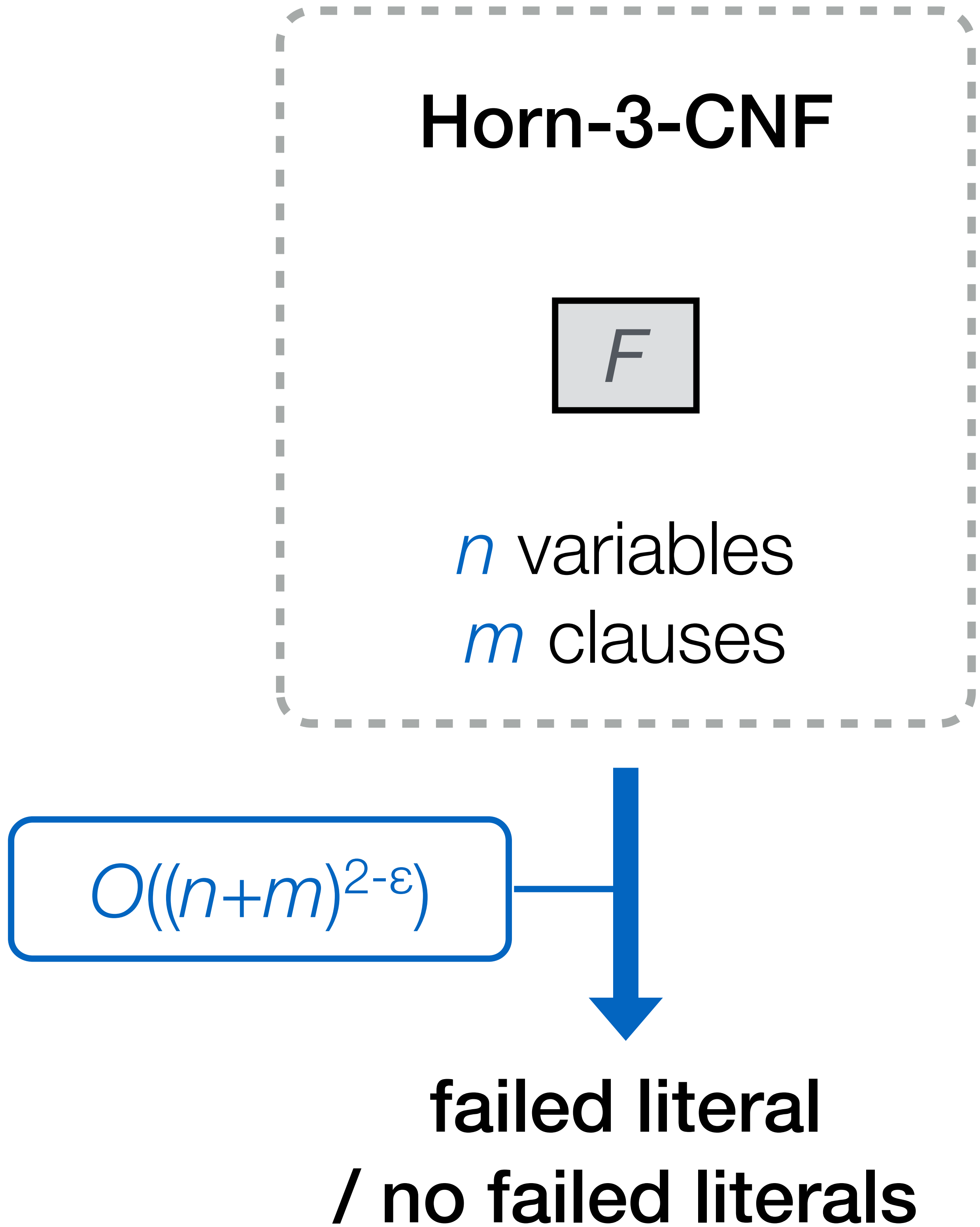
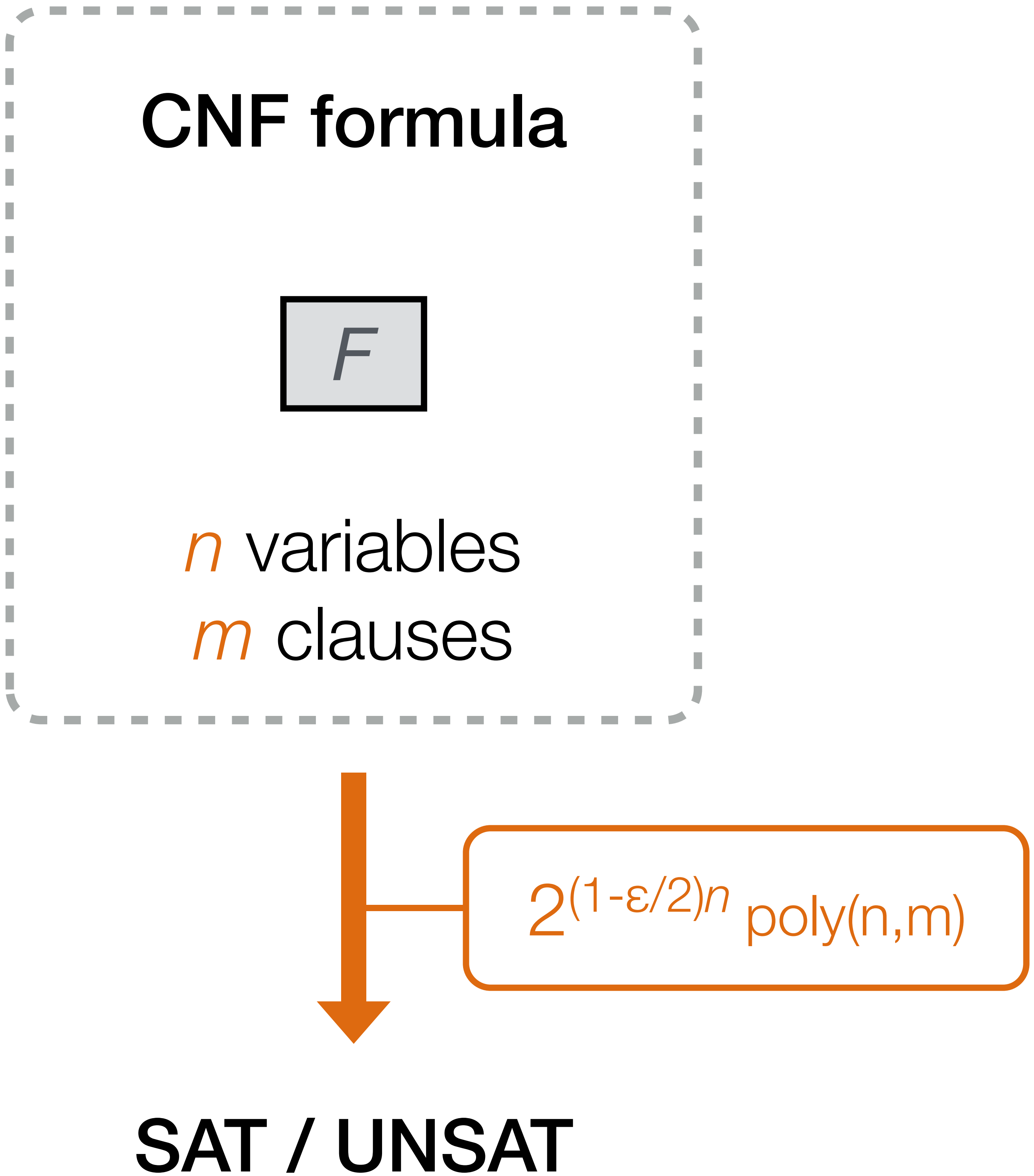
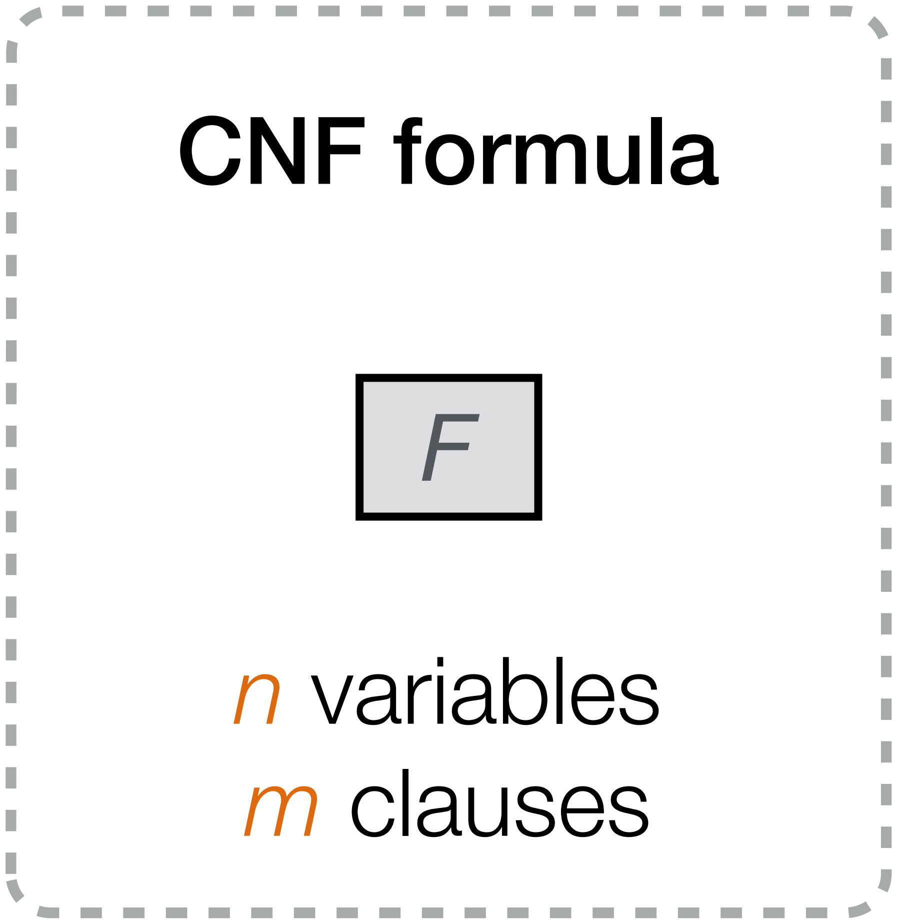
$$2^{(1-\varepsilon/2)n} \text{poly}(n,m)$$

