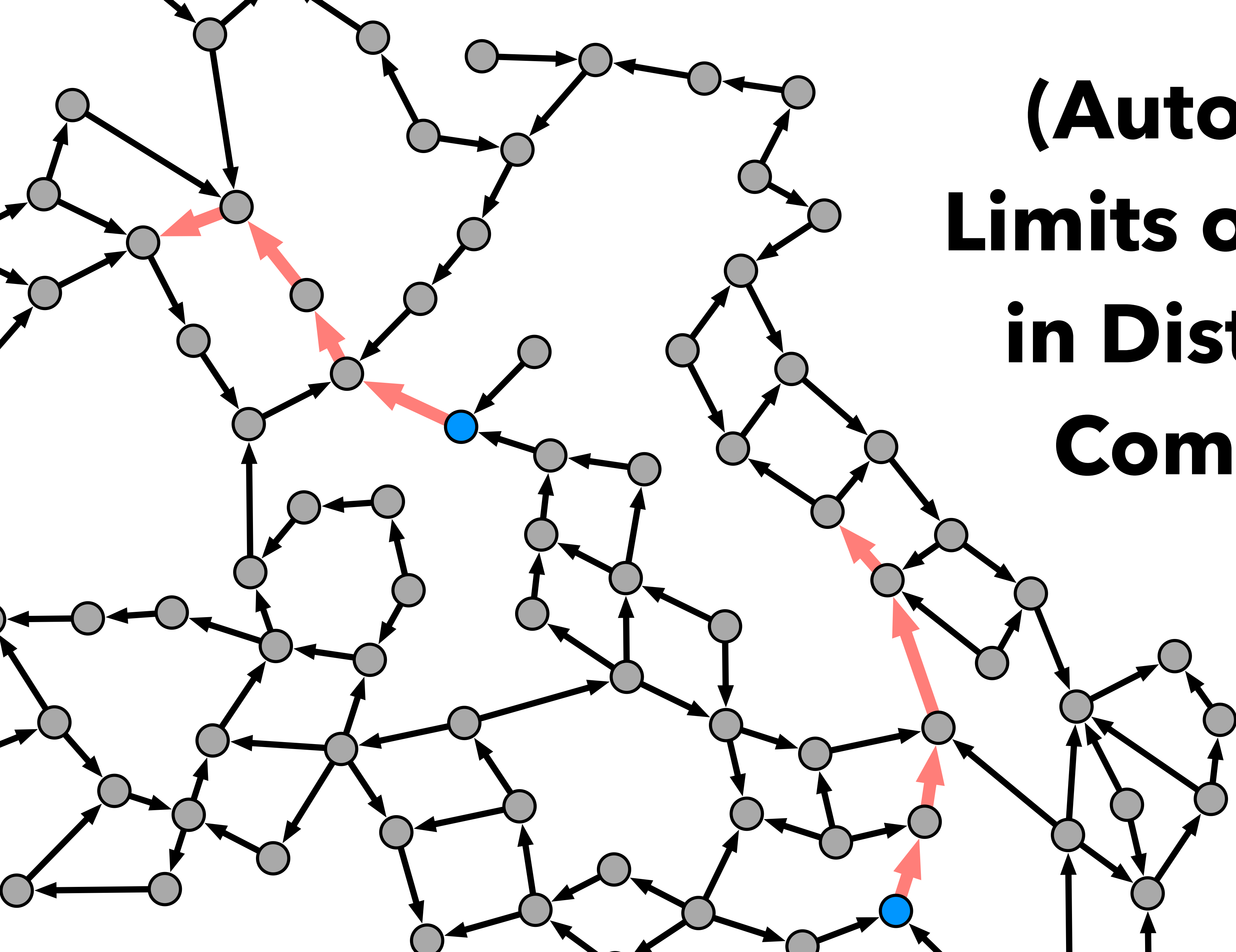


# (Automated) Limits of Locality in Distributed Computing



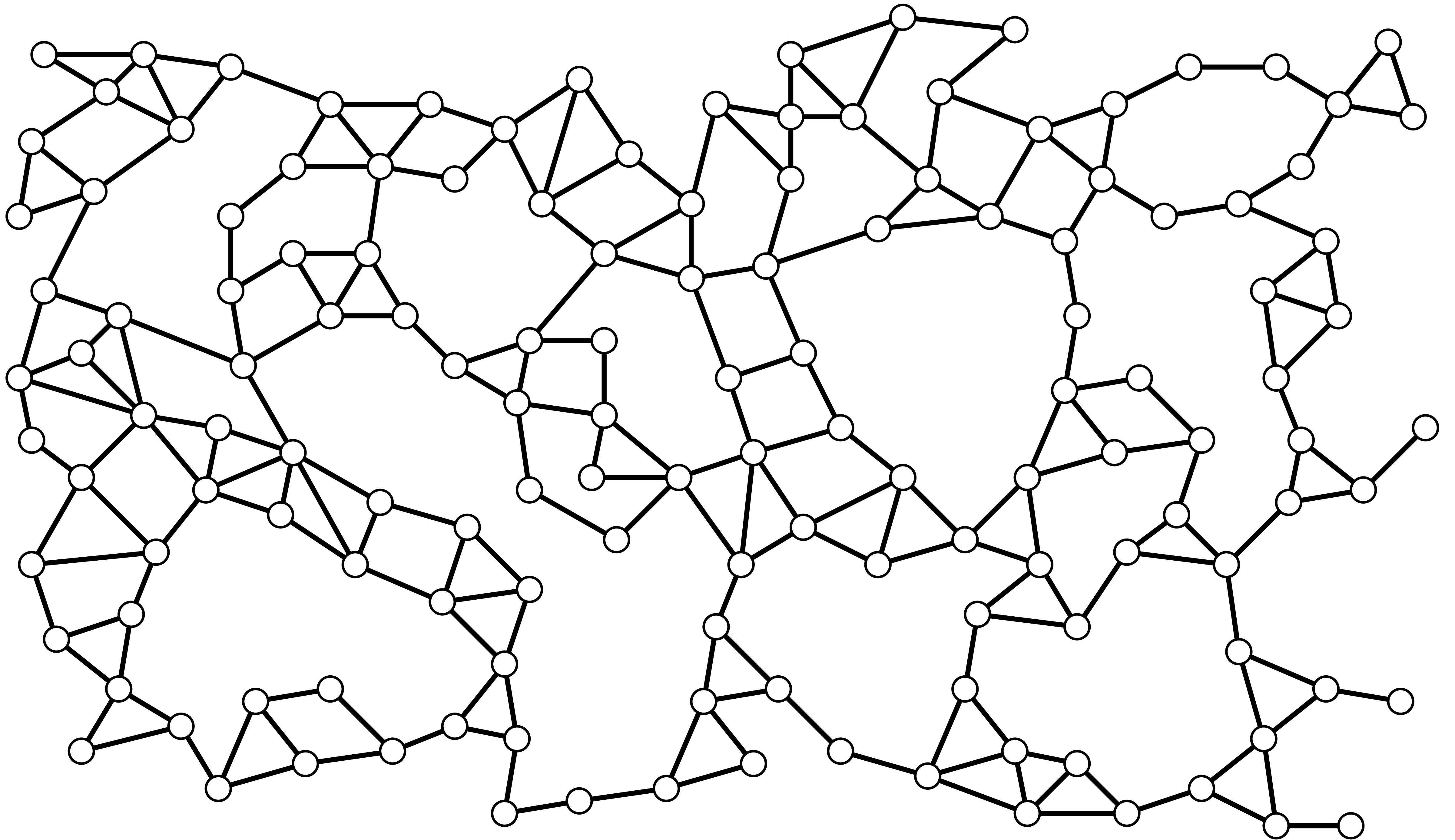
Aalto CS Research Day 2019  
Juho Hirvonen  
Dennis Olivetti

