Exercise Round 6

The deadline of this exercise round is **December 8, 2016**. The solutions will be gone through during the exercise session at 14:15-16:00. The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

The exercise markings should be posted to the ”Exercise markings for session” forum on the MyCourses web page by 14:15 on the exercise day. Alternatively you can return PDF solutions to the ”Mailbox for pre-exercise reports” on the MyCourses page before 14:15 on the exercise day.

Exercise 1 (Kalman filter and RTS smoother for OU)

Consider the Ornstein–Uhlenbeck model

$$\begin{aligned} \mathrm{d}x &= -\lambda\, x\, \mathrm{d}t + \mathrm{d}\beta, \\ y_k &= x(t_k) + \varepsilon_k, \end{aligned} \tag{1}$$

with $\lambda = 1/2$, $q = 1$, $x(0) \sim \mathrm{N}(0, P_\infty)$, $\varepsilon_k \sim \mathrm{N}(0,1)$, where P_∞ is the stationary variance of the SDE.

(a) Simulate data from the model by using Euler–Maruyama with step size $\Delta t = 1/100$ over the time period $[0, 10]$, and generate noisy measurements only at the time steps $t_j = j$, for $j = 1, 2, \ldots, 10$.

(b) Implement a Kalman filter to the model. Plot the simulated data, the observed values, and the filter mean in the same figure.

(c) Implement an RTS smoother to the problem. Plot the simulated data, the observed values, and the smoother mean in the same figure.

(d) How would you compute the smoothing solution at an arbitrary t?

Exercise 2 (Continuous-time filtering)

(a) Write down the Kushner–Stratonovich equation for the model

$$\begin{aligned} \mathrm{d}x &= -\lambda\, x\, \mathrm{d}t + \mathrm{d}\beta, \\ \mathrm{d}y &= x\, \mathrm{d}t + \mathrm{d}\eta, \end{aligned} \tag{2}$$

where β and η are independent standard Brownian motions.

(b) Write down the corresponding Zakai equation.

(c) Write down the Kalman–Bucy filter for the model.

(d) Show that the filters in (a), (b), and (c) are equivalent.

Exercise 3 (Continuous-time approximate non-linear filtering)

Consider the model

$$\begin{aligned}\mathrm{d}x &= \tanh(x)\,\mathrm{d}t + \mathrm{d}\beta, \\ \mathrm{d}y &= \sin(x)\,\mathrm{d}t + \mathrm{d}\eta,\end{aligned} \tag{3}$$

where β and η are independent Brownian motions with diffusions $Q = 1$ and $R = 0.01$, respectively.

(a) Write down the extended Kalman–Bucy filter for this model.

(b) Simulate data from the model over a time span $[0, 5]$ with $\Delta t = 1/100$, and try implementing the filtering method numerically. How does it work?