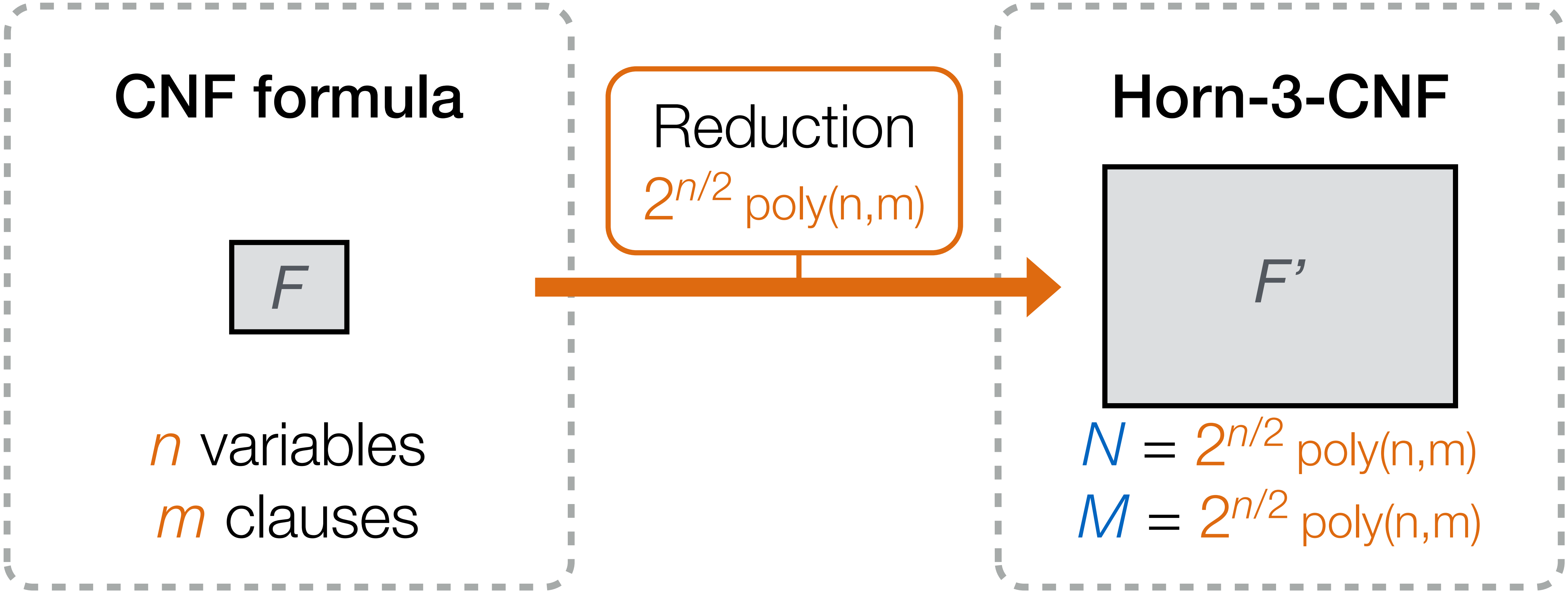
on formulas of unrestricted clause length.

