Exercise Round 5

The deadline of this exercise round is **December 1, 2016**. The solutions will be gone through during the exercise session at 14:15-16:00. The problems should be *solved before the exercise session*, and during the session those who have completed the exercises may be asked to present their solutions on the board/screen.

The exercise markings should be posted to the "Exercise markings for session" forum on the MyCourses web page by 14:15 on the exercise day. Alternatively you can return PDF solutions to the "Mailbox for pre-exercise reports" on the MyCourses page before 14:15 on the exercise day.

Exercise 1 (A strong stochastic Runge–Kutta method)

Consider a simple strong order 1.0 method with the following extended Butcher tableau (see the lecture notes for details):

$$
\begin{array}{c|ccc|ccc|ccc}
0 & & & & & & & & & \\
0 & 0 & & & 0 & & & & & \\
0 & 0 & 0 & & 0 & 0 & & & & \\
\hline
0 & & & & & & & & & \\
0 & 0 & & & 1 & & & & & \\
0 & 0 & 0 & & -1 & 0 & & & & \\
\hline
 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2}
\end{array}
\tag{1}
$$

(a) Write down the iteration equations required for evaluating the method corresponding to the table in Equation (1).

(b) Consider the Duffing van der Pol oscillator model:

$$
\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1\,(\alpha - x_1^2) - x_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ x_1 \end{pmatrix} \mathrm{d}\beta, \tag{2}
$$

where $\beta(t)$ is a one-dimensional Brownian motion ($q = 0.5^2$) and $\alpha = 1$. Use the method you just constructed for drawing trajectories starting from $x_2(0) = 0$ and $x_1(0) = -4, -3.9, \ldots, -2$. Use a time span $[0, 10]$. Plot the results in the (x_1, x_2) plane.

(c) Experiment with different step sizes $\Delta t = 2^{-k}, k = 0, 2, 4, 6$ and visually compare the trajectories produced by the method implemented in (b) to the Euler–Maruyama scheme.

Exercise 2 (A weak stochastic Runge–Kutta method)

Consider the following two-dimensional SDE:

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}x_1 \\ \frac{3}{2}x_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \frac{1}{10}x_1 & 0 \\ 0 & \frac{1}{10}x_2 \end{pmatrix} \mathrm{d}\boldsymbol{\beta}, \tag{3}$$

where $\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t))$ such that $\beta_i(t)$ is a standard Brownian motion. The initial value is $\mathbf{x}(0) = (1/10\ ,\ 1/10)$.

(a) Implement the Euler–Maruyama scheme for this problem.

(b) Implement the following weak order 2.0 Runge–Kutta method for this problem (following Alg. 6.4 in the lecture notes):

$$\begin{array}{c|ccc|ccc|ccc}
0 & & & & & & & & & \\
\frac{2}{3} & \frac{2}{3} & & & 1 & & & & & \\
\frac{2}{3} & -\frac{1}{3} & 1 & & 0 & 0 & & & & \\
\hline
0 & & & & & & & & & \\
1 & 1 & & & 1 & & & & & \\
1 & 1 & 0 & & -1 & 0 & & & & \\
\hline
0 & & & & & & & & & \\
0 & 0 & & & 1 & & & & & \\
0 & 0 & 0 & & -1 & 0 & & & & \\
\hline
 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{2} \\
\cline{2-10}
 & & & & -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} & -\frac{1}{2}
\end{array}$$

(c) Simulate 1000 trajectories from the SDE with Euler–Maruyama and the weak order 2.0 Runge–Kutta method. Use step sizes $\Delta t = 2^{-k}, k = 0, 1, \ldots, 6$. Compare your results to the expected value given by

$$\mathrm{E}[x_i(t)] = \frac{1}{10} \exp\left(\frac{3}{2}\, t\right)$$

for $i = 1, 2$, and plot the absolute errors as a function of step size.

Exercise 3 (Stochastic flow)

Consider the following SDE ($d = 2, m = 4$) describing stochastic flow on a torus:

$$\mathrm{d}\mathbf{x} = \mathbf{L}(\mathbf{x})\, \mathrm{d}\boldsymbol{\beta},$$

where $\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t))$ such that $\beta_i(t)$ is a standard Brownian motion. The diffusion is given such that the columns in $\mathbf{L}(\mathbf{x})$ are (use

$\alpha = 1$):

$$\mathbf{L}^1(\mathbf{x}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \sin(x_1), \qquad \mathbf{L}^2(\mathbf{x}) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \cos(x_1),$$

$$\mathbf{L}^3(\mathbf{x}) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \sin(x_2), \qquad \mathbf{L}^4(\mathbf{x}) = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \end{pmatrix} \cos(x_2).$$

(a) Consider a set of initial points $\mathbf{x}(0)$ on a uniform 15×15 grid on $[0, 2\pi] \times [0, 2\pi]$. Use the Euler–Maruyama method with the same realization of Brownian motion (reset the random seed) for each trajectory, and a step size of $\Delta t = 2^{-4}$. Plot what the solution looks like at $t = 0.5, 1.0, 2.0, 4.0$ (consider x_i modulo 2π for staying on the torus).

(b) Implement the weak order 2.0 Runge–Kutta scheme presented in the lecture notes (Alg. 6.5), and repeat the above experiment.