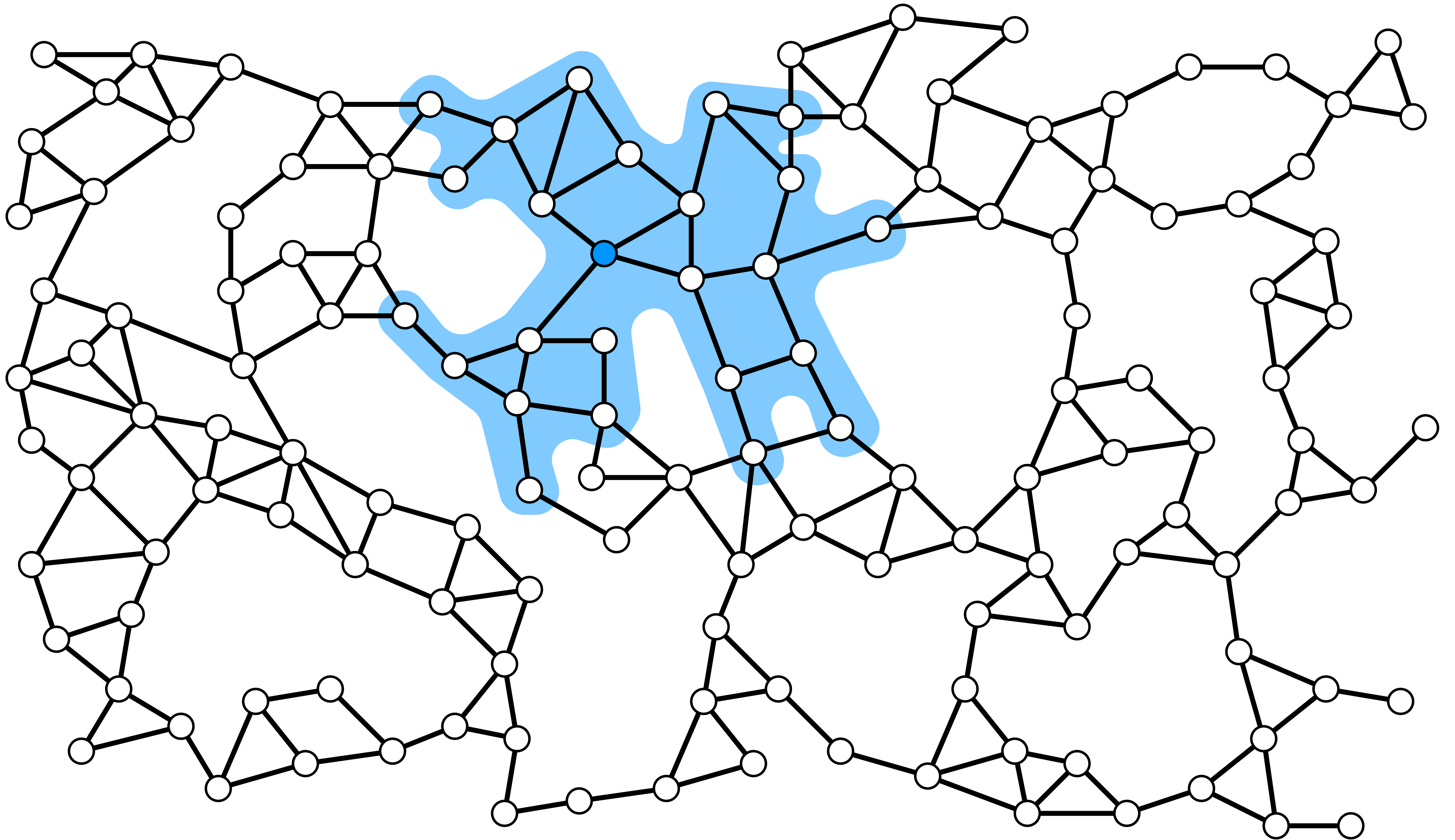


# Distributed Algorithms

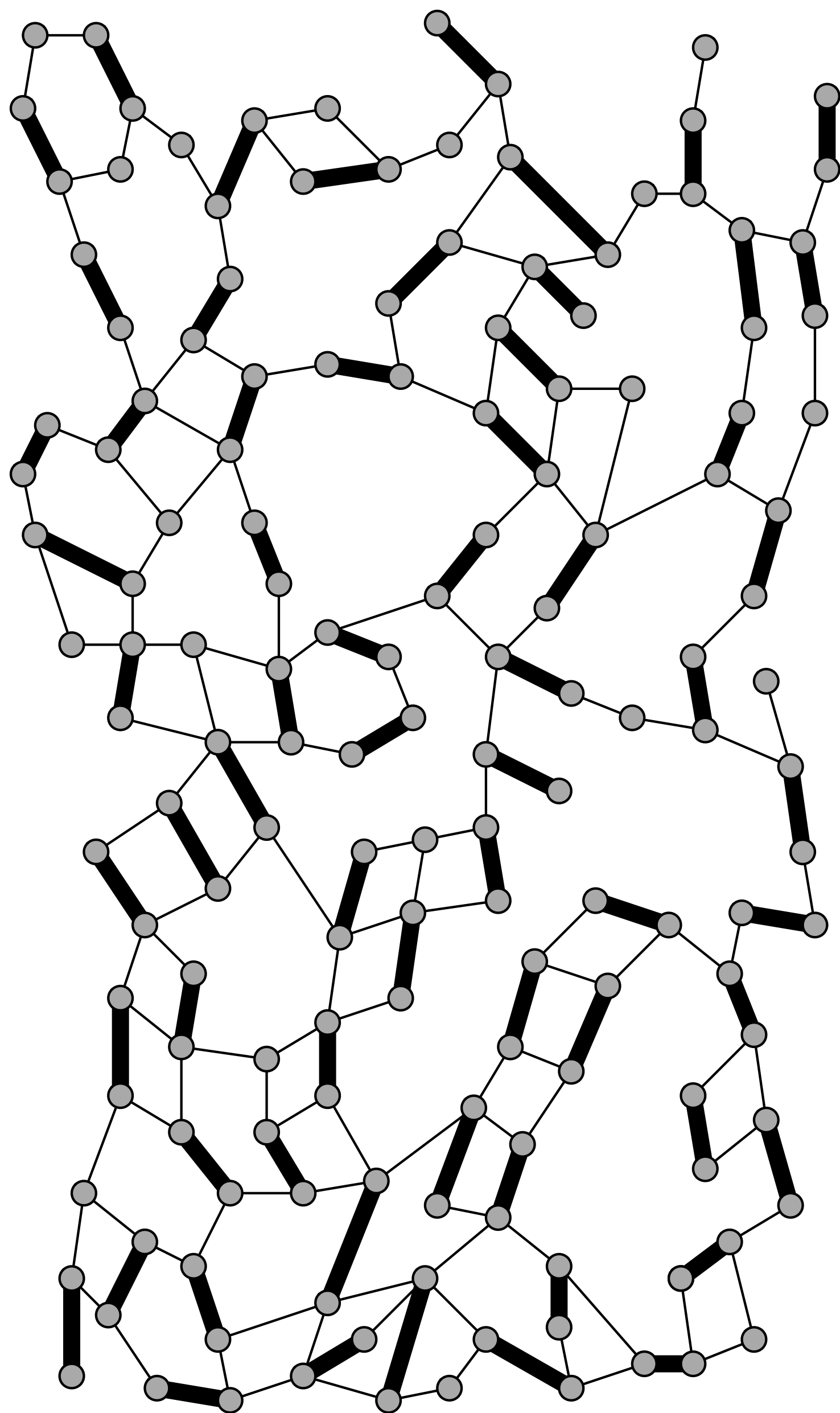




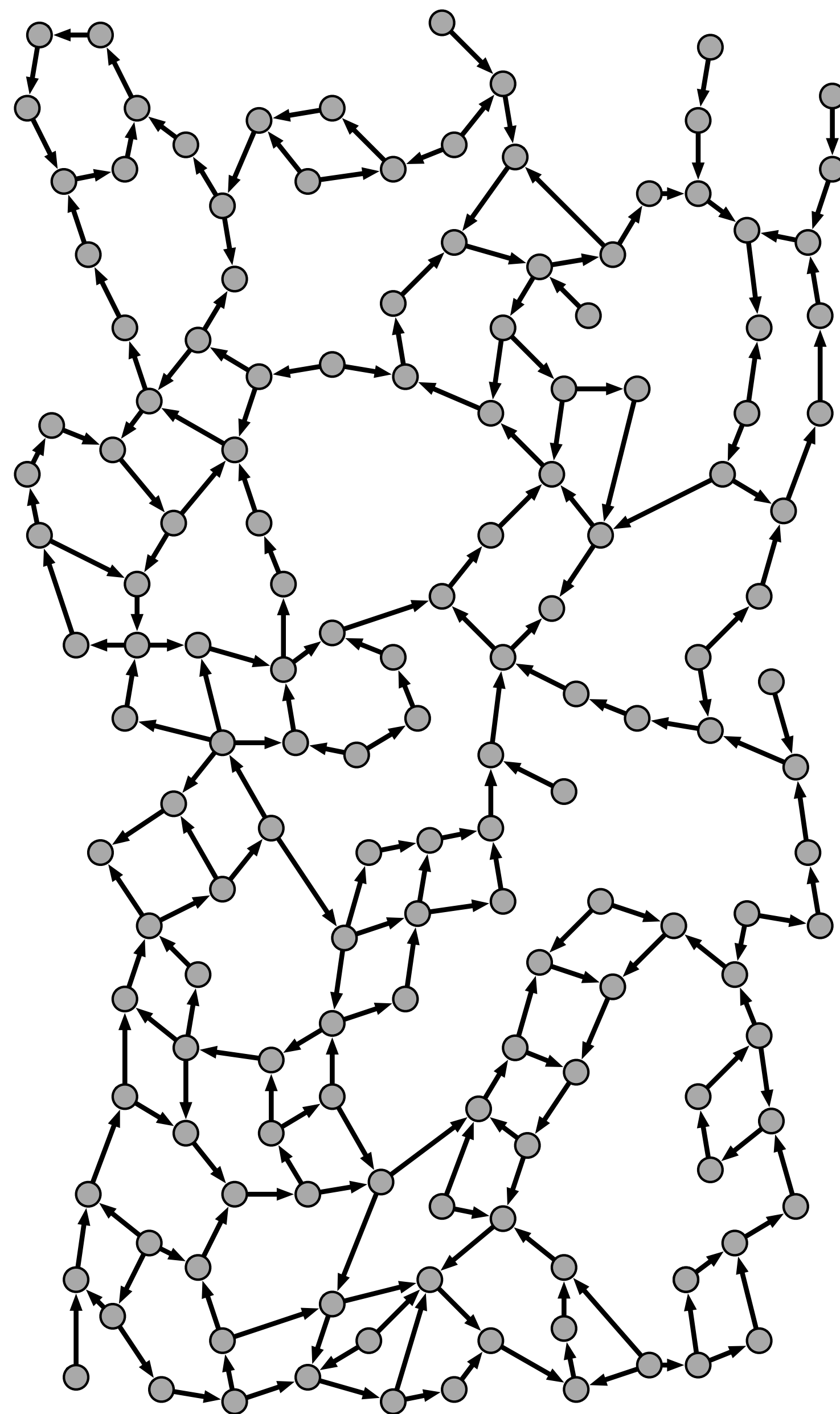