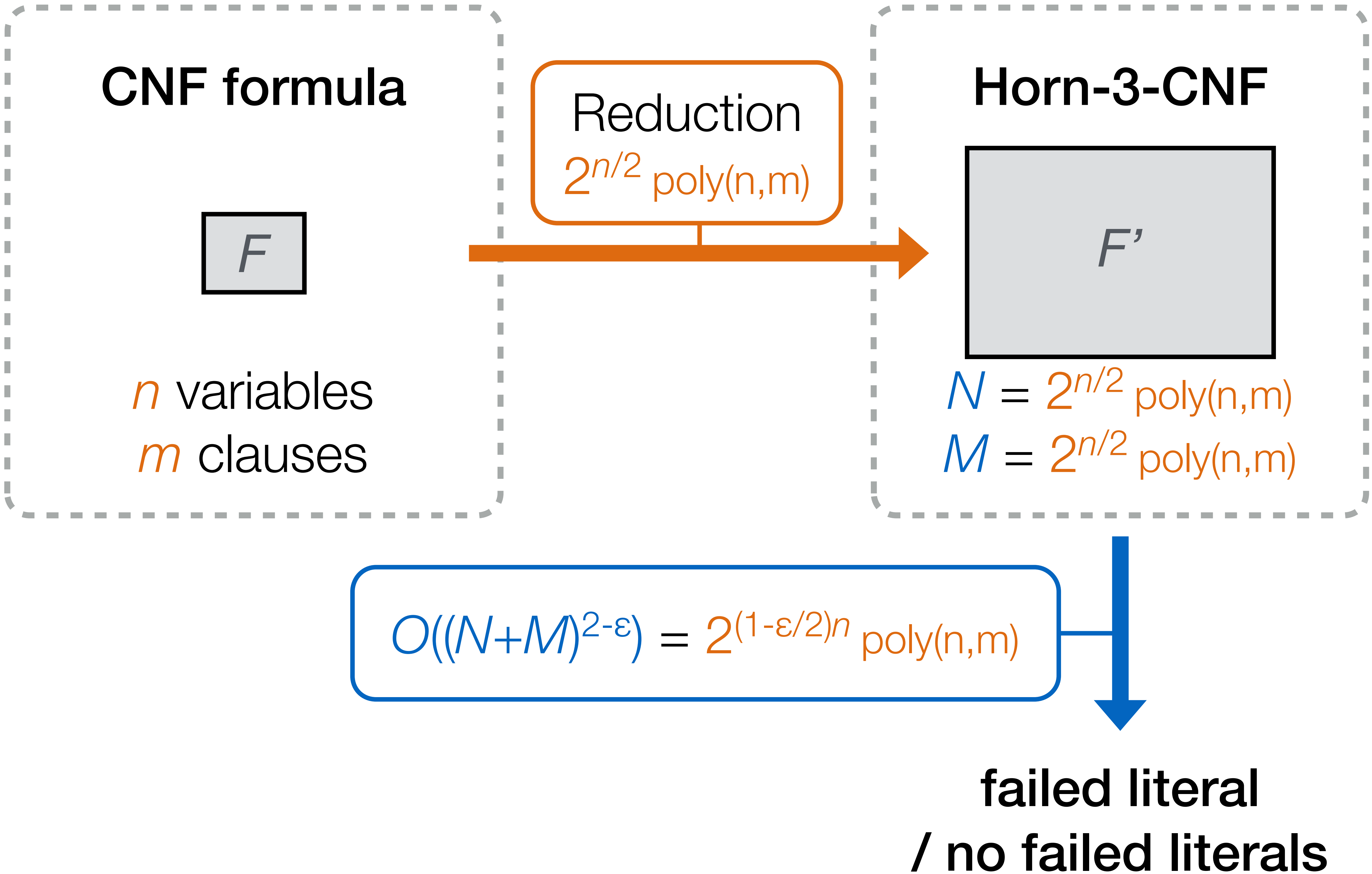


Horn-3-CNF
F
n variables
m clauses
O((n+m)2-ε)
failed literal
/ no failed literals

