A posteriori error analysis for the Morley plate element

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Introduction

► Thin structures (shells, plates, membranes, beams) are the main building blocks in modern structural design.

► Beside the classical fields as civil engineering, the variety of applications have strongly increased also in many other fields as aeronautics, biomechanics, surgical medicine or microelectronics.

► In particular, new applications arise when thin structures are combined with functional, smart or composite materials (shape memory alloys, piezo-electric ceramics etc.).

► Increasing demands for accuracy and productivity have created a need for adaptive (automated, efficient, reliable) computational methods for thin structures.
Kirchhoff plate bending model

We consider **bending** of a **thin planar structure** occupied by

\[ \mathcal{P} = \Omega \times (-\frac{t}{2}, \frac{t}{2}), \]

where \( \Omega \subset \mathbb{R}^2 \) denotes the **midsurface** of the plate and \( t \ll \text{diam}(\Omega) \) denotes the **thickness** of the plate.

**Kinematical assumptions** for the dimension reduction:
- Straight fibres normal to the midsurface remain straight and normal.
- Fibres normal to the midsurface do not stretch.
- The midsurface moves only in the vertical direction.
Deformations

Under these assumptions, with the deflection $w$, the displacement field $\mathbf{u} = (u_x, u_y, u_z)$ takes the form

$$u_x = -z \frac{\partial w(x, y)}{\partial x}, \quad u_y = -z \frac{\partial w(x, y)}{\partial y}, \quad u_z = w(x, y).$$

The corresponding deformation is defined by the symmetric linear strain tensor

$$\varepsilon(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T),$$

in the component form as

$$e_{xx} = -z \frac{\partial^2 w}{\partial x^2}, \quad e_{yy} = -z \frac{\partial^2 w}{\partial y^2}, \quad e_{zz} = 0,$$

$$e_{xy} = -z \frac{\partial^2 w}{\partial x \partial y}, \quad e_{xz} = 0, \quad e_{yz} = 0.$$
Stress resultants

Next, we define the stress resultants, the moments and the shear forces:

\[
M = \begin{pmatrix}
M_{xx} & M_{xy} \\
M_{yx} & M_{yy}
\end{pmatrix}
\quad \text{with} \quad M_{ij} = -\int_{-t/2}^{t/2} z \sigma_{ij} \, dz , \quad i, j = x, y ,
\]

\[
Q = \begin{pmatrix}
Q_x \\
Q_y
\end{pmatrix}
\quad \text{with} \quad Q_i = \int_{-t/2}^{t/2} \sigma_{iz} \, dz , \quad i = x, y ,
\]

where the stress tensor is assumed to be symmetric:

\[
\sigma_{ij} = \sigma_{ji} , \quad i, j = x, y, z.
\]
Equilibrium equations and boundary conditions

The principle of virtual work gives, with the load resultant $F$, the equilibrium equation

$$\text{div} \text{div} M = F \quad \text{with} \quad \text{div} M + Q = 0.$$ 

and the boundary conditions

\begin{align*}
w &= 0, \quad \nabla w \cdot n = 0 & \text{on } \Gamma_C, \\
w &= 0, \quad n \cdot Mn = 0 & \text{on } \Gamma_S, \\
n \cdot Mn &= 0, \quad \frac{\partial^2}{\partial s^2} (s \cdot Mn) + n \cdot \text{div} M = 0 & \text{on } \Gamma_F, \\
(s_1 \cdot Mn_1)(c) &= (s_2 \cdot Mn_2)(c) & \forall c \in V,
\end{align*}

where the indices 1 and 2 refer to the sides of the boundary angle at a corner point $c$ on the free boundary $\Gamma_F$. 
Constitutive assumptions

- The material of the plate is assumed to be
  — linearly elastic (defined by the generalized Hooke’s law)
  — homogeneous (independent of the coordinates $x, y, z$)
  — isotropic (independent of the coordinate system).

- Furthermore, we assume that the transverse normal stress vanishes: $\sigma_{zz} = 0$. 
Variational formulation

Let the deflection \( w \) belong to the Sobolev space

\[
W = \{ v \in H^2(\Omega) \mid v = 0 \text{ on } \Gamma_C \cup \Gamma_S, \ \nabla v \cdot n = 0 \text{ on } \Gamma_C \},
\]

where \( n \) indicates the unit outward normal to the boundary \( \Gamma \).