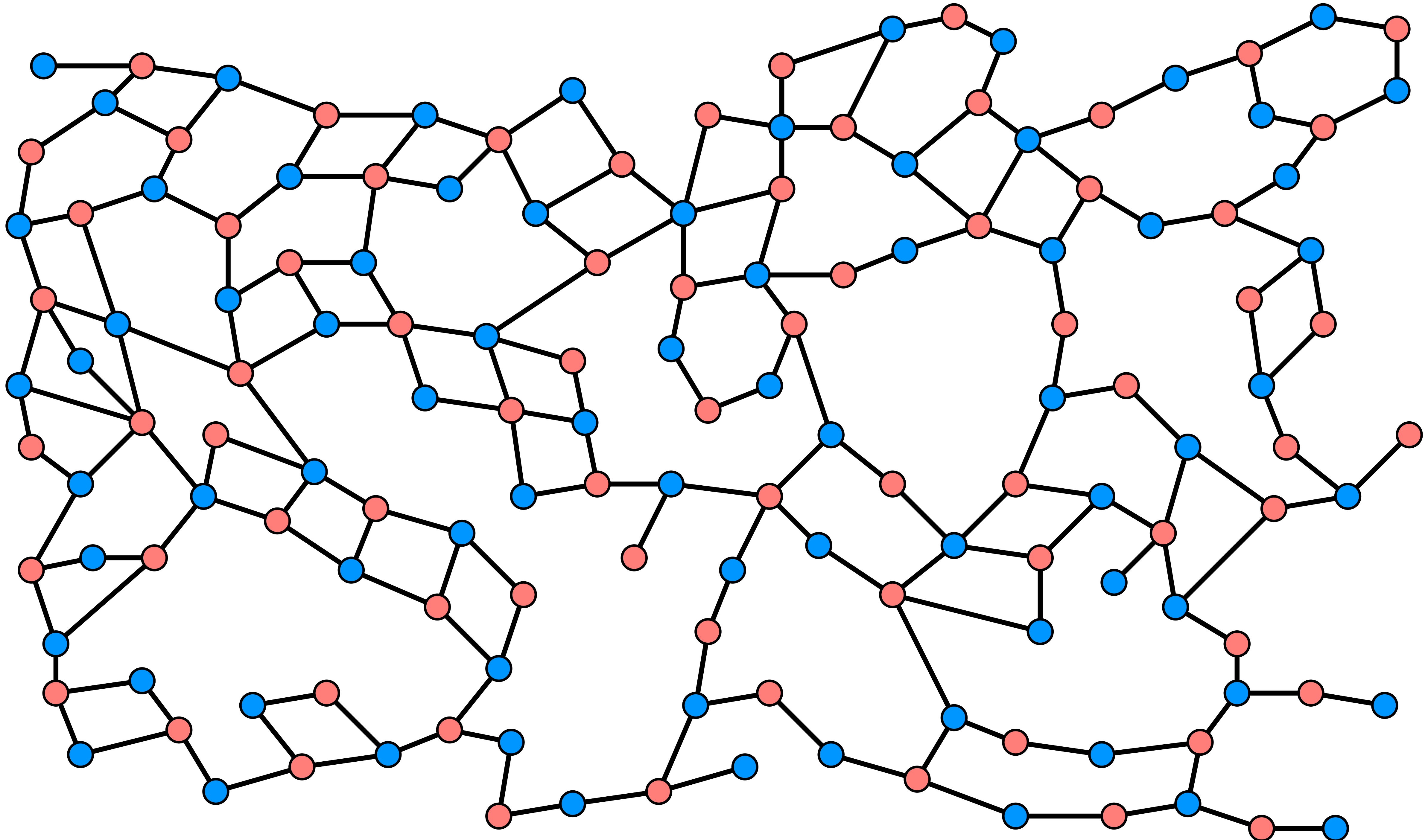
**maximal matching**

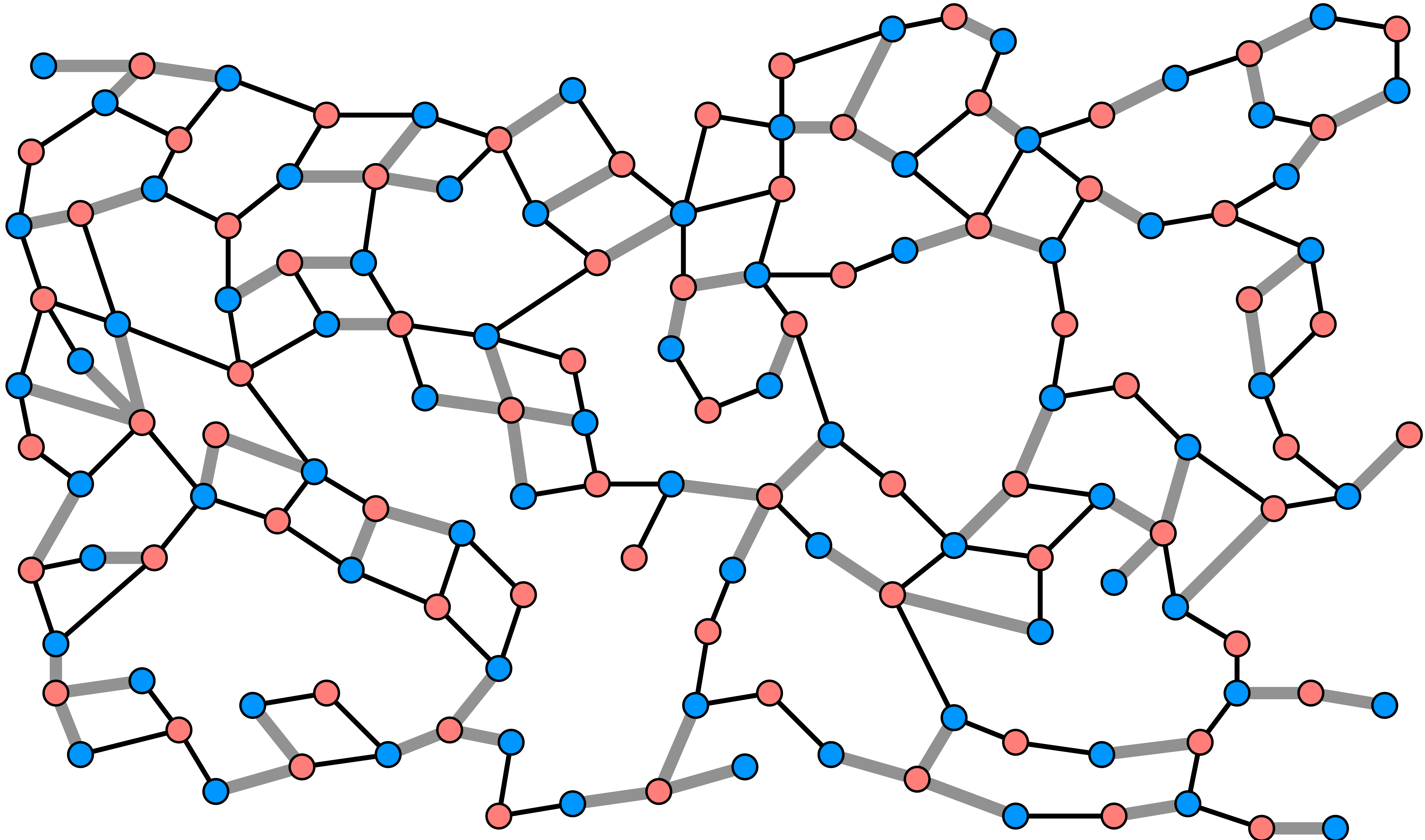