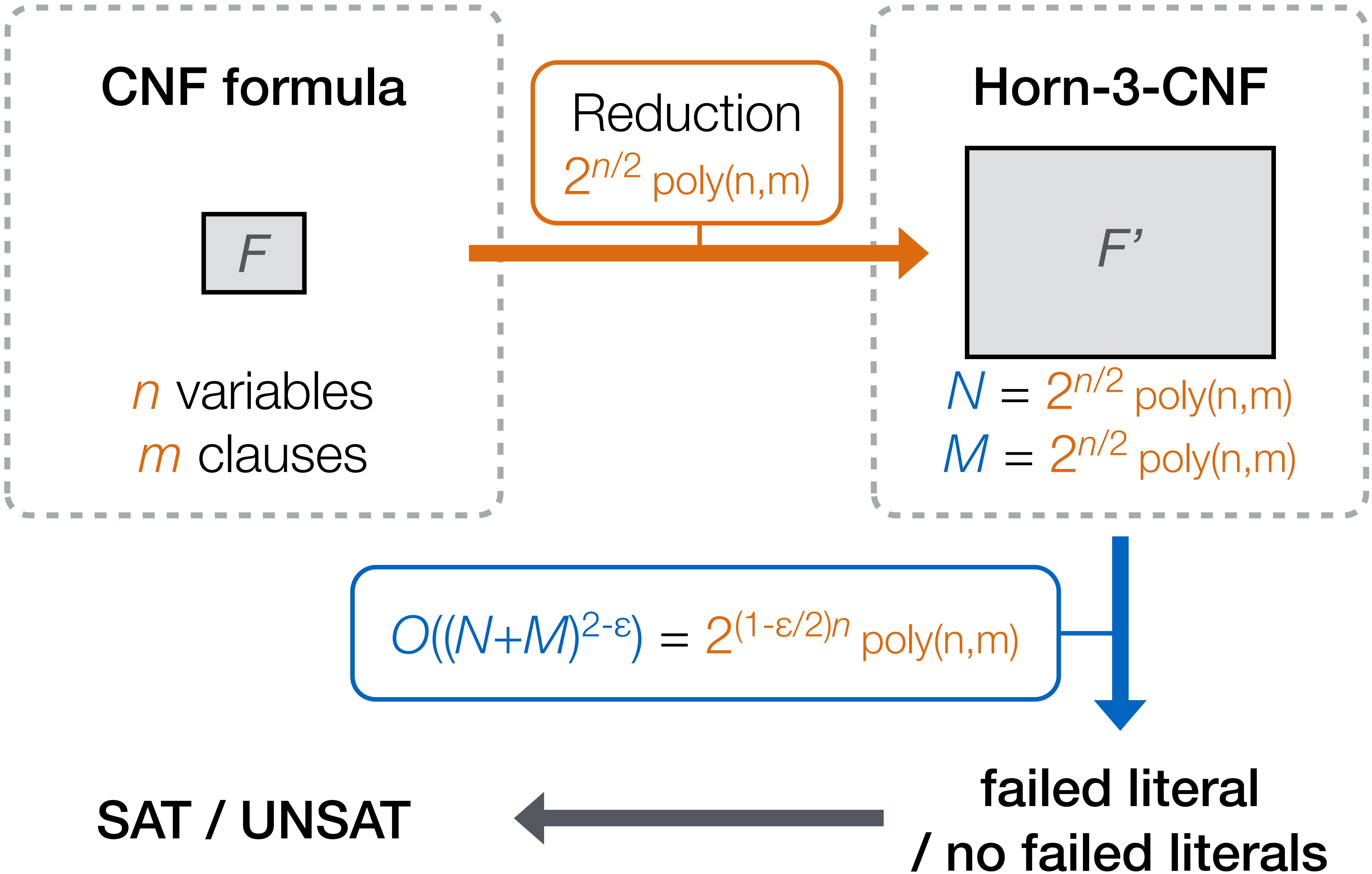

CNF formula
F
n variables
m clauses
$2^{(1-\varepsilon/2)n}$ poly(n,m)
SAT / UNSAT


CNF formula
F
n variables
m clauses


CNF formula
F
n variables
m clauses
Reduction
2^{n/2} poly(n,m)
Horn-3-CNF
F'
N = 2^{n/2} poly(n,m)
M = 2^{n/2} poly(n,m)



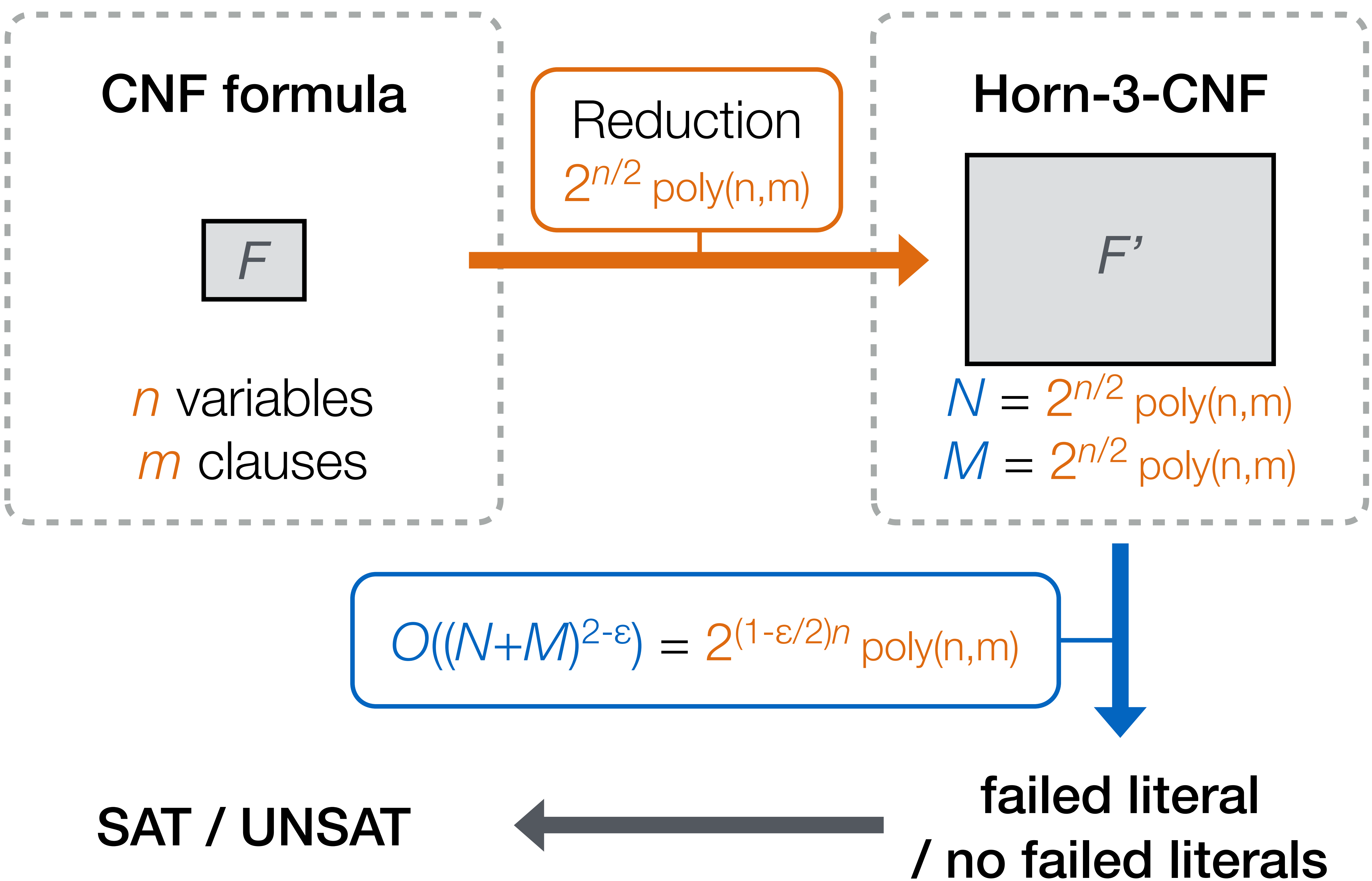
CNF formula
F
n variables
m clauses
Reduction
2^{n/2} poly(n,m)
Horn-3-CNF
F'
N = 2^{n/2} poly(n,m)
M = 2^{n/2} poly(n,m)
O((N+M)^{2-ε}) = 2^{(1-ε/2)n} poly(n,m)
failed literal
/ no failed literals