Problem. Variational formulation: Find \( w \in W \) such that

\[
(E\varepsilon(\nabla w), \varepsilon(\nabla v)) = (f, v) \quad \forall v \in W,
\]

with the elasticity tensor \( E \) defined as

\[
E\varepsilon = \frac{E}{12(1+\nu)} \left( \varepsilon + \frac{\nu}{1-\nu} \text{tr}(\varepsilon)I \right) \quad \forall \varepsilon \in \mathbb{R}^{2\times2},
\]

with Young’s modulus \( E \) and the Poisson ratio \( \nu \).
Morley finite element formulation

Let $E$ denote an edge of a triangle $K$ in a triangulation $T_h$. We define the discrete space for the deflection as

$$W_h = \{ v \in M_{2,h} \mid \int_E [\nabla v \cdot n_E] = 0 \quad \forall E \in \mathcal{E}_h^i \cup \mathcal{E}_h^c \},$$

where $M_{2,h}$ denotes the space of the second order piecewise polynomial functions on $T_h$ which are

— continuous at the vertices of all the internal triangles and

— zero at all the triangle vertices of $\Gamma_C \cup \Gamma_S$.

Finite element method. Morley: Find $w_h \in W_h$ such that

$$\sum_{K \in T_h} (E\varepsilon(\nabla w_h), \varepsilon(\nabla v))_K = (f, v) \quad \forall v \in W_h.$$
A priori error estimate

The method is stable and convergent with respect to the following discrete norm on the space $W_h + H^2$:

$$\|v\|_h^2 := \sum_{K \in T_h} |v|_{2,K}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-3} \|v\|_{0,E}^2 + \sum_{E \in \mathcal{E}_h} h_E^{-1} \left\| \frac{\partial v}{\partial n_E} \right\|_{0,E}^2,$$

**Proposition.** (Shi 90, Ming and Xu 06) Assuming that $\Gamma = \Gamma_C$ there exists a positive constant $C$ such that

$$\|w - w_h\|_h \leq C h \left( |w|_{H^3(\Omega)} + h \|f\|_{L^2(\Omega)} \right).$$

The numerical results indicate the same convergence rate for general boundary conditions as well.
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A posteriori error estimates

▶ We use the following notation: $\llbracket \cdot \rrbracket$ for jumps (and traces), $h_E$ and $h_K$ for the edge length and the element diameter.

Interior error indicators

▶ For all the elements $K$ in the mesh $\mathcal{T}_h$,

$$\tilde{\eta}_K^2 := h_K^4 \| f \|_{0,K}^2,$$

and for all the internal edges $E \in \mathcal{I}_h$,

$$\eta_E^2 := h_E^{-3} \| \llbracket w_h \rrbracket \|_{0,E}^2 + h_E^{-1} \| \llbracket \frac{\partial w_h}{\partial n_E} \rrbracket \|_{0,E}^2.$$
Boundary error indicators

▶ The boundary of the plate is divided into clamped, simply supported and free parts:

\[ \Gamma := \partial \Omega = \Gamma_C \cup \Gamma_S \cup \Gamma_F. \]

▶ For edges on the clamped and simply supported boundaries \( \Gamma_C \) and boundary \( \Gamma_S \), respectively,

\[ \eta_{E,C}^2 := h_E^{-3} || [w_h] ||_{0,E}^2 + h_E^{-1} || \left[ \frac{\partial w_h}{\partial n_E} \right] ||_{0,E}^2, \]

\[ \eta_{E,S}^2 := h_E^{-3} || [w_h] ||_{0,E}^2. \]
Error indicators — local and global

For any element $K \in \mathcal{T}_h$, let the local error indicator be

$$
\eta_K := \left( \tilde{\eta}_K^2 + \frac{1}{2} \sum_{E \in \mathcal{I}_h} \eta_E^2 + \sum_{E \in \mathcal{C}_h} \eta_{E,C}^2 + \sum_{E \in \mathcal{S}_h} \eta_{E,S}^2 \right)^{1/2},
$$

with the notation

— $\mathcal{I}_h$ for the collection of all the internal edges,
— $\mathcal{C}_h$ and $\mathcal{S}_h$ for the collections of all the boundary edges on $\Gamma_C$ and $\Gamma_S$, respectively.