**sinkless orientation**

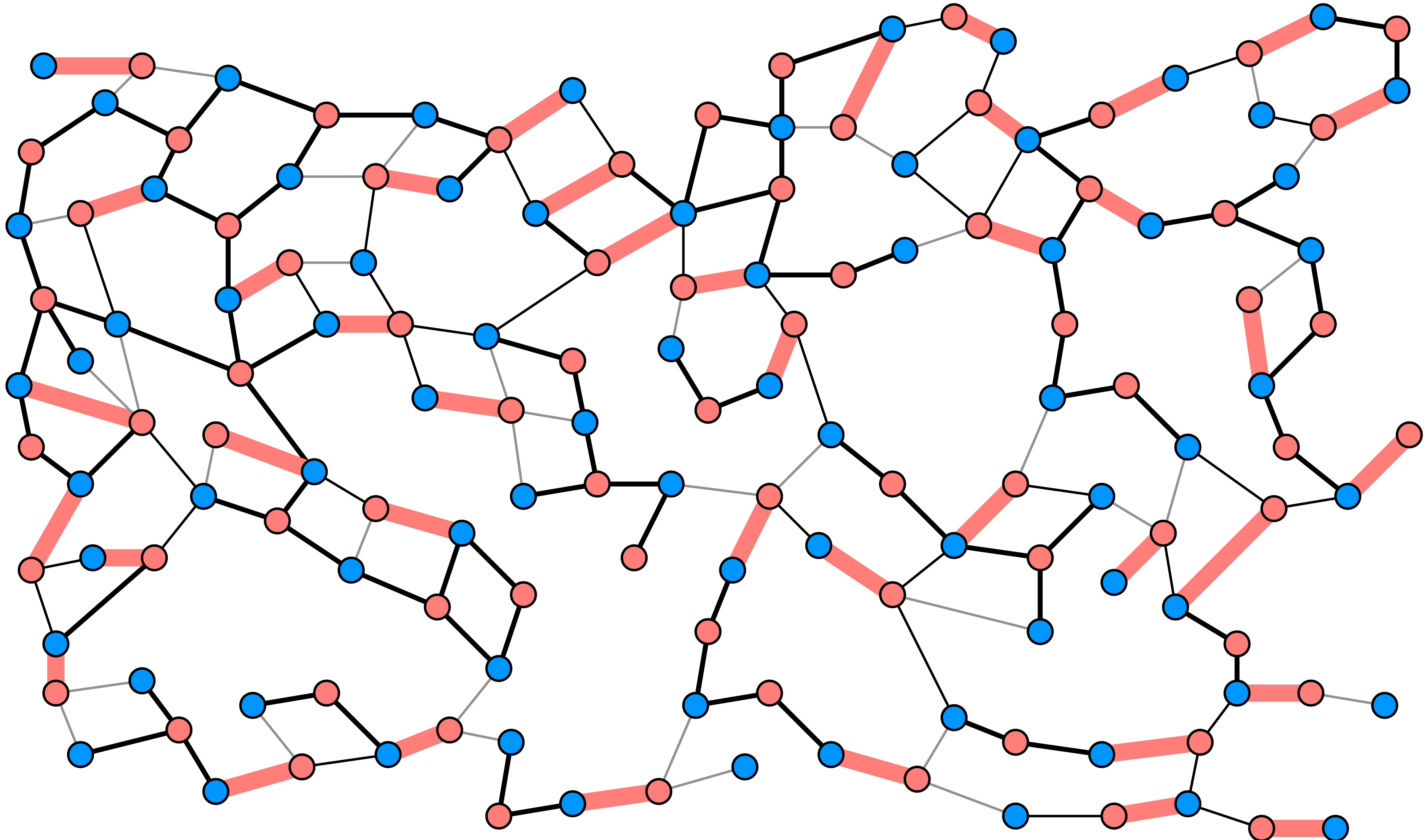


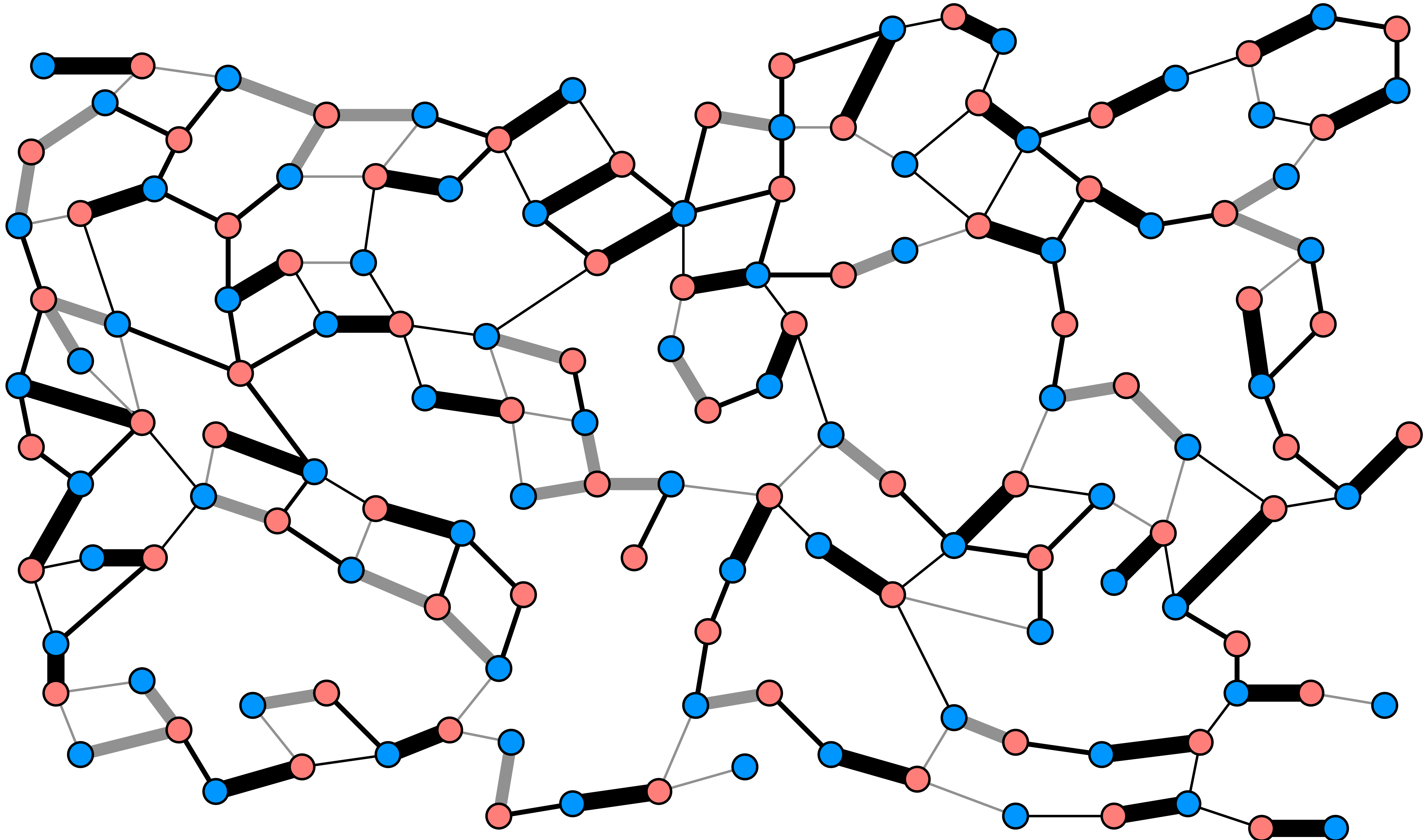
# **Proposal algorithm: maximal matching**