CNF formula
F
n variables
m clauses
Reduction
2^{n/2} poly(n,m)
Horn-3-CNF
F'
N = 2^{n/2} poly(n,m)
M = 2^{n/2} poly(n,m)
O((N+M)^{2-ε}) = 2^{(1-ε/2)n} poly(n,m)
failed literal
/ no failed literals
SAT / UNSAT

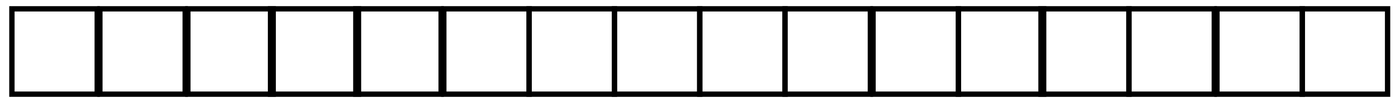
3b.

Reduction from CNF-SAT to Failed Literal Elimination


CNF formula
F
n variables
m clauses
Reduction
$2^{n/2}$ poly(n,m)
Horn-3-CNF
F'
$N = 2^{n/2}$ poly(n,m)
$M = 2^{n/2}$ poly(n,m)
$O((N+M)^{2-\varepsilon}) = 2^{(1-\varepsilon/2)n}$ poly(n,m)
failed literal
/ no failed literals
SAT / UNSAT