The global error indicator is defined as

$$
\eta_h := \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2}.
$$
Upper bound — Reliability

Theorem. *Reliability*: There exists a positive constant $C$ such that

$$\|w - w_h\|_h \leq C\eta_h.$$ 

Lower bound — Efficiency

Theorem. *Efficiency*: For any element $K$, there exists a positive constant $C_K$ such that

$$\eta_K \leq C_K \left(\|w - w_h\|_{h,K} + h_K^2\|f - f_h\|_{0,K}\right).$$

Efficiency is proved by standard arguments; reliability needs a new Clément-type interpolant and a new Helmholtz-type decomposition.
Techniques for the analysis  
— Helmholtz decomposition

**Lemma.** Let $\sigma$ be a second order tensor field in $L^2(\Omega; \mathbb{R}^{2\times2})$.  
Then, there exist $\psi \in W$, $\rho \in L^2_0(\Omega)$ and $\phi \in [\tilde{H}^1(\Omega)]^2$ such that

$$
\sigma = E\varepsilon(\nabla \psi) + \rho + \text{Curl} \phi, \quad \text{with} \quad \rho = \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}.
$$

$$
\|\psi\|_{H^2(\Omega)} + \|\rho\|_{L^2(\Omega)} + \|\phi\|_{H^1(\Omega)} \leq C\|\sigma\|_{L^2(\Omega)}.
$$

Here $\tilde{H}^m(\Omega)$, $m \in \mathbb{N}$, indicate the quotient space of $H^m(\Omega)$ where the seminorm $\|\cdot\|_{H^m(\Omega)}$ is null.

In analysis, Lemma is applied to the tensor field $E\varepsilon(\nabla (w - w_h)).$
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Numerical results

- We have implemented the method in the open-source finite element software *Elmer* developed by CSC – the Finnish IT Center for Science.

- The software provides error balancing strategy and complete remeshing for triangular meshes.

- We have used test problems with convex rectangular domains – and with known exact solutions – for investigating the effectivity index for the error estimator derived.

- Non-convex domains we have used for studying the adaptive performance and robustness of the method.
Effectivity index $\eta = \frac{\eta_h}{\|w - w_h\|_h}$

![Graphs showing uniform and adaptive refinements with clamped, simply supported, and free boundaries included.](image)

Figure 1: *Left*: uniform refinements; *Right*: adaptive refinements. Clamped (squares), simply supported (circles) and free (triangles) boundaries included.
Adaptively refined mesh — Error estimator
Simply supported L-corner

Figure 2: Simply supported L-shaped domain: Distribution of the error estimator for two adaptive steps.
Figure 3: Simply supported L-shaped domain: Convergence of the error estimator for the uniform refinements and adaptive refinements; Solid lines for global, dashed lines for maximum local ones.
Adaptively refined mesh — Error estimator
Clamped L-corner

Figure 4: Simply supported L-shaped domain with a clamped L-corner: Distribution of the error estimator for two adaptive steps.
Uniform vs. Adaptive — Convergence

Figure 5: Clamped L-corner: Convergence of the error estimator for the uniform refinements and adaptive refinements; Solid lines for global, dashed lines for maximum local ones.
Adaptively refined mesh — Error estimator
Simply supported M-domain

Figure 6: Simply supported M-shaped domain: Distribution of the error estimator for two adaptive steps.
Figure 7: Simply supported M-shaped domain: Convergence of the error estimator for the uniform refinements and adaptive refinements; Solid lines for global, dashed lines for maximum local ones.
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Advantages

- **Reliability**: computable (non-guaranteed due to $C$) global upper bound for the error.

- **Efficiency**: computable (non-guaranteed due to $C_K$) local lower bound.

- **Robustness**: $C_K$ independent of the mesh size, data and the solution.

- **Computational costs**: small (local) compared to solving the problem itself.
Disadvantages

▶ **Residual based** error estimates in the **energy norm** only — no estimates for other quantities of interest.

▶ **Method dependent**: applicable for the Morley element only — although the techniques can be generalized.

▶ Valid only for **static** problem with **transversal loading** and **isotropic, homogeneous, linearly elastic** material — so far.
References