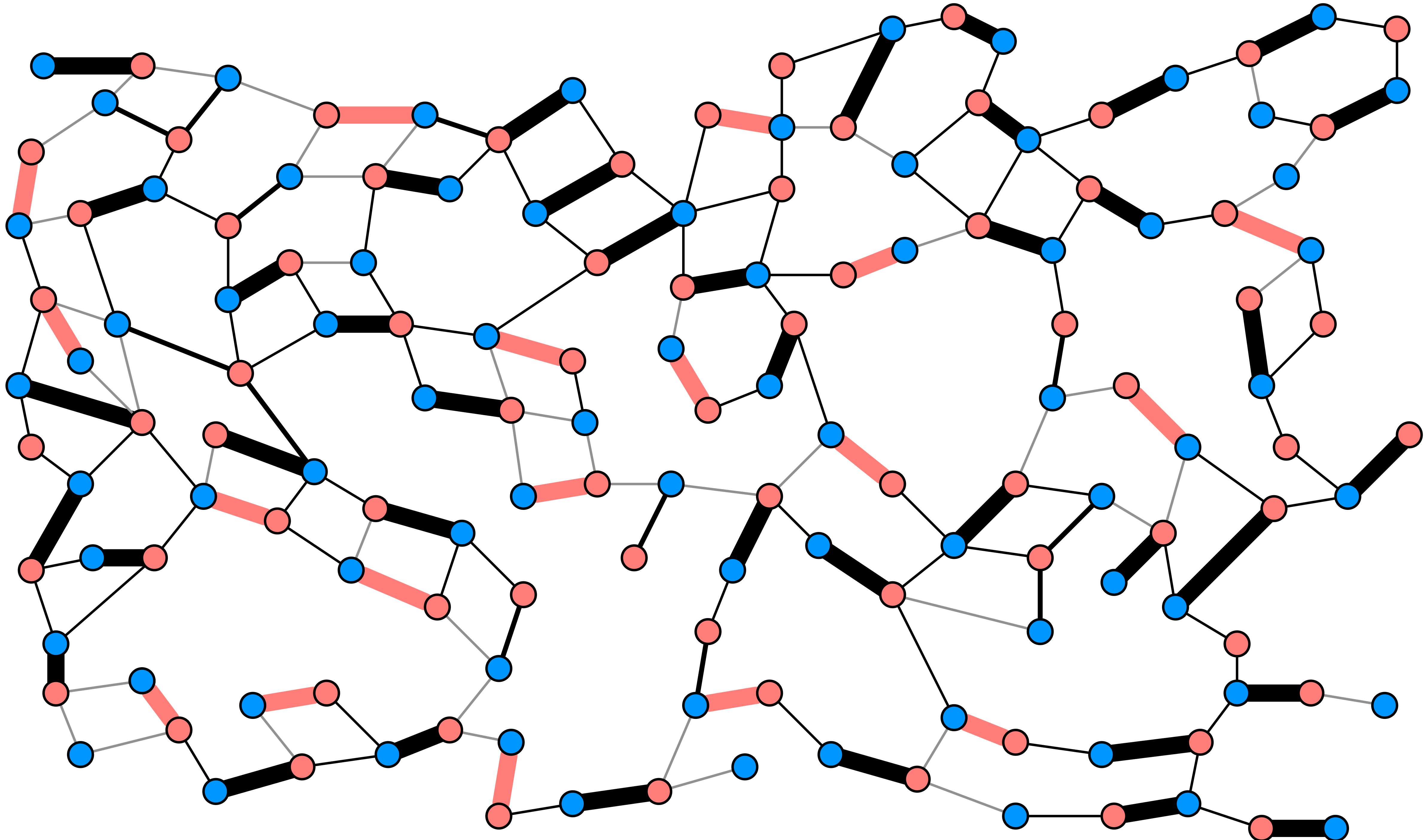




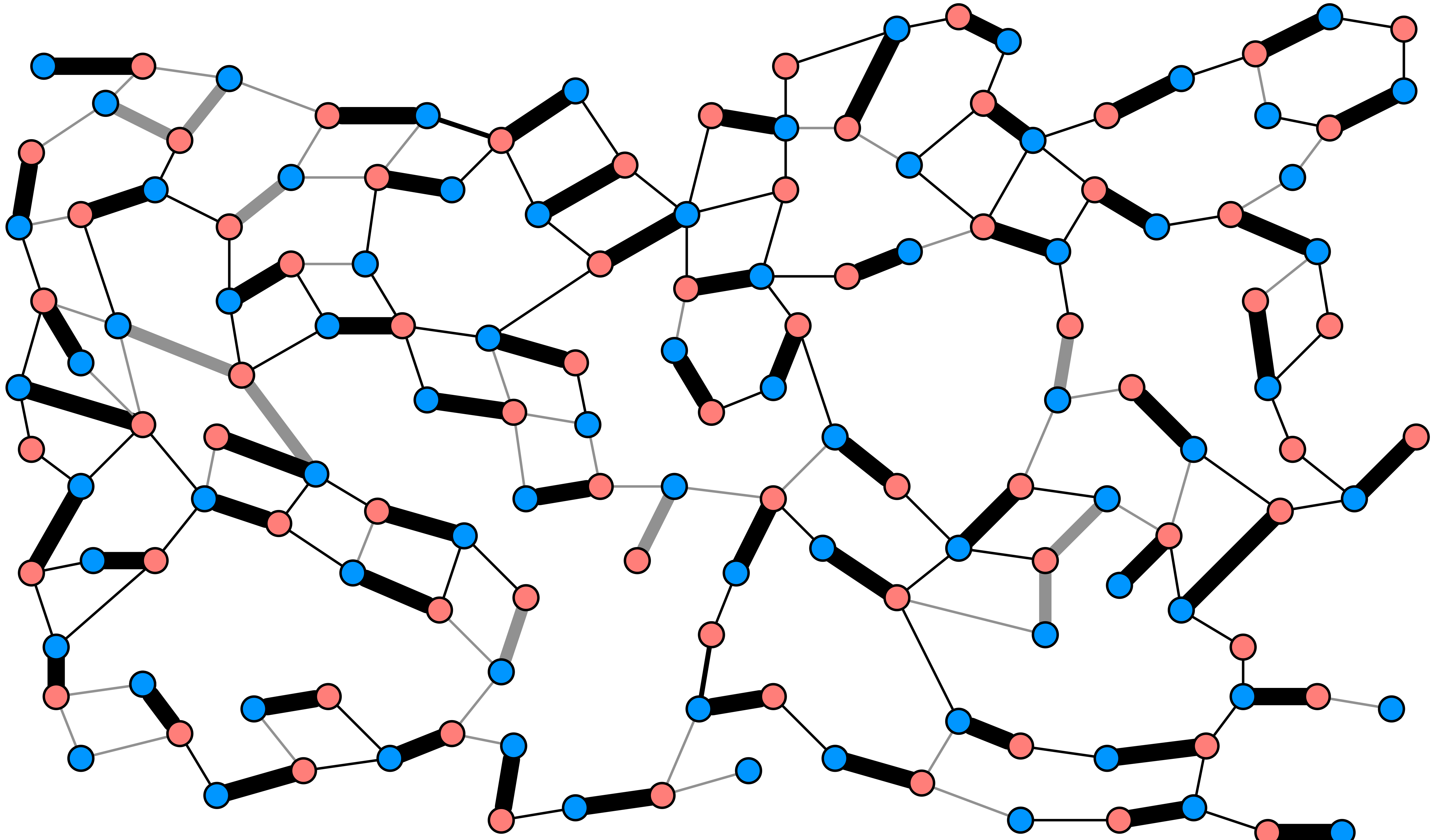


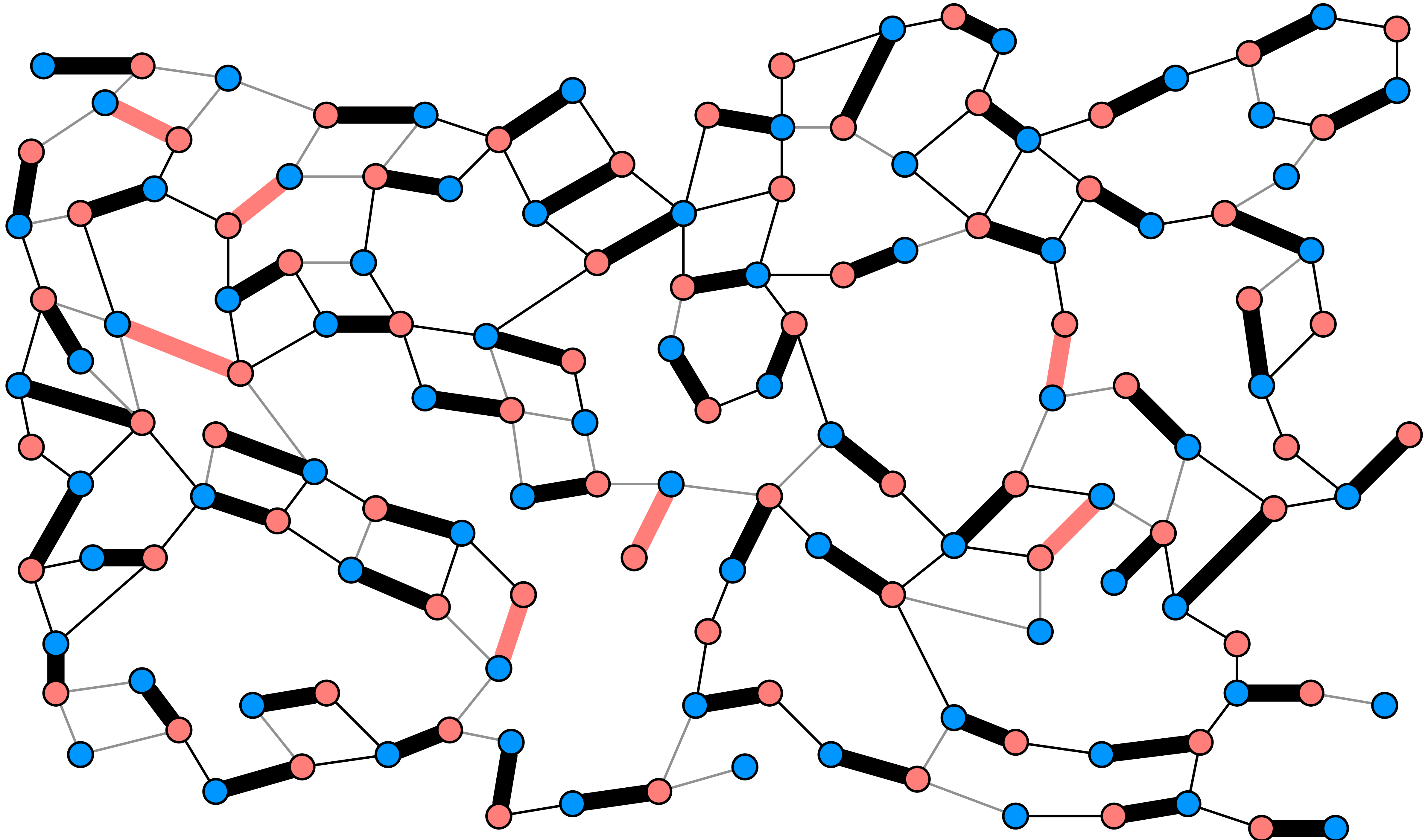






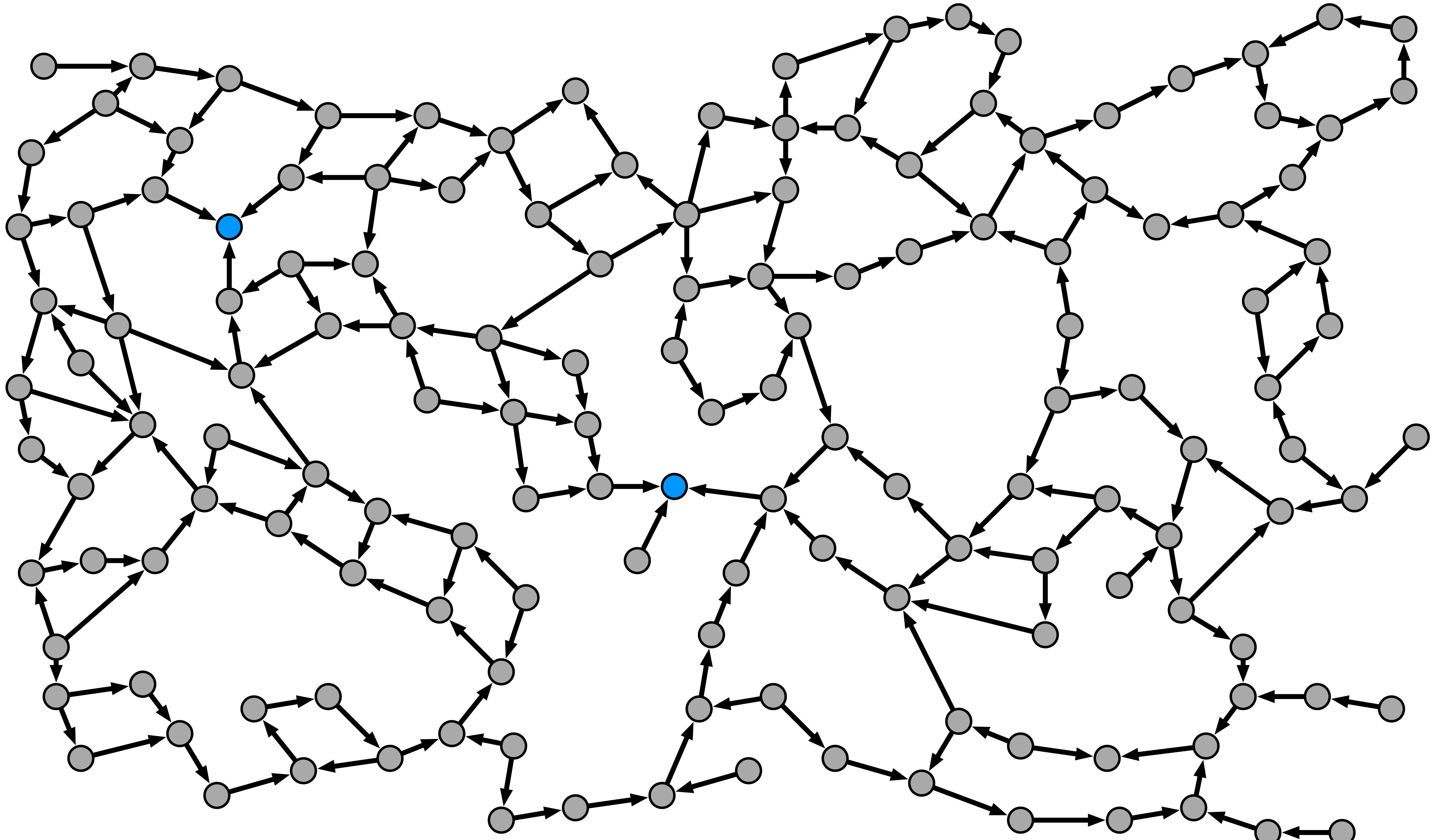