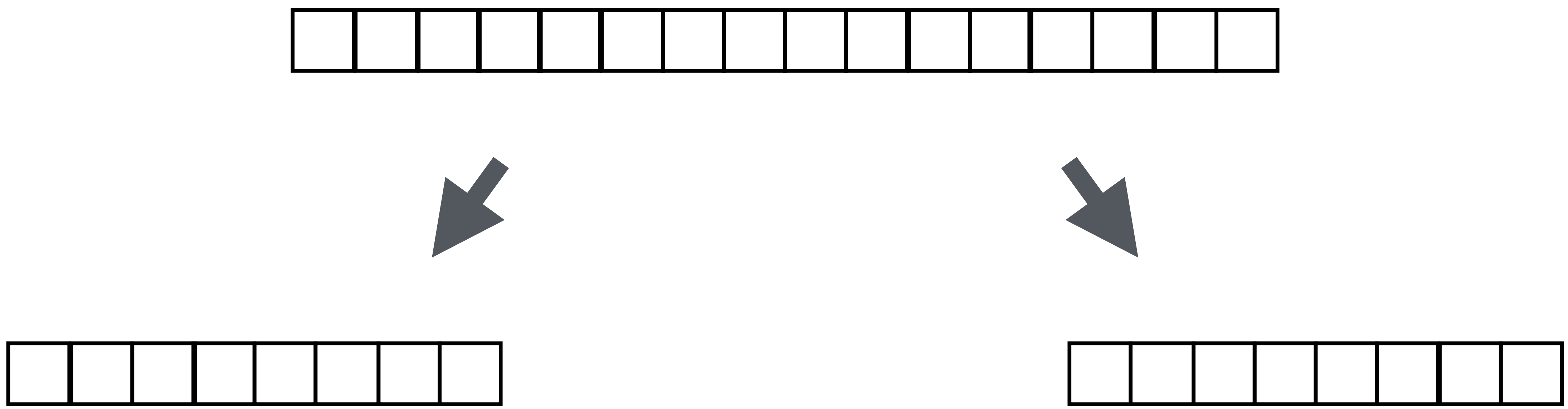
input formula F

n variables



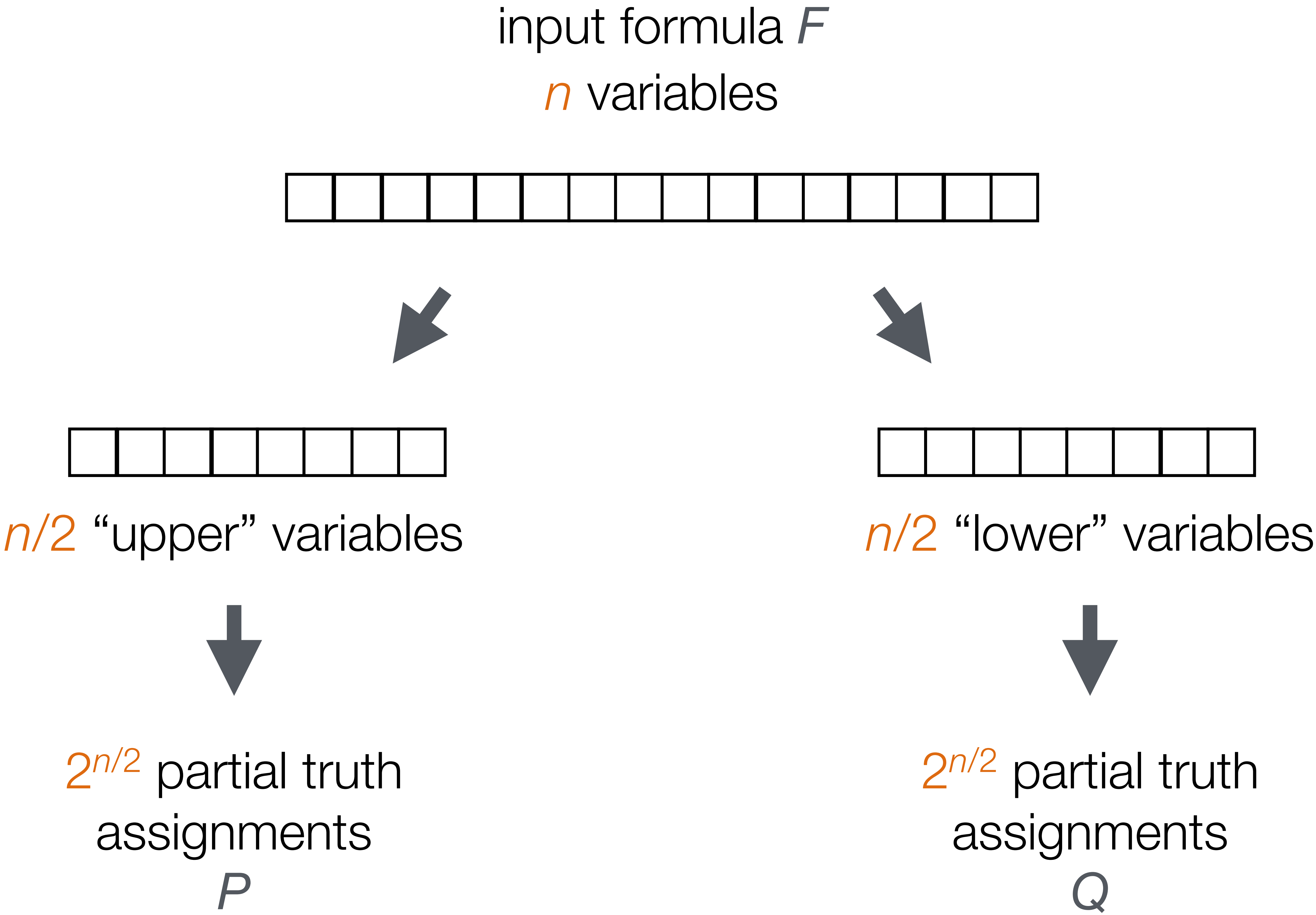
input formula F

n variables

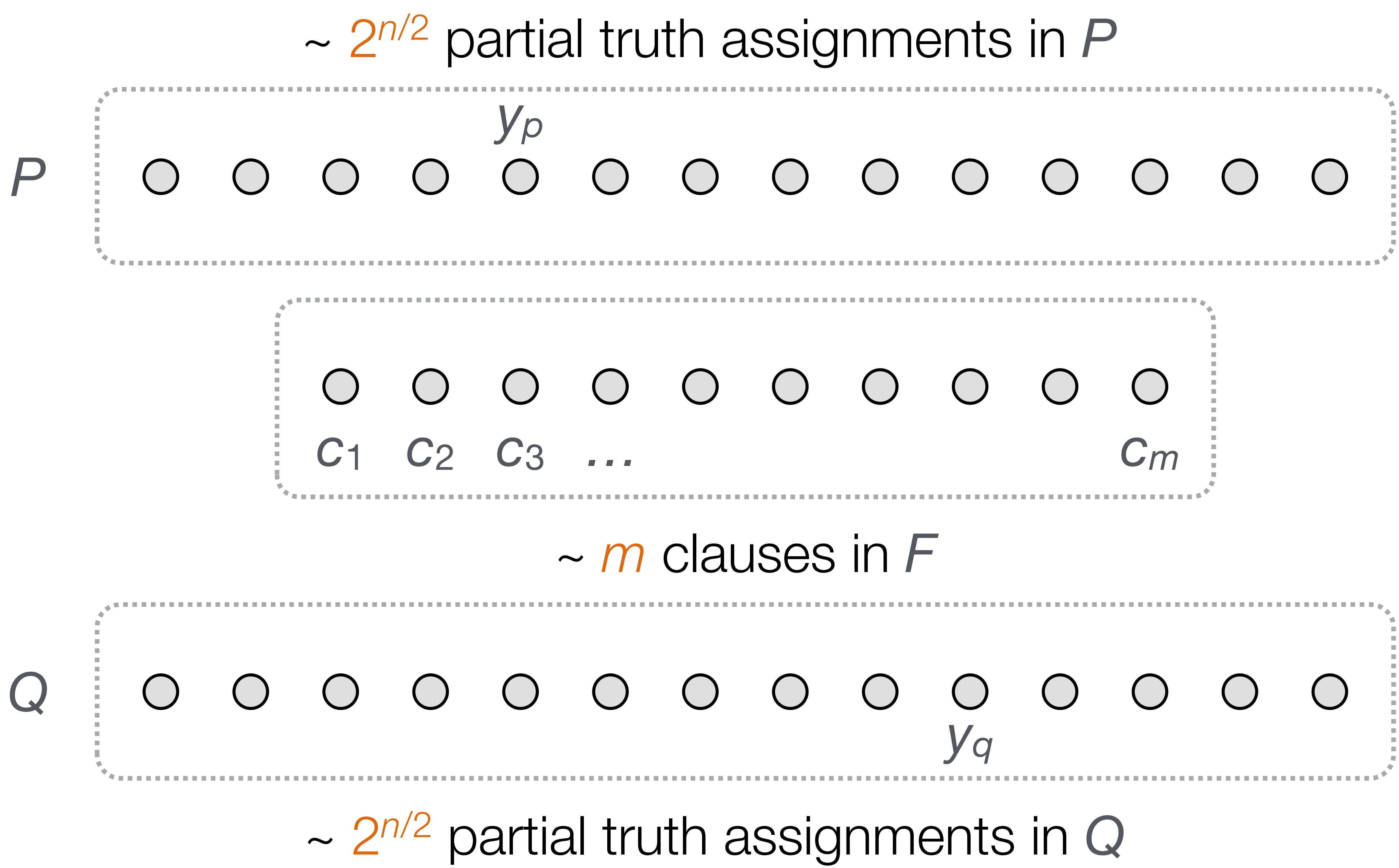


$n/2$ “upper” variables

$n/2$ “lower” variables

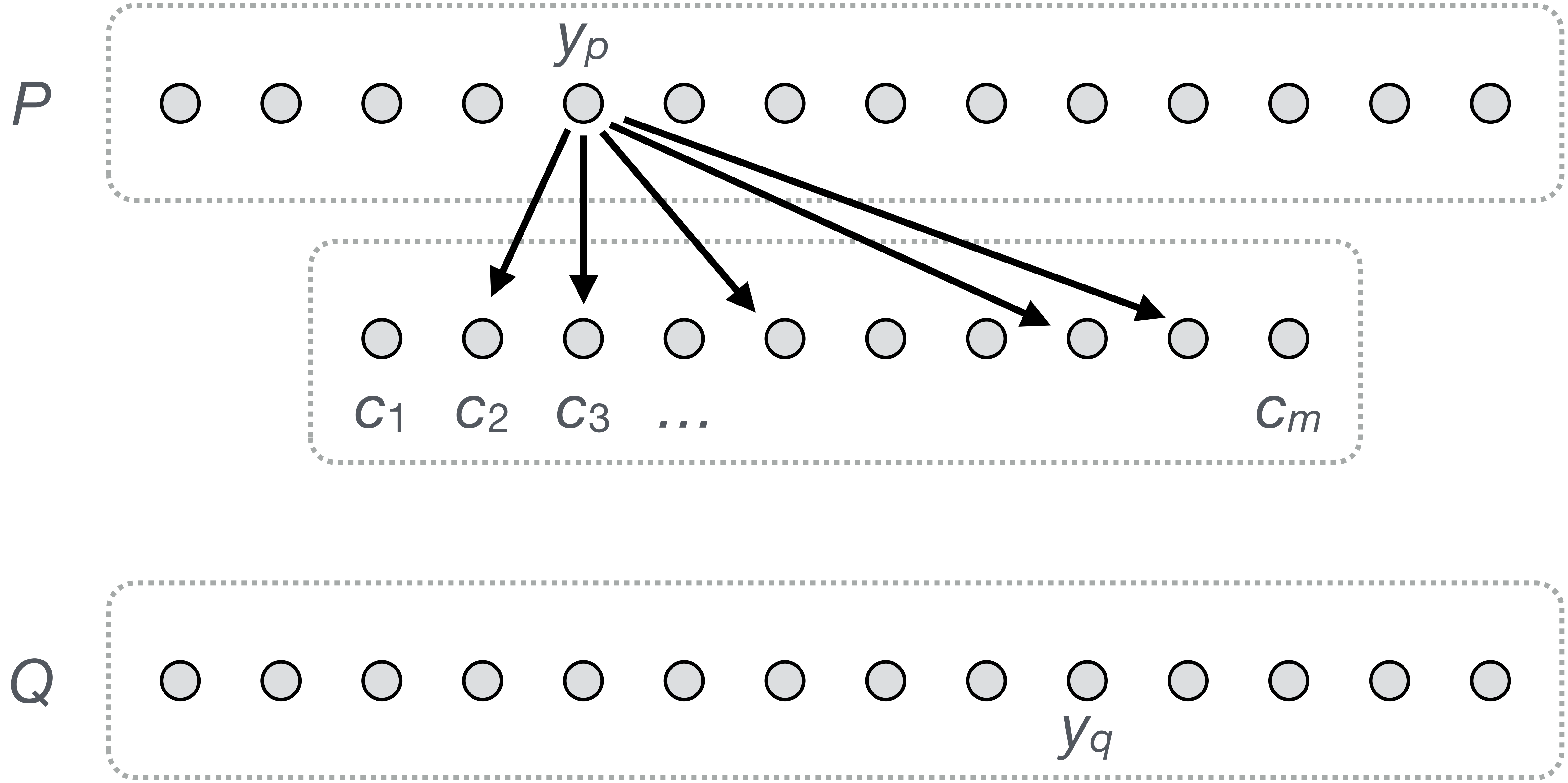