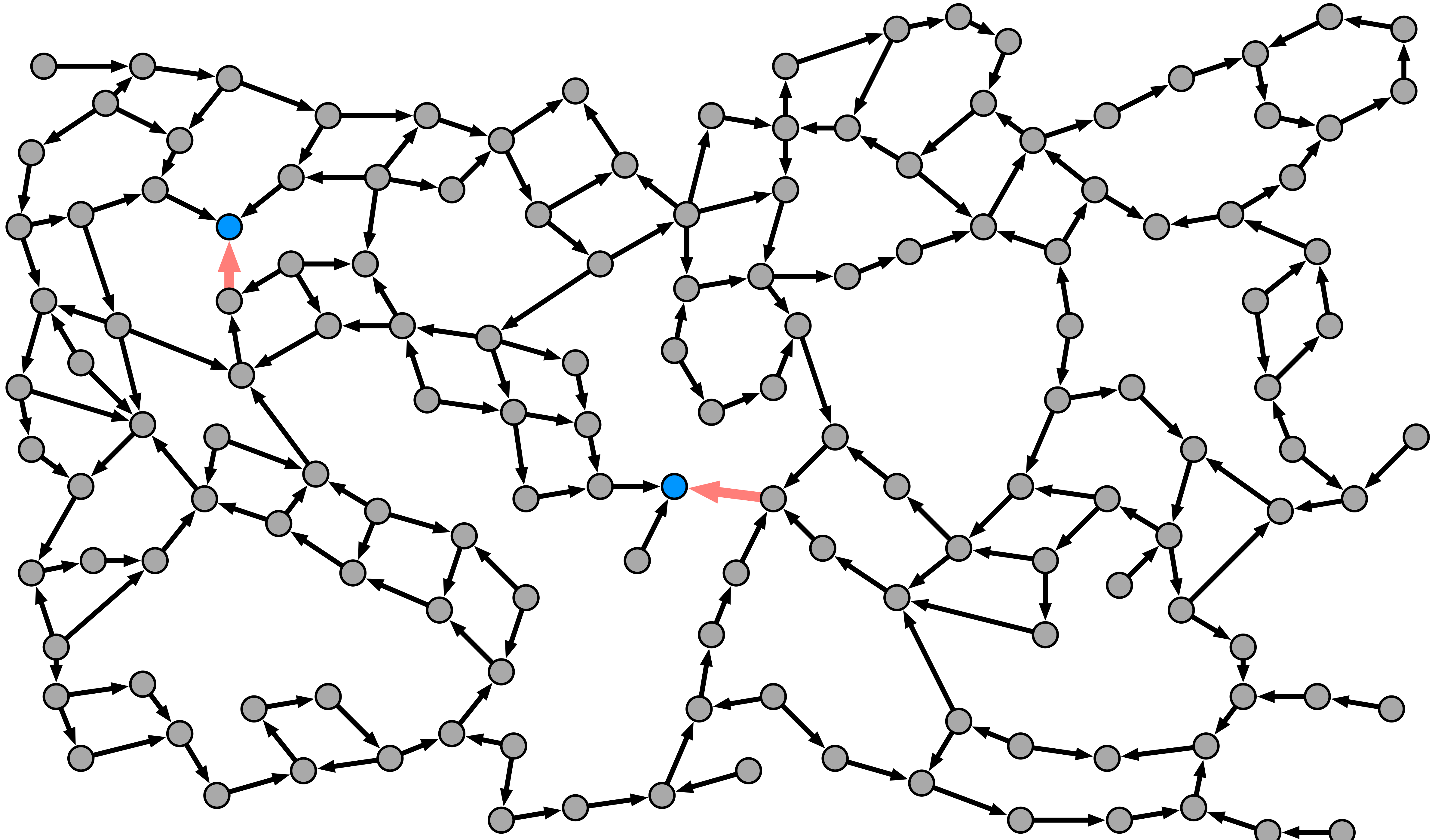


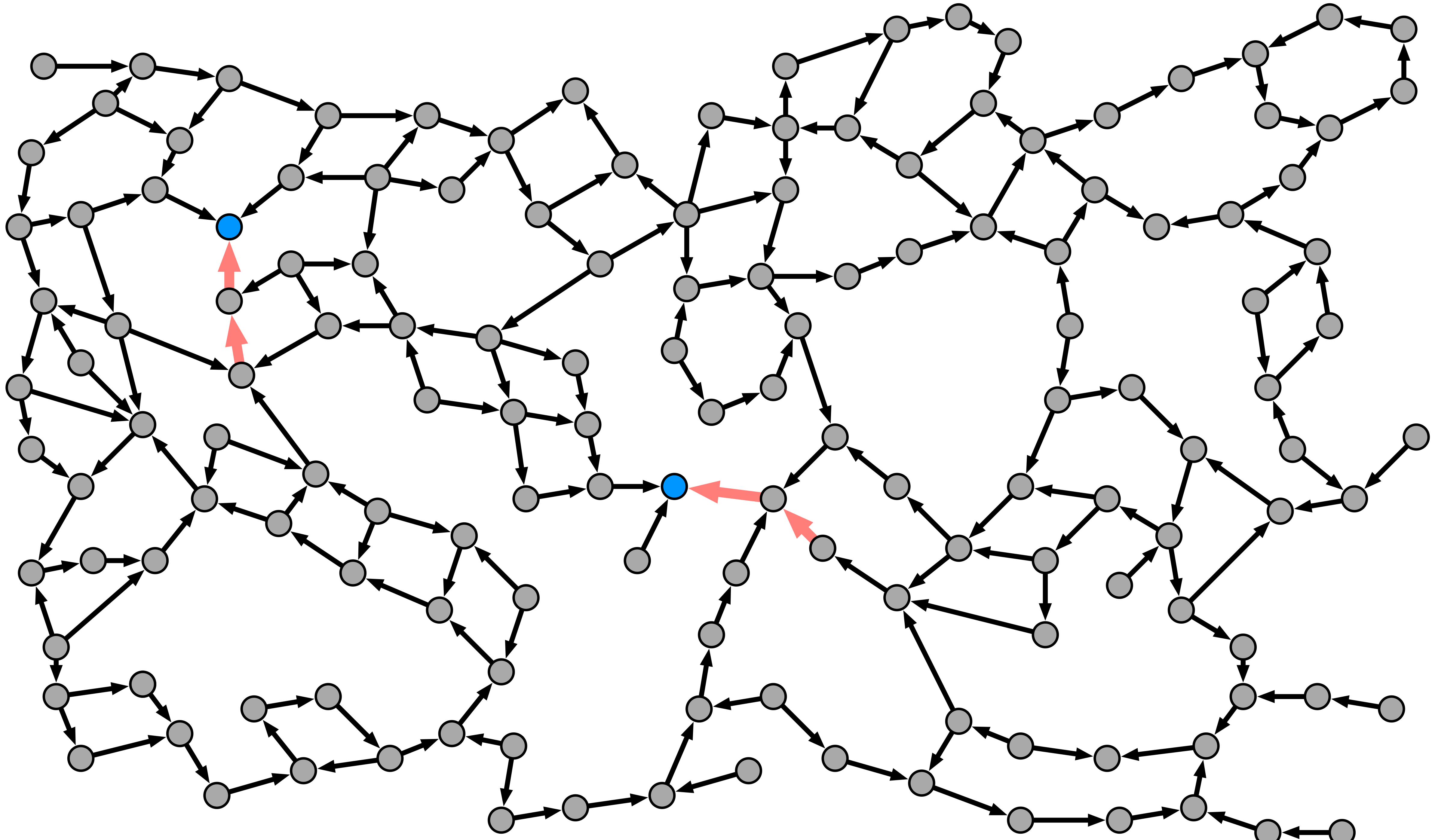


# **Finding a sinkless orientation**

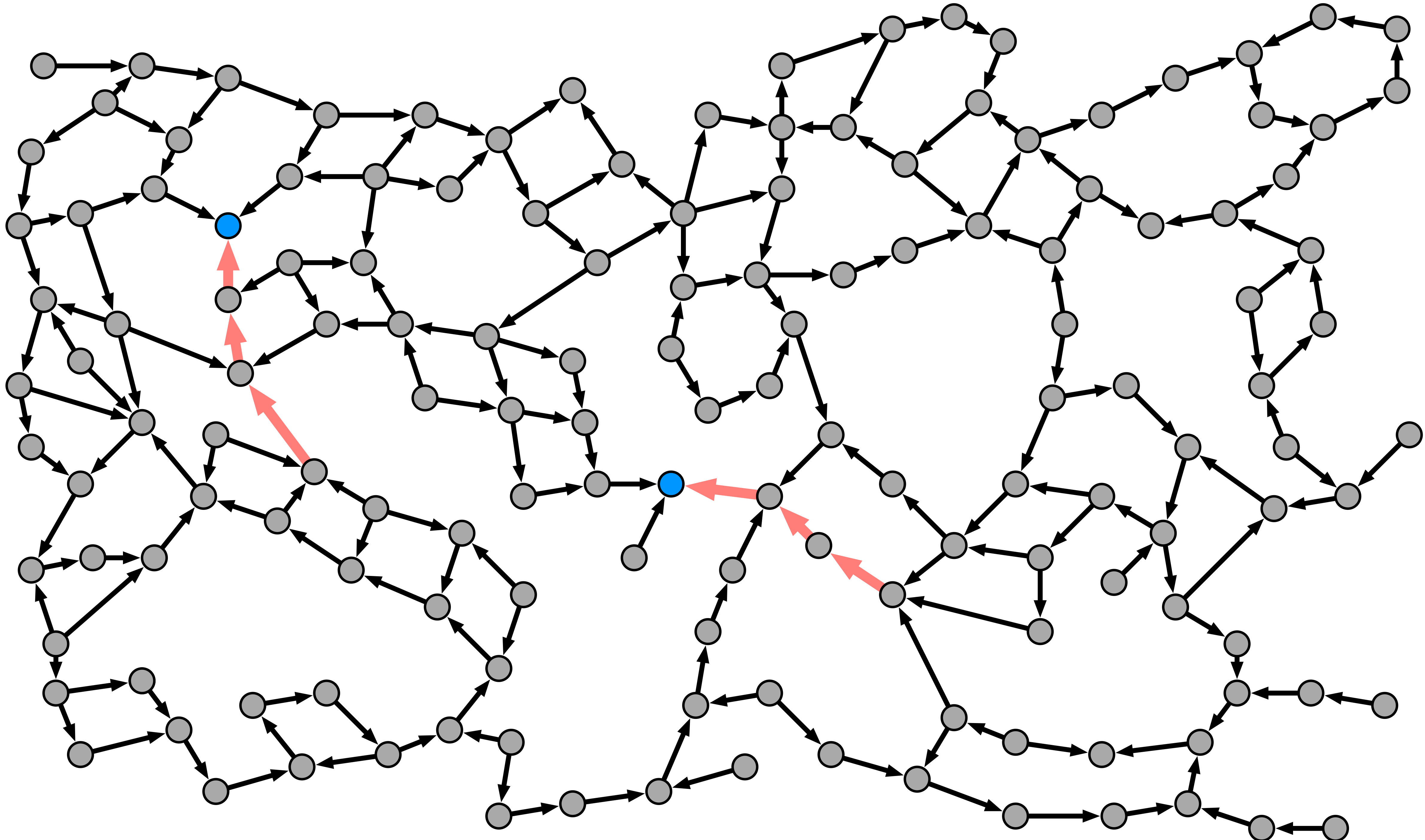


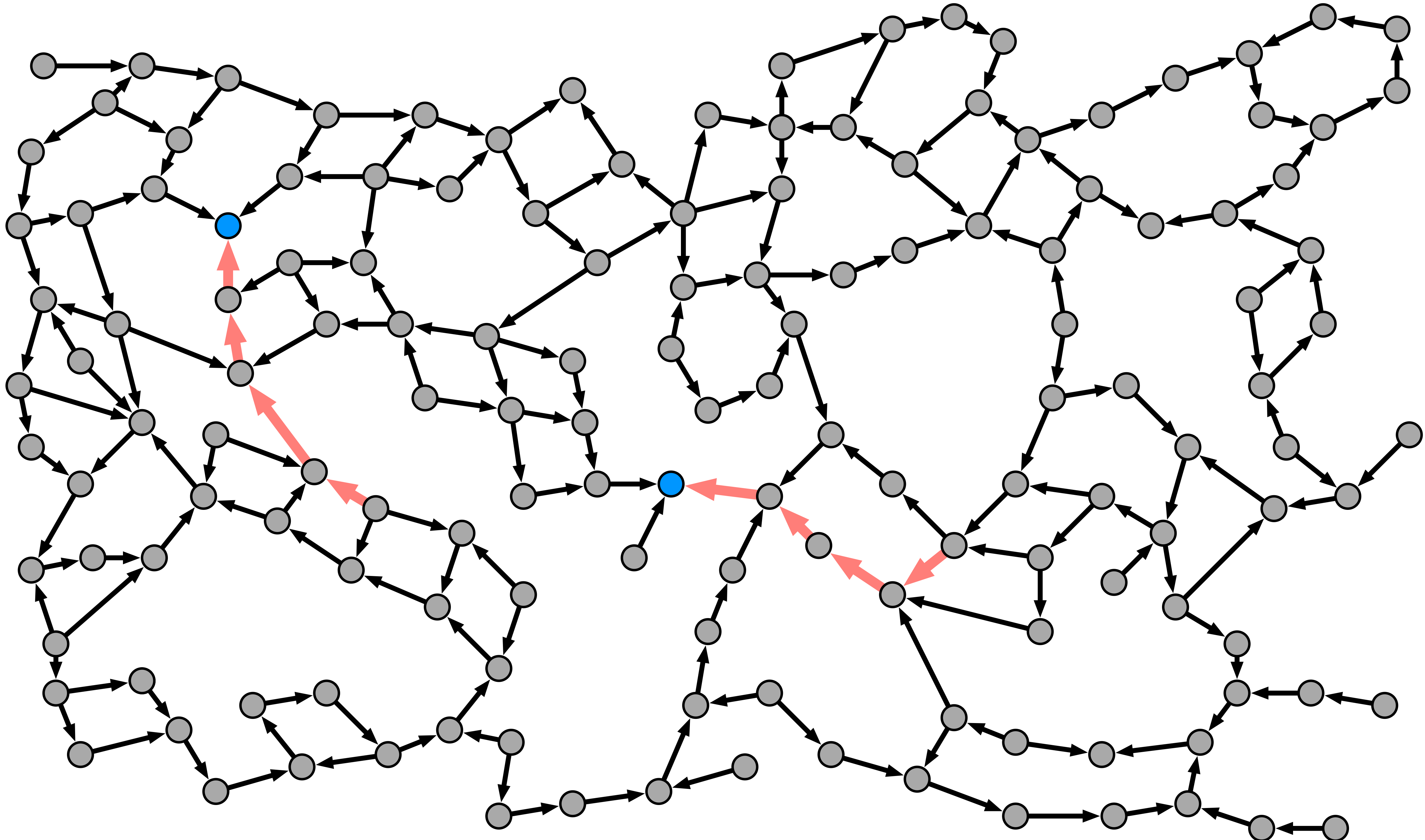


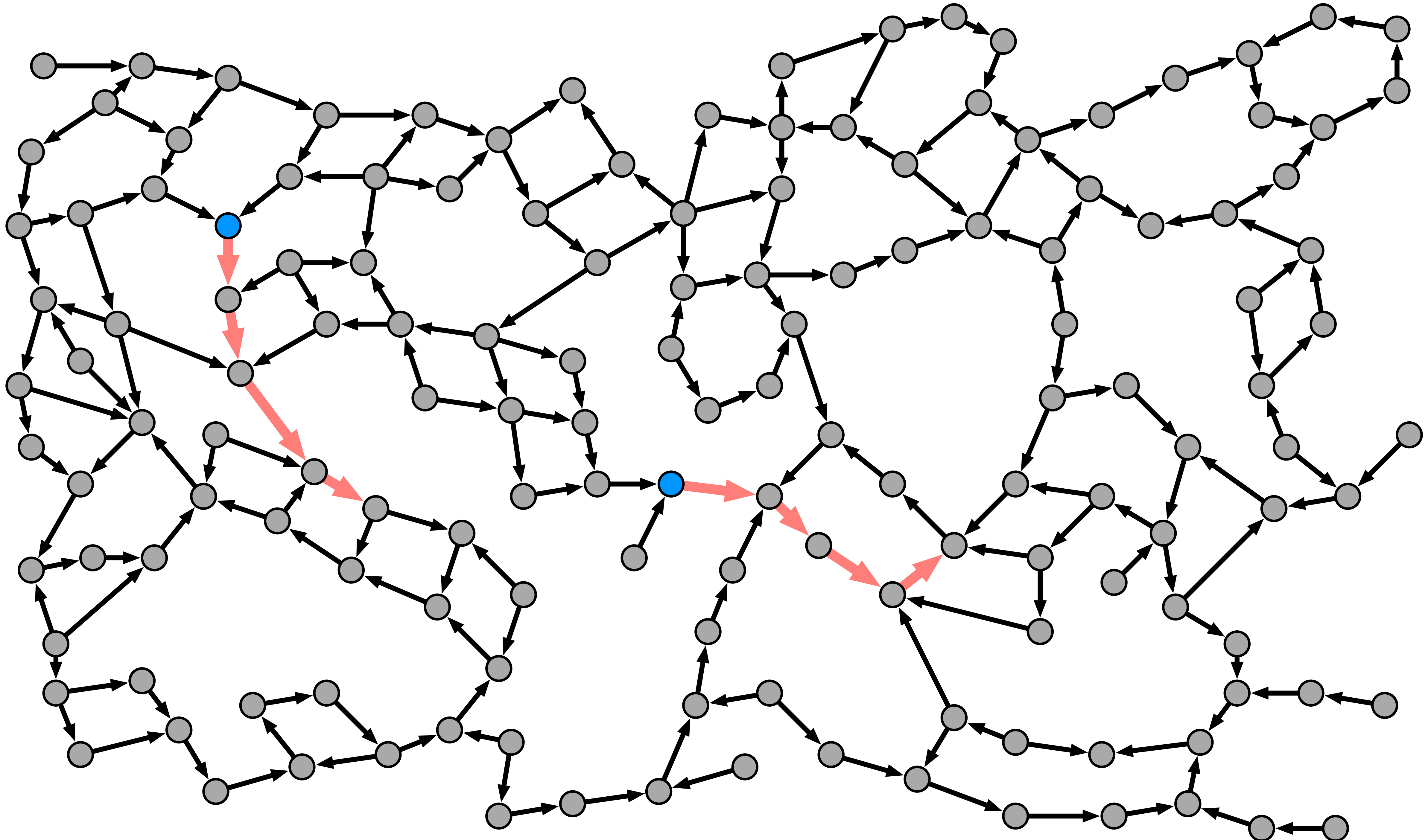














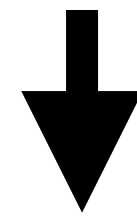
# Best possible?