input formula F
n variables
n/2 “upper” variables
n/2 “lower” variables
2^{n/2} partial truth assignments
P
2^{n/2} partial truth assignments
Q

constructing the output formula *F'*



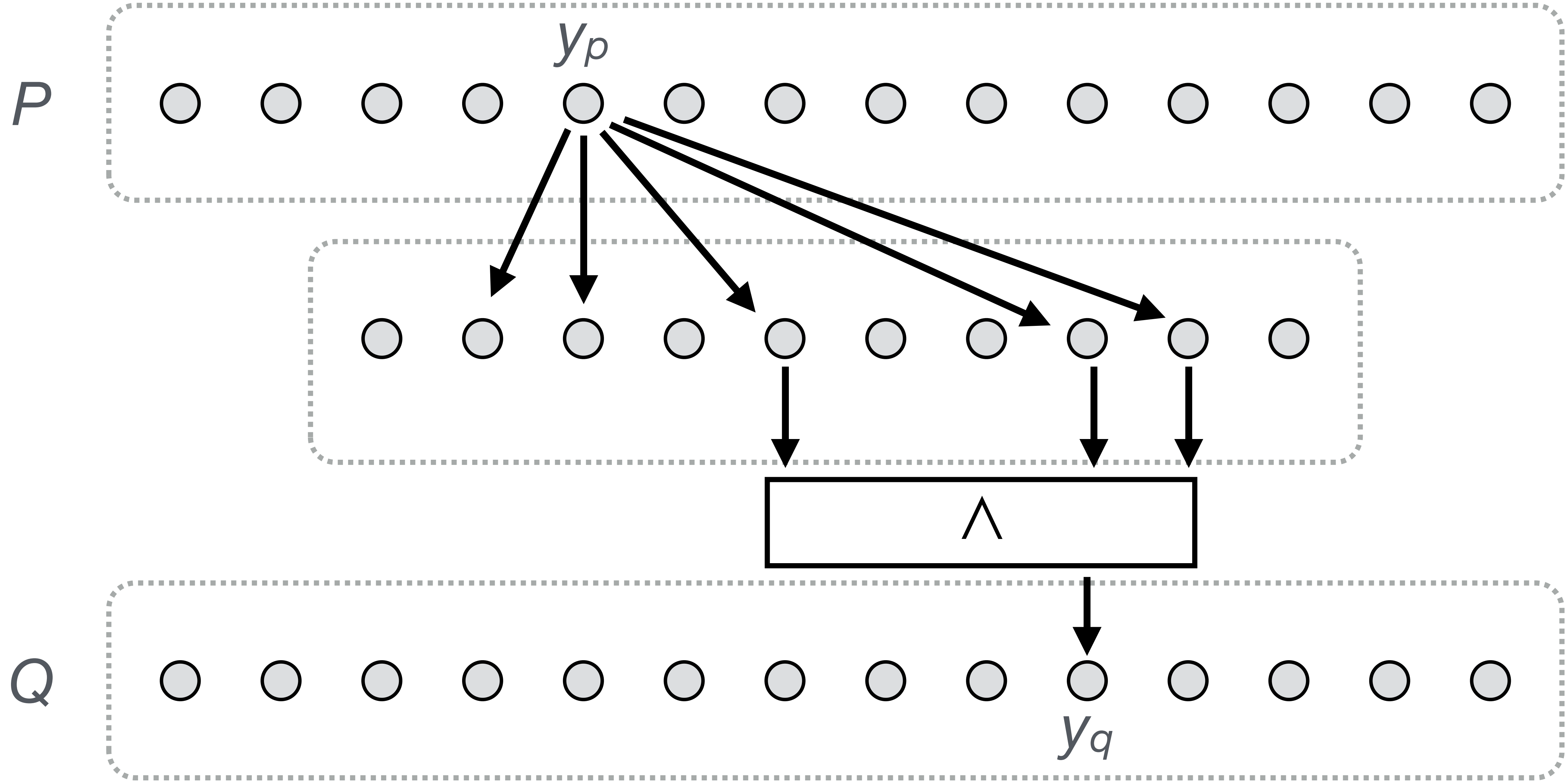
constructing the output formula F'

add clause $(y_p \rightarrow c_i)$ if $p(C_i) = 1$



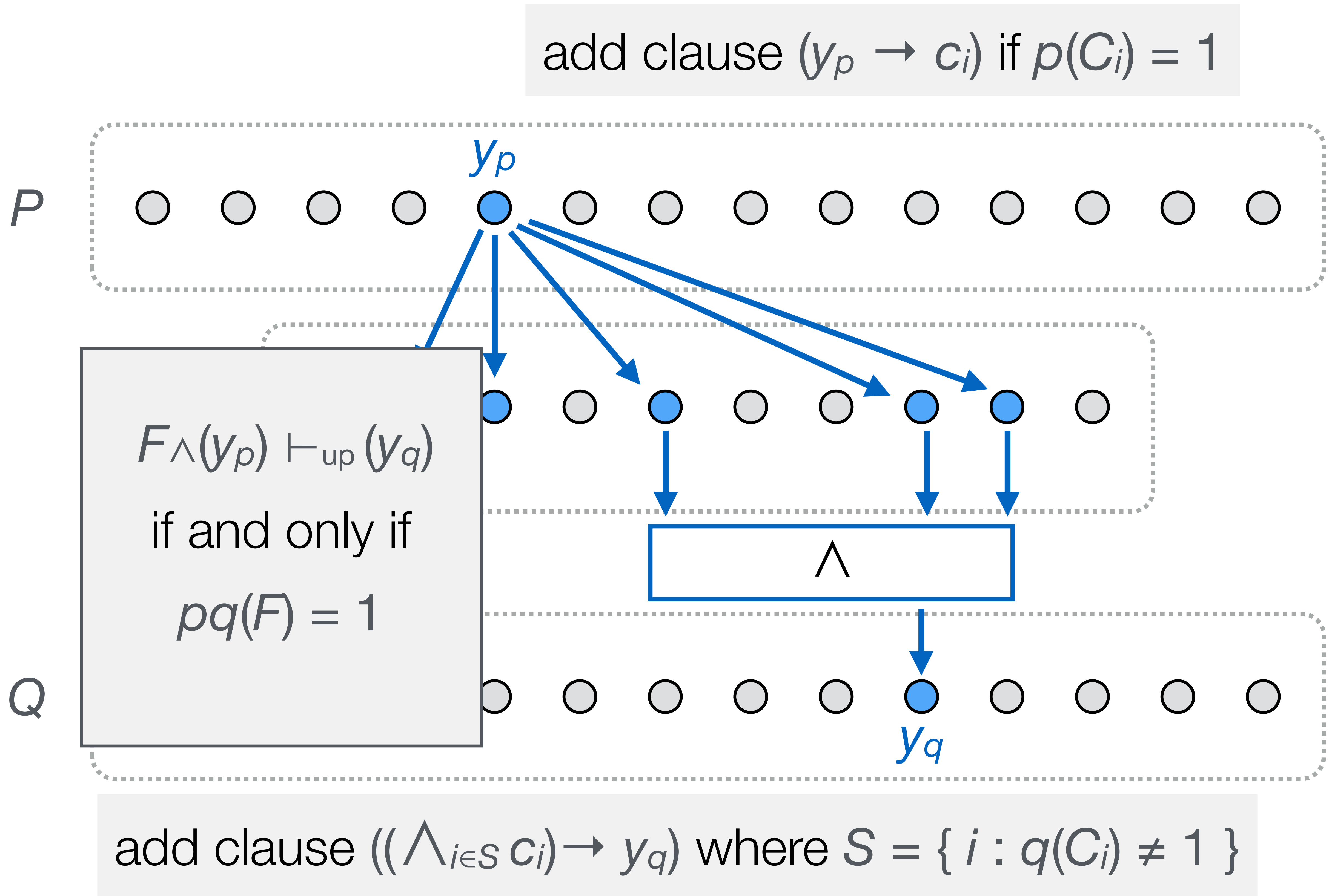
constructing the output formula F'

add clause $(y_p \rightarrow c_i)$ if $p(C_i) = 1$



add clause $((\bigwedge_{i \in S} c_i) \rightarrow y_q)$ where $S = \{ i : q(C_i) \neq 1 \}$

constructing the output formula *F'*



4.

Extensions and Open Questions

Extensions of the Main Result

- **Assuming SETH, for any $\varepsilon > 0$ we cannot solve**
  - **asymmetric tautology existence** on Horn-3-CNFs in time $O((n+m)^{2-\varepsilon})$
  - **asymmetric literal existence** on Horn-3-CNFs in time $O((n+m)^{2-\varepsilon})$
  - **singleton arc consistency** on (3,2)-CSPs in time $O((n+m)^{2-\varepsilon})$
- ***k*-step lookahead lower bound?**
  - Fix values for k variables, do unit propagation
  - We can *probably* show lower bound vs. $O((n+m)^{k+1-\varepsilon})$ time

Open Questions

- **Failed literal existence on 2-CNFs?**
  - CNF version requires clause length 3
  - CSP version requires domain size 3
  - Maybe one **can** do better on 2-CNFs?
- **Failed literal elimination fixpoint?**
  - Lower bound $O((n+m)^{3-\varepsilon})$?

Thank you!

Questions, comments?

Definitions:
Asymmetric Tautologies and Literals

- clause $C = (\ell_1 \vee \ldots \vee \ell_k) \in F$ is an **asymmetric tautology** if $(F \setminus C) \wedge (\neg \ell_1) \wedge \ldots \wedge (\neg \ell_k) \vdash_{\text{up}} (\ell'), (\neg \ell')$ for some $\ell' \in F$
    - replace F with $F \setminus \mathrm{C}$

- literal ℓ in a clause $C \in F$ is an **asymmetric literal** if $F \wedge (\ell) \vdash_{\text{up}} (\ell')$ for some $\ell' \in C \setminus \{\ell\}$
    - replace F with $(F \setminus C) \wedge (C \setminus \ell)$