How **local** can distributed algorithms be?

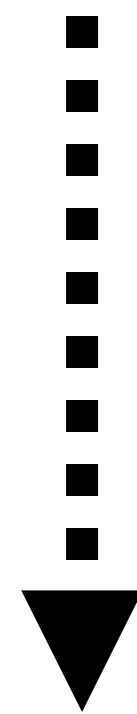
- Proposal algorithm is optimal (FOCS 2019 best paper)  
Balliu, Brandt, Hirvonen, Olivetti, Rabie, and Suomela,  
[arXiv:1901.02441](https://arxiv.org/abs/1901.02441)
- Sinkless orientation is hard (STOC 2016)  
Brandt, Fischer, Hirvonen, Keller, Lempiäinen, Rybicki, Suomela, and Uitto  
[arXiv:1511.00900](https://arxiv.org/abs/1511.00900)

# Speedup simulation

Assume problem  $\Pi_0$ : can be solved in  $T$  rounds



*There exists* problem  $\Pi_1$ : can be solved in  $T-1$  rounds



mechanical transform

*There exists* problem  $\Pi_T$ : can be solved in  $0$  rounds

Contradiction?

# 2016: state of the art

holds, and otherwise that the coin

Observe that  $c \in C(u)$  if and only if  $A_c(u) = N^t(e)$ . Let  $\mathcal{E}_c$  be the event that  $c \in C(u)$ .

**Lemma 18.** *Let  $\mathcal{E}_c$  be the event that  $c \in C(u)$ . Then  $\Pr[\mathcal{E}_c] \leq K$ .*

*Proof.* Let  $\mathcal{E}_c$  be the event that  $c \in C(u)$ . Then  $\Pr[\mathcal{E}_c] \leq K$ .

Since by Lemma 18,  $\Pr[\mathcal{E}_c] \leq K$ .

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**Definition 17.** *A sinkless orientation algorithm  $B$  is said to be  $\ell$ -sinkless if for every node  $u$ , the probability that  $B$  produces a sink at  $u$  is at most  $\ell$ .*

That is,  $\Pr[u \text{ is a sink}] \leq \ell$ .

**Lemma 19.** *Let  $B$  be an  $\ell$ -sinkless orientation algorithm that runs in time  $t$  such that the probability that  $B$  produces a forbidden configuration at any edge  $e$  is at most  $p$ . Then  $B$  is  $\ell$ -sinkless.*

*Proof.* Let  $B$  be an  $\ell$ -sinkless orientation algorithm that runs in time  $t$  such that the probability that  $B$  produces a forbidden configuration at any edge  $e$  is at most  $p$ . Then  $B$  is  $\ell$ -sinkless.

As the probability that  $B$  produces a sink at  $u$  is at most  $\ell$ , and we have  $\Pr[u \text{ is a sink}] \leq \ell$ .

which completes the proof.

that there is a sink at  $u$  or for which  $B$  produces a forbidden configuration.

Our strategy is to show that  $B$  is  $\ell$ -sinkless.

**Lemma 19.** *Suppose  $B'$  is a sinkless orientation algorithm that runs in time  $t$  such that the probability that any node  $u$  is a sink is at most  $\ell$ . Then there exists a sinkless colouring algorithm  $B''$  that runs in time  $t-1$  such that the probability for any edge  $e = \{u, v\}$  having a forbidden configuration  $B''(u) = \psi(e) = B''(v)$  is less than  $4\ell^{1/4}$ .*

*Proof.* Let  $B''$  as defined earlier and consider an edge  $e = \{u, v\}$ . If algorithm  $B''$  outputs a forbidden configuration  $B''(u) = \psi(e) = B''(v)$ , then either  $C'(u) \cup C'(v) = \emptyset$  or  $\psi(e) \in C'(u) \cap C'(v)$  holds. We will now bound the probability of both events.

Observe that before fixing any random bits, the probability of having a bad radius- $(t-1)$  neighbourhood is the same for all nodes, as all radius- $(t-1)$  node neighbourhoods are identical. Let  $S = \Pr[N^{t-1}(u) \text{ is bad}]$  be this probability. By union bound and Lemma 18 we get that

$$\begin{aligned} \Pr[C'(u) \cup C'(v) = \emptyset] &\leq \Pr[C'(u) = \emptyset] + \Pr[C'(v) = \emptyset] \\ &\leq \Pr[N^{t-1}(u) \text{ is bad}] + \Pr[N^{t-1}(v) \text{ is bad}] \\ &\leq 2S. \end{aligned}$$

From Lemma 16 we get that

$$\Pr[\psi(e) \in C'(u) \cap C'(v)] \leq 2L.$$

Using the union bound and the above, we get that the probability of a forbidden configuration is

$$\Pr[B''(u) = \psi(e) = B''(v)] \leq 2S + 2L.$$

To prove the claim, observe that from Definition 17 and the assumption that  $B'$  produces a sink at  $u$  with probability at most  $\ell$ , it follows that

$$\ell \geq \Pr[u \text{ is a sink}] \geq \Pr[u \text{ is a sink} \mid N^{t-1}(u) \text{ is bad}] \cdot \Pr[N^{t-1}(u) \text{ is bad}] > SL^3.$$

Therefore,  $\ell > SL^3$ . By setting  $L = \ell^{1/4}$  we get that  $S < \ell^{1/4}$  implying  $2S + 2L < 4\ell^{1/4}$ .  $\square$

**4.3 The Speedup Lemma**

The following is an immediate consequence of Lemma 14 and Lemma 19.

**Lemma 20.** *Suppose  $B$  is a sinkless colouring algorithm that runs in time  $t$  such that for any edge  $e$  the probability that  $B$  produces a forbidden configuration at  $e$  is at most  $p$ . Then there is a sinkless colouring algorithm  $B'$  that runs in  $t-1$  rounds such that it produces a forbidden configuration at any edge with probability less than  $4 \cdot 6^{1/4} \cdot p^{1/12}$ .*

*Proof.* Fix the coin flips in  $N^{t-1}(u)$  and assume  $N^{t-1}(u)$  is nice. For the sake of contradiction, suppose  $C'(u) = \emptyset$ . Now by definition of  $C'(u)$  we have

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# 2016: state of the art

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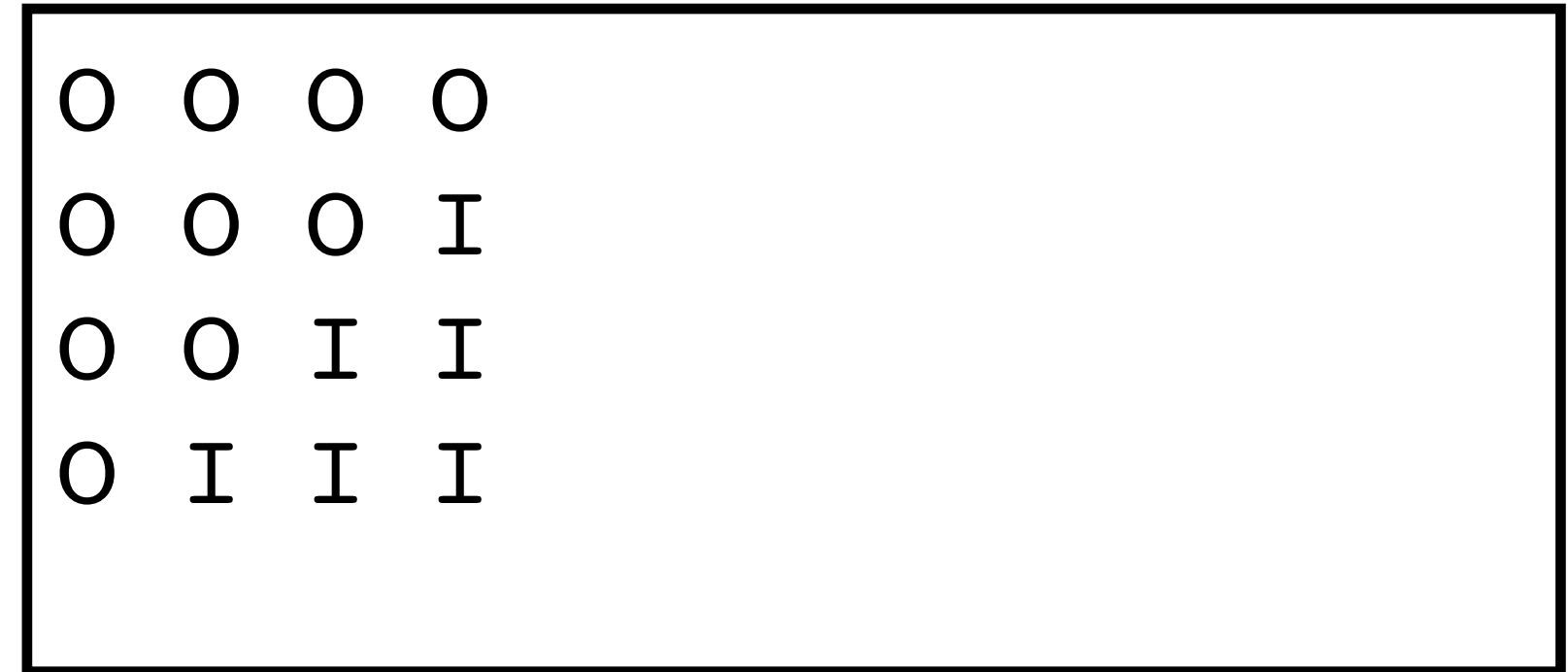
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$C^t(u) = \{\psi(e) : \Pr[B^t(e) = u \leftarrow v \mid N^{t-1}(u)] \leq L\},$

# 2019: state of the art



speedup

intermediate complexity,  $\Omega(\log